On neighborhood and partial sums problem for generalized Sakaguchi type functions

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Abstract In the present investigation, we introduce a new class $k - \tilde{U}\tilde{S}_\eta^\lambda(\mu, \gamma, t)$ of analytic functions in the open unit disc $\mathcal{U}$ with negative coefficients. The object of the present paper is to determine coefficient estimates, neighborhoods and partial sums for functions $f(z)$ belonging to this class.

Keywords analytic function · uniformly starlike function · coefficient estimate · neighborhood problem · partial sums

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1 Introduction

Let $\mathcal{A}$ denote the family of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ of functions that are univalent in $\mathcal{U}$.

For $f \in \mathcal{A}$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

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their convolution (or Hadamard product), denoted by \((f * g)\), is defined as
\[(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z), \quad (z \in U). \tag{1.3}\]

Note that \(f * g \in \mathcal{A}\).

A function \(f \in \mathcal{A}\) is said to be in \(k - \mathcal{US}(\gamma)\), the class of \(k\)-uniformly starlike functions of order \(\gamma\), \(0 \leq \gamma < 1\), if satisfies the condition
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \gamma, \quad (k \geq 0), \tag{1.4}\]
and a function \(f \in \mathcal{A}\) is said to be in \(k - \mathcal{UC}(\gamma)\), the class of \(k\)-uniformly convex functions of order \(\gamma\), \(0 \leq \gamma < 1\), if satisfies the condition
\[
\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| + \gamma, \quad (k \geq 0). \tag{1.5}\]

Uniformly starlike and uniformly convex functions were first introduced by Goodman [8] and then studied by various authors. It is known that \(f \in k - \mathcal{US}(\gamma)\) or \(f \in k - \mathcal{UC}(\gamma)\) if and only if \(1 + \frac{zf''(z)}{f'(z)}\) or \(\frac{zf'(z)}{f(z)}\), respectively, takes all the values in the conic domain \(\mathcal{R}_{k,\gamma}\) which is included in the right half plane given by
\[
\mathcal{R}_{k,\gamma} := \left\{ w = u + iv \in \mathbb{C} : u > k \sqrt{(u - 1)^2 + v^2 + \gamma}, \right. \left. k \geq 0 \text{ and } \gamma \in [0,1) \right\}. \tag{1.6}\]

Denote by \(\mathcal{P}(P_{k,\gamma})\), \((k \geq 0, 0 \leq \gamma < 1)\) the family of functions \(p\), such that \(p \in \mathcal{P}\), where \(\mathcal{P}\) denotes well-known class of Caratheodory functions. The function \(P_{k,\gamma}\) maps the unit disk conformally onto the domain \(\mathcal{R}_{k,\gamma}\) such that \(1 \in \mathcal{R}_{k,\gamma}\) and \(\partial \mathcal{R}_{k,\gamma}\) is a curve defined by the equality
\[
\partial \mathcal{R}_{k,\gamma} := \left\{ w = u + iv \in \mathbb{C} : u^2 = (k \sqrt{(u - 1)^2 + v^2 + \gamma})^2, \right. \left. k \geq 0 \text{ and } \gamma \in [0,1) \right\}. \tag{1.7}\]

From elementary computations we see that (1.7) represents conic sections symmetric about the real axis. Thus \(\mathcal{R}_{k,\gamma}\) is an elliptic domain for \(k > 1\), a parabolic domain for \(k = 1\), a hyperbolic domain for \(0 < k < 1\) and the right half plane \(u > \gamma\), for \(k = 0\).

In [13], Sakaguchi defined the class \(\mathcal{S}_s\) of starlike functions with respect to symmetric points as follows:

Let \(f \in \mathcal{A}\). Then \(f\) is said to be starlike with respect to symmetric points in \(U\) if and only if
\[
\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad (z \in U). \nonumber\]

Recently, Owa et. al. [10] defined the class \(\mathcal{S}_s(\alpha, t)\) as follows:
\[
\Re \left\{ \frac{(1 - t)zf'(z)}{f(z) - f(tz)} \right\} > \alpha, \quad (z \in U), \nonumber\]
∂R_{\lambda, \gamma} \text{ domain}

where \(0 \leq \alpha < 1, |t| \leq 1, t \neq 1\). Note that \(S_s(0, -1) = S_s\) and \(S_s(\alpha, -1) = S_s(\alpha)\) is called Sakaguchi function of order \(\alpha\).

