WPZI rings and strong regularity

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Received: 21.I.2013 / Revised: 12.VI.2013 / Accepted: 13.VI.2013

Abstract In this paper, we study the strong regularity of left $SF$ rings and obtain the following results: Let $R$ be a left $SF$ ring. If $R$ satisfies one of the following conditions, then $R$ is a strongly regular ring: 1) $R$ is a left WPZI ring; 2) $R$ is a right WPZI ring; 3) $R$ is a right weakly semicommutative ring; 4) $R$ is a semicommutative ring; 5) $R$ is a reversible ring.

Keywords WPZI rings · $SF$ rings · strongly regular rings · semicommutative rings · reduced rings · weakly semicommutative rings

Mathematics Subject Classification (2010) 16A30 · 16A50 · 16E50 · 16D30

Introduction

All rings considered in this paper are associative rings with identity, and all modules are unital. The symbols $N(R)$, $Z_l(R)$ and $Z_r(R)$ will stand respectively for the set of all nilpotent elements, left and right singular ideal of $R$. For any nonempty subset $X$ of $R$, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the set of right annihilators of $X$ and the set of left annihilators of $X$, respectively. In particular, if $X = \{a\}$, we write $l(X) = l(a)$ and $r(X) = r(a)$.

A ring $R$ is called (von Neumann) regular (cf. Goodearl [2]) if for every $a \in R$ there exists $b \in R$ such that $a = aba$. A ring $R$ is strongly regular (cf. Rege [6]) if for every $a \in R$ there exists $b \in R$ such that $a = a^2b$. A ring $R$ is called reduced (cf. Ramamurthi [5]) if $R$ has no nonzero nilpotent elements. It is well known that $R$ is a strongly regular ring if and only if $R$ is a reduced regular ring. A ring $R$ is called MELT (resp., MERT) if every maximal essential left (resp., right) ideal of $R$ is an ideal. According to Ramamurthi [5], a ring $R$ is called left (resp., right) $SF$ if each simple left (resp., right) $R$–module is flat. It is known that regular rings are left and right $SF$ rings. Ramamurthi [5] initiated the study of $SF$ rings and the question whether an $SF$ ring is necessarily regular. For several years, $SF$ rings have
been studied by many authors and the regularity of SF rings which satisfy certain additional conditions is showed (cf. Ramamurthi [5], Rege [6], Yue Chi Ming [9–11], Zhang and Du [14,15], Zhang [12,13], Zhou and Wang [17,18], Zhou [19]). But the question remains open. Yue Chi Ming [11] proved the strong regularity of right SF rings whose complement left ideals are ideals, and he proposed the following question: Is \( R \) strongly regular if \( R \) is a left SF rings whose complement left ideals are ideals? Zhang and Du [14] affirmatively answered the question. Zhou and Wang [17] proved that if \( R \) is a right SF rings whose all maximal essential right ideals are GW–ideals, then \( R \) is a regular ring. Zhang [13] proved that if \( R \) is an MELT and right SF rings, then \( R \) is a regular ring. Zhou [19] proved that if \( R \) is a left SF rings whose complement left (right) ideals are W–ideals, then \( R \) is a strongly regular ring.

Following Zhou and Wang [17], a left ideal \( L \) of a ring \( R \) is called GW–ideal, if for any \( a \in L \), there exists a positive integer \( n \) such that \( a^n R \subseteq L \). Clearly, every ideal is GW–ideal, but the converse is not true, in general, by Zhou and Wang ([17], Example 1.2).

According to Zhou [19], a left ideal \( L \) of a ring \( R \) is called a weakly ideal (W–ideal), if for any \( 0 \neq a \in L \), there exists \( n \geq 1 \) such that \( a^n \neq 0 \) and \( a^n R \subseteq L \). A right ideal \( K \) of a ring \( R \) is defined similarly to be a weakly ideal. Clearly, ideals are W–ideals and W–ideals are GW–ideals, but the converses are not true, in general, by Zhou [19].

According to Cohn [1], a ring \( R \) is called reversible if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \), and \( R \) is said to be semicommutative (Zhao and Yang [16]) if \( ab = 0 \) implies \( aRb = 0 \).

A ring \( R \) is called left (resp., right) WPZI if for any \( 0 \neq a \in R \), there exists \( n \geq 1 \) such that \( a^n \neq 0 \) and \( l(a^n) \) (resp., \( r(a^n) \)) is a W–ideal of \( R \).

Clearly, semicommutative rings are left and right WPZI rings.

The first purpose of this paper is to study the (strong) regularity of left SF–rings in terms of WPZI rings. We obtain the following main results:

1) Left WPZI left SF–rings are strongly regular;
2) Right WPZI left SF–rings are strongly regular.

So some known results appeared in Rege [6] are extended.

1 Some properties of WPZI rings

According to Hwang, Jeon and Park [3], a ring \( R \) is called NCI if \( N(R) = 0 \) or there exists a nonzero ideal of \( R \) contained in \( N(R) \). Clearly, NI rings (that is, \( N(R) \) forms an ideal of \( R \)) are NCI, but the converse is not true, in general, by Hwang, Jeon and Park [3].

Following Wei and Chen [8], left \( R \)–module \( M \) is called nil–injective if for any \( a \in N(R) \), every left \( R \)–homomorphism \( Ra \) to \( M \) extends to \( R \). Evidently, \( YJ \)–injective modules are nil–injective, but the converse is not true, in general, by Wei and Chen [8].