The linear multiplier differential operator \(D_{\lambda, \mu}^n f\) was defined by the authors in (see [11]) as follows:

\[
\begin{align*}
D_{\lambda, \mu}^0 f(z) &= f(z) \\
D_{\lambda, \mu}^1 f(z) &= D_{\lambda, \mu} f(z) = \lambda \mu z^2 f'(z) + (\lambda - \mu) z f(z) + (1 - \lambda + \mu) f(z) \\
D_{\lambda, \mu}^2 f(z) &= D_{\lambda, \mu} \left( D_{\lambda, \mu}^1 f(z) \right) \\
&\vdots \\
D_{\lambda, \mu}^n f(z) &= D_{\lambda, \mu} \left( D_{\lambda, \mu}^{n-1} f(z) \right)
\end{align*}
\]

where \(0 \leq \mu \leq \lambda \leq 1\) and \(\eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\). Later, the operator \(D_{\lambda, \mu}^n f\) was extended for \(\lambda \geq \mu \geq 0\) by the authors in (see [5]).

If \(f\) is given by (1.1) then from the definition of the operator \(D_{\lambda, \mu}^n f(z)\) it is easy to see that

\[
D_{\lambda, \mu}^n f(z) = z + \sum_{n=2}^{\infty} \Phi^n(\lambda, \mu, n) a_n z^n
\]

where

\[
\Phi^n(\lambda, \mu, n) = [1 + (\lambda \mu n + \lambda - \mu)(n-1)]^n.
\]

It should be remarked that the operator \(D_{\lambda, \mu}^n\) is a generalization of many other linear operators considered earlier. In particular, for \(f \in \mathcal{A}\) we have the following:

- \(D_{1, 0}^n f(z) \equiv D^n f(z)\) the operator investigated by SĂLĂGEAN (see [14]).
- \(D_{\lambda, 0}^n f(z) \equiv D^n f(z)\) the operator studied by AL-OBoudI (see [1]).
Now, by making use of the differential operator $D_{\lambda,\mu}^\eta$, we define a new subclass of functions belonging to the class $A$.

**Definition 1.1** A function $f(z) \in A$ is said to be in the class $k-US_\eta^s(\lambda, \mu, \gamma, t)$ if for all $z \in U$,

$$\Re \left\{ \frac{(1-t)z \left( D_{\lambda,\mu}^\eta f(z) \right)'}{D_{\lambda,\mu}^\eta f(z) - D_{\lambda,\mu}^\eta f(tz)} \right\} \geq k \left| \frac{(1-t)z \left( D_{\lambda,\mu}^\eta f(z) \right)'}{D_{\lambda,\mu}^\eta f(z) - D_{\lambda,\mu}^\eta f(tz)} - 1 \right| + \gamma$$

for $\lambda \geq \mu \geq 0$, $\eta, k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$.

Furthermore, we say that a function $f(z) \in k-US_\eta^s(\lambda, \mu, \gamma, t)$ is in the subclass $k-\tilde{US}_s^\eta(\lambda, \mu, \gamma, t)$ if $f(z)$ is of the following form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, \; n \in \mathbb{N}). \quad (1.10)$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k-\tilde{US}_s^\eta(\lambda, \mu, \gamma, t)$.

**Remark 1.1** Through our present investigation, we tacitly assume that the parametric constraints listed (1.9).

### 2 Coefficient bounds of the function class $k-\tilde{US}_s^\eta(\lambda, \mu, \gamma, t)$

Firstly, we shall need the following lemmas.

**Lemma 2.1** Let $w = u + iv$. Then $\Re w \geq \alpha$ if and only if $|w - (1 + \alpha)| \leq |w + (1 - \alpha)|$.

**Lemma 2.2** Let $w = u + iv$ and $\alpha, \gamma$ are real numbers. Then $\Re w > \alpha|w - 1| + \gamma$ if and only if $\Re \{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma$.

**Theorem 2.3** The function $f(z)$ defined by (1.10) is in the class $k-\tilde{US}_s^\eta(\lambda, \mu, \gamma, t)$ if and only if

$$\sum_{n=2}^{\infty} \phi^\eta(\lambda, \mu, n) |n(k+1) - u_n (k+\gamma)| a_n \leq 1 - \gamma,$$

where $\lambda \geq \mu \geq 0$, $\eta, k \geq 0$, $|t| \leq 1$, $t \neq 1$, $0 \leq \gamma < 1$, $u_n = 1 + t + ... + t^{n-1}$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{1 - \gamma}{\phi^\eta(\lambda, \mu, n) |n(k+1) - u_n (k+\gamma)|} z^n.$$
Proof. By Definition 1.1, we get
\[
\Re \left\{ \frac{(1-t)z \left( D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} \right\} \geq k \left| \frac{(1-t)z \left( D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} - 1 \right| + \gamma.
\]

Then by Lemma 2.2, we have
\[
\Re \left\{ \frac{(1-t)z \left( D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} \left( 1 + ke^{i\theta} \right) - ke^{i\theta} \right\} \geq \gamma, \quad -\pi < \theta \leq \pi
\]
or equivalently
\[
\Re \left\{ \frac{(1-t)z \left( D_{\lambda,\mu}^n f(z) \right)'(1+ke^{i\theta}) - ke^{i\theta}[D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)]}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} \right\} \geq \gamma. \quad (2.2)
\]

Let \( F(z) = (1-t)z(D_{\lambda,\mu}^n f(z))'(1+ke^{i\theta}) - ke^{i\theta}[D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)] \) and \( E(z) = D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz) \). By Lemma 2.1, (2.2) is equivalent to
\[
|F(z) + (1-\gamma)E(z)| \geq |F(z) - (1+\gamma)E(z)|, \text{ for } 0 \leq \gamma < 1.
\]

But
\[
|F(z) + (1-\gamma)E(z)| = \left| (1-t) \left\{ (2-\gamma)z - \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)(n+u_n(1-\gamma))a_n z^n \right. \right.
\]
\[
\left. -ke^{i\theta} \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)(n-u_n) a_n z^n \right\} \right|
\]
\[
\geq |1-t| \left\{ (2-\gamma)|z| - \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)|n+u_n(1-\gamma)|a_n |z|^n \right.
\]
\[
\left. -k \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)|n-u_n|a_n |z|^n \right\}.
\]

Also
\[
|F(z) - (1+\gamma)E(z)| = \left| (1-t) \left\{ -\gamma z - \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)(n-(1+\gamma)u_n)a_n z^n \right. \right.
\]
\[
\left. -ke^{i\theta} \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)(n-u_n) a_n z^n \right\} \right|
\]
\[
\leq |1-t| \left\{ \gamma|z| + \sum_{n=2}^{\infty} \Phi^\eta(\lambda,\mu,n)|n-u_n(1+\gamma)|a_n |z|^n \right\}.
\]
\[ +k \sum_{n=2}^{\infty} \Phi^\eta(\lambda, \mu, n) |n-u_n| a_n |z|^n \}\]

and so

\[ |F(z) + (1-\gamma)E(z)| - |F(z) - (1+\gamma)E(z)| \geq |1-t| \left\{ 2(1-\gamma) |z| - \sum_{n=2}^{\infty} \Phi^\eta(\lambda, \mu, n) \left[ n + u_n (1-\gamma) \right] 
+ |n-u_n (1+\gamma)| + 2k |n-u_n| a_n |z|^n \right\} \]

\[ \geq 2(1-\gamma) |z| - \sum_{n=2}^{\infty} 2\Phi^\eta(\lambda, \mu, n) n(k+1) - u_n (k+\gamma) |a_n |z|^n \geq 0 \]

or

\[ \sum_{n=2}^{\infty} \Phi^\eta(\lambda, \mu, n) n(k+1) - (k+\gamma) u_n |a_n | \leq 1 - \gamma. \]

Conversely, suppose that (2.1) holds. Then we must show

\[ \Re \left\{ \left( 1-t \right) z \left( D^n_{\lambda,\mu} f(z) \right)' (1+k e^{i\theta}) - k e^{i\theta} \left[ D^n_{\lambda,\mu} f(z) - D^n_{\lambda,\mu} f(tz) \right] \right\} \geq \gamma. \]

Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), the above inequality reduces to

\[ \Re \left\{ \left( 1-\gamma \right) - \sum_{n=2}^{\infty} \Phi^\eta(\lambda, \mu, n) \left( n(1+k e^{i\theta}) - u_n (\gamma+k e^{i\theta}) \right) a_n r^{n-1} \right\} \geq 0. \]

Since \( \Re (-e^{i\theta}) \geq -|e^{i\theta}| = -1 \), the above inequality reduces to

\[ \Re \left\{ \left( 1-\gamma \right) - \sum_{n=2}^{\infty} \Phi^\eta(\lambda, \mu, n) \left( n(1+k) - u_n (\gamma+k) \right) a_n r^{n-1} \right\} \geq 0. \]

Letting \( r \to 1^- \), we have desired conclusion. \( \square \)

**Corollary 2.4** If \( f(z) \in k-\mathcal{US}_s^\eta(\lambda, \mu, t) \), then

\[ a_n \leq \frac{1-\gamma}{\Phi^\eta(\lambda, \mu, n) |n(k+1) - u_n (k+\gamma)|}, \]

where \( \lambda \geq \mu \geq 0, \eta, k \geq 0, |t| \leq 1, t \neq 1, 0 \leq \gamma < 1, u_n = 1 + t + ... + t^{n-1}. \)
3 Neighborhood of the function class $k - \tilde{US}_s^n(\lambda, \mu, \gamma, t)$

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by GOODMAN [7], RUSCHEWEYH [12], ALTINTAŞ ET AL. ([2], [3]) and others including SRIVASTAVA ET AL. ([17], [18], [19]), ORHAN [9], DENIZ ET AL. [6], CATAŞ [4].

**Definition 3.1** Let $\lambda \geq \mu \geq 0, \eta, k \geq 0, |t| \leq 1$, $t \neq 1, 0 \leq \gamma < 1, \alpha \geq 0$, $u_n = 1 + t + \ldots + t^{n-1}$. We define the $\alpha$–neighborhood of a function $f \in A$ and denote by $\mathcal{N}_a(f)$ consisting of all functions $g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in S \ (b_n \geq 0, n \in \mathbb{N})$ satisfying

$$
\sum_{n=2}^{\infty} \frac{\Phi^\eta(\lambda, \mu, n)}{1 - \gamma} |n(k+1) - u_n (k+\gamma)| |a_n - b_n| \leq \alpha.
$$

**Theorem 3.2** Let $f \in k - \tilde{US}_s^n(\lambda, \mu, \gamma, t)$ and for all real $\theta$ we have $\gamma(e^{i\theta} - 1) - 2e^{i\theta} \neq 0$. For any complex number $\varepsilon$ with $|\varepsilon| < \alpha \ (\alpha \geq 0)$, if $f$ satisfies the following condition

$$
\frac{(1-t)z(D^\eta_{\lambda,\mu}f(z))'(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)(D^\eta_{\lambda,\mu}f(z) - D^\eta_{\lambda,\mu}f(tz))}{(1-t)z(D^\eta_{\lambda,\mu}f(z))'(1 + ke^{i\theta}) + (1 - ke^{i\theta} - \gamma)(D^\eta_{\lambda,\mu}f(z) - D^\eta_{\lambda,\mu}f(tz))} < 1,
$$

$(-\pi < \theta < \pi)$ for any complex number $s$ with $|s| = 1$, we have

$$
\frac{(1-t)z(D^\eta_{\lambda,\mu}f(z))'(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma)(D^\eta_{\lambda,\mu}f(z) - D^\eta_{\lambda,\mu}f(tz))}{(1-t)z(D^\eta_{\lambda,\mu}f(z))'(1 + ke^{i\theta}) + (1 - ke^{i\theta} - \gamma)(D^\eta_{\lambda,\mu}f(z) - D^\eta_{\lambda,\mu}f(tz))} \neq s.
$$

In other words, we must have

$$
(1-s)(1-t)z(D^\eta_{\lambda,\mu}f(z))'(1 + ke^{i\theta}) - (ke^{i\theta} + 1 + \gamma + s(ke^{i\theta} - 1 + \gamma))(D^\eta_{\lambda,\mu}f(z) - D^\eta_{\lambda,\mu}f(tz)) \neq 0
$$

which is equivalent to

$$
z - \sum_{n=2}^{\infty} \frac{\Phi^\eta(\lambda, \mu, n)((n-u_n)(1+ke^{i\theta}-ske^{i\theta}) - s(n+u_n) - u_n \gamma(1-s))}{\gamma(s-1) - 2s} z^n \neq 0. \text{ However, } f \in k - \tilde{US}_s^n(\lambda, \mu, \gamma, t) \text{ if and only if } \frac{f(z)}{h(z)} = 0, z \in U - \{0\} \text{ where } h(z) =
$$

$$
z - \sum_{n=2}^{\infty} c_n z^n, \text{ and } c_n = \frac{\Phi^\eta(\lambda, \mu, n)((n-u_n)(1+ke^{i\theta}-ske^{i\theta}) - s(n+u_n) - u_n \gamma(1-s))}{\gamma(s-1) - 2s}
$$

we note that

$$
|c_n| \leq \frac{\Phi^\eta(\lambda, \mu, n)|n(1+k) - u_n(k+\gamma)|}{1 - \gamma}
$$
since \( \frac{f(z) + \varepsilon z}{1 + \varepsilon} \in k - \mathcal{US}^n(\lambda, \mu, \gamma, t) \), therefore \( z^{-1} \left( \frac{f(z) + \varepsilon z}{1 + \varepsilon} \right) \neq 0 \), which is equivalent to

\[
\frac{(f * h)(z)}{(1 + \varepsilon)z} + \frac{\varepsilon}{1 + \varepsilon} \neq 0.
\] (3.1)

Now suppose that \( \frac{|(f * h)(z)|}{z} < \alpha \). Then by (3.1), we must have

\[
\left| \frac{(f * h)(z) + \varepsilon}{(1 + \varepsilon)z} \right| \geq \frac{\varepsilon}{1 + \varepsilon} - \frac{1}{1 + \varepsilon} \left| \frac{(f * h)(z)}{z} \right| > |\varepsilon| - \alpha \geq 0,
\]

this is a contradiction by \( |\varepsilon| < \alpha \) and however, we have \( \frac{|(f * h)(z)|}{z} \geq \alpha \). If \( g(z) = z - \sum_{n=2}^{\infty} b_n z^n \in \mathcal{N}_{\alpha}(f) \), then

\[
\alpha - \left| \frac{(g * h)(z)}{z} \right| \leq \left| \frac{((f - g) * h)(z)}{z} \right| \leq \sum_{n=2}^{\infty} |a_n - b_n| \left| \frac{c_n}{z^n} \right| < \sum_{n=2}^{\infty} \Phi^\eta(\lambda, \mu, n) \left| \frac{n(n + 1) - u_n(k + \gamma)}{1 - \gamma} \right| |a_n - b_n| \leq \alpha.
\]

\[\square\]

4 Partial sums of the function class \( k - \mathcal{US}^n(\lambda, \mu, \gamma, t) \)

In this section, applying methods used by Silverman [15] and Silvia [16], we investigate the ratio of a function of the form (1.10) to its sequence of partial sums \( f_m(z) = z + \sum_{n=2}^{m} a_n z^n \).

**Theorem 4.1** If \( f \) of the form (1.1) satisfies the condition (2.1), then

\[
\mathfrak{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{\delta_{m+1}}
\] (4.1)

and

\[
\delta_n \geq \begin{cases} 1, & n = 2, 3, \ldots, m \\ \delta_{m+1}, & n = m + 1, m + 2, \ldots \end{cases}
\] (4.2)

where

\[
\delta_{m+1} = \left| \frac{\Phi^\eta(\lambda, \mu, k) |n(k + 1) - u_n(k + \gamma)|}{1 - \gamma} \right|.
\] (4.3)

The result in (4.1) is sharp for every \( m \), with the extremal function

\[
f(z) = z + \frac{z^{m+1}}{\delta_{m+1}}.
\] (4.4)
Proof. Define the function \( w(z) \), we may write
\[
\begin{align*}
\frac{1 + w(z)}{1 - w(z)} &= \delta_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left( 1 - \frac{1}{\delta_{m+1}} \right) \right\} \\
&= \left\{ \frac{1 + \sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{m} a_n z^{n-1}} \right\}.
\end{align*}
\]
(4.5)

Then, from (4.5) we can obtain
\[
w(z) = \frac{\delta_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^{m} a_n z^{n-1} + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}.
\]
and
\[
|w(z)| \leq \frac{\delta_{m+1} \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^{m} a_n - \delta_{m+1} \sum_{n=m+1}^{\infty} a_n}.
\]

Now \(|w(z)| \leq 1\) if
\[
2 \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 2 - 2 \sum_{n=2}^{m} a_n,
\]
which is equivalent to
\[
\sum_{n=2}^{m} a_n + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 1.
\]
(4.6)

It is suffices to show that the left hand side of (4.6) is bounded above by \( \sum_{n=2}^{\infty} \delta_n a_n \), which is equivalent to
\[
\sum_{n=2}^{m} (\delta_n - 1) a_n + \sum_{n=m+1}^{\infty} (\delta_n - \delta_{m+1}) a_n \geq 0.
\]

To see that the function given by (4.4) gives the sharp result, we observe that for \( z = r e^{i\pi/n} \),
\[
\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{\delta_{m+1}}.
\]
(4.7)

Taking \( z \to 1^- \), we have
\[
\frac{f(z)}{f_m(z)} = 1 - \frac{1}{\delta_{m+1}}.
\]

This completes the proof of Theorem 4.1.

We next determine bounds for \( f_m(z)/f(z) \).

**Theorem 4.2** If \( f \) of the form (1.1) satisfies the condition (2.1), then
\[
\Re \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{\delta_{m+1}}{1 + \delta_{m+1}}.
\]
(4.8)
The result is sharp with the function given by (4.4).
Proof. We may write
\[
\frac{1 + w(z)}{1 - w(z)} = (1 + \delta_{m+1}) \left\{ f_m(z) - \frac{\delta_{m+1}}{1 + \delta_{m+1}} f(z) \right\} = \left\{ \frac{1 + \sum_{n=2}^{m} a_n z^{n-1} - \delta_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right\}, \tag{4.9}
\]
where
\[
w(z) = \frac{(1 + \delta_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^{m} a_n z^{n-1} - (1 - \delta_{m+1}) \sum_{n=m+1}^{\infty} a_n z^{n-1}},
\]
and
\[
|w(z)| \leq \frac{(1 + \delta_{m+1}) \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^{m} a_n + (1 - \delta_{m+1}) \sum_{n=m+1}^{\infty} a_n} \leq 1. \tag{4.10}
\]
This last inequality is equivalent to
\[
\sum_{n=2}^{m} a_n + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 1. \tag{4.11}
\]
It is suffices to show that the left hand side of (4.11) is bounded above by \(\sum_{n=2}^{\infty} \delta_n a_n\), which is equivalent to
\[
\sum_{n=2}^{m} (\delta_n - 1) a_n + \sum_{n=m+1}^{\infty} (\delta_n - \delta_{m+1}) a_n \geq 0.
\]
This completes the proof of Theorem 4.2. \(\square\)

We next turn to ratios involving derivatives.

**Theorem 4.3** If \(f\) of the form (1.1) satisfies the condition (2.1), then
\[
\Re \left\{ \frac{f'(z)}{f(m)(z)} \right\} \geq 1 - \frac{m + 1}{\delta_{m+1}}, \tag{4.12}
\]
\[
\Re \left\{ \frac{f_m'(z)}{f'(z)} \right\} \geq \frac{\delta_{m+1}}{1 + m + \delta_{m+1}} \tag{4.13}
\]
where
\[
\delta_n \geq \begin{cases} 1, & n = 1, 2, 3, ..., m \\ n \frac{\delta_{m+1}}{m+1}, & n = m + 1, m + 2, ...
\end{cases}
\]
and \(\delta_n\) is defined by (4.3). The estimates in (4.12) and (4.13) are sharp with the extremal function given by (4.4).
Proof. Firstly, we will give proof of (4.12). We write
\[
\frac{1 + w(z)}{1 - w(z)} = \frac{\delta_{m+1}}{1 + \sum_{n=2}^{m+1} an z^{n-1}}
\]
where
\[
w(z) = \frac{\delta_{m+1}}{2 + 2 \sum_{n=2}^{m+1} an z^{n-1}}.
\]
and
\[
|w(z)| \leq \frac{\delta_{m+1}}{2 - 2 \sum_{n=2}^{m+1} an}.
\]
Now \(|w(z)| \leq 1\) if and only if
\[
\sum_{n=2}^{m} an + \frac{\delta_{m+1}}{m+1} \sum_{n=m+1}^{\infty} an \leq 1,
\]
(4.14)
since the left hand side of (4.14) is bounded above by is bounded above by \(\sum_{n=2}^{\infty} \delta_n a_n\).
The proof of (4.13) follows the pattern of that in Theorem (4.2).
This completes the proof of Theorem 4.3. \(\square\)

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References
Proof. Firstly, we will give proof of (4.12). We write

\[ 1 + w(z). \]

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