**Proposition 1.1** (1) Left or right WPZI rings are Abelian.
(2) Left or right WPZI rings are NCI.
(3) Let \( R \) be a left or right WPZI ring. If every singular simple left \( R \)–module is nil–injective, then \( R \) is a reduced ring.
(4) If \( R \) is a left or right WPZI ring, then \( N_2(R) = \{ a \in R | a^2 = 0 \} \subseteq P(R) \).
Proof. (1) Let $R$ be a left WPZI ring and $e \in E(R)$. Then there exists $n \geq 1$ such that $l(e) = l(e^n)$ is a $W$–ideal of $R$. Since $1 - e \in l(e)$, there exists $m \geq 1$ such that $(1 - e)^m \neq 0$ and $(1 - e)^m R \subseteq l(e)$. Therefore we obtain $(1 - e)Re = 0$ for each $e \in E(R)$, so $R$ is an Abelian ring. Similarly, we can show that right WPZI rings are Abelian.

(2) If $N(R) \neq 0$, then there exists $0 \neq a \in N(R)$. Let $n \geq 1$ be such that $a^n = 0$ and $a^{n-1} \neq 0$. Since $R$ is a left WPZI ring, there exists $m \geq 1$ such that $a^m \neq 0$ and $l(a^m)$ is a $W$–ideal. Clearly, $n > m$ and $0 \neq a^{n-m} \in l(a^m)$. Since $l(a^m)$ is a $W$–ideal, there exists $l \geq 1$ such that $(a^{n-m})^l \neq 0$ and $(a^{n-m})^l R \subseteq l(a^m)$. If $(n - m)l \geq m$, then $Ra^{(n-m)l}$ is a nonzero nilpotent ideal of $R$. If $(n - m)l < m$, then $Ra^{(n-m)l}$ is a nonzero nilpotent ideal of $R$. Hence $R$ is a $NCI$ ring.

Similarly, we can show that right WPZI rings are $NCI$.

(3) Let $a^2 = 0$. If $a \neq 0$, then there exists a maximal left ideal $M$ of $R$ such that $l(a) \subseteq M$. If $M$ is not essential in $R$, then $M = l(e)$ for some $e \in E(R)$. Thus $ae = 0$ because $a \in l(a) \subseteq M$. By (1), $R$ is an Abelian ring, so $ea = 0$. This gives $e \in l(a) \subseteq l(e)$, a contradiction. Hence $M$ is an essential left ideal of $R$, so $R/M$ is a singular simple left $R$–module. By hypothesis, $R/M$ is a nil–injective left $R$–module. Let $f : Ra \rightarrow R/M$ defined by $f(ra) = r + M$. Then $f$ is a well defined left $R$–homomorphism, so there exists a left $R$–homomorphism $g : R \rightarrow R/M$ such that $g(a) = f(a)$. Hence there exists $e \in R$ such that $1 + M = f(a) = g(a) = ag(1) = ac + M$. Since $R$ is a left or right WPZI ring, $aRa = 0$. Thus $ac \in l(a) \subseteq M$. This leads to $1 \in M$, which is a contradiction. Hence $a = 0$.

(4) It follows from the proof of (3). \square

A ring $R$ is called directly finite if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is well known that Abelian rings are directly finite. Hence left or right WPZI rings are directly finite by Proposition 1.1. According to HWANG, JEON and PARK [3], $NCI$ rings need not be directly finite. Hence $NCI$ rings need neither be left nor right WPZI.

A ring $R$ is called left $NV$ if every singular simple left $R$–module is nil–injective. Clearly, left $NV$–rings and reduced rings are left $NV$. Since reduced rings are reversible and reversible rings are semicommutative, by Proposition 1.1, we have the following corollary.

**Corollary 1.2** The following conditions are equivalent for a ring $R$:

1. $R$ is a reduced ring;
2. $R$ is a reversible left $NV$ ring;
3. $R$ is a semicommutative left $NV$ ring;
4. $R$ is a left WPZI left $NV$ ring;
5. $R$ is a right WPZI left $NV$ ring.

**Kim, Nam** and **Kim** ([4], Theorem 4) proved that if $R$ is a semicommutative ring whose every simple singular left module is $YJ$–injective, then $R$ is a reduced weakly regular ring. Hence, by Corollary 1.2, we have the following corollary.

**Corollary 1.3** Let $R$ be a left or right WPZI ring. If every singular simple left $R$–module is $YJ$–injective, then $R$ is a reduced weakly regular ring.

**Wei** ([7], Theorem 16) proved that a ring $R$ is a strongly regular ring if and only if $R$ is a semicommutative $MELT$ ring whose singular simple left modules are $YJ$–injective. Hence, by Corollary 1.2, we have the following corollary.
Corollary 1.4 A ring $R$ is a strongly regular ring if and only if $R$ is a MELT left or right WPZI ring whose every singular simple left module is $YJ$–injective.

It is well known that a ring $R$ is a reduced ring if and only if $R$ is a semiprime semicommutative ring. On the other hand, semiprime left (right) WPZI rings are reversible (in fact, if $ab = 0$, then $(ba)^2 = 0$. If $R$ is a left WPZI ring, then $((ba)$ is a $W$–ideal, so $baRba = 0$. Since $R$ is a semiprime ring, $ba = 0$). So, we have the following proposition:

Proposition 1.5 The following conditions are equivalent for a ring $R$:

1. $R$ is a reduced ring;
2. $R$ is a semiprime left WPZI ring;  
3. $R$ is a semiprime right WPZI ring.

2 Strong regularity of $SF$–rings

Rege ([6], Remark 3.13) pointed out that if $R$ is a reduced left (right) $SF$ ring, then $R$ is a strongly regular ring. We can extend this result to right WPZI rings.

Proposition 2.1 Let $R$ be a left $SF$ ring. If $R$ is right WPZI, then $R$ is strongly regular.

Proof. Assume that $a \in R$. If $a = 0$, we are done. If $a \neq 0$, then there exists $n \geq 1$ such that $a^n \neq 0$ and $r(a^n)$ is a $W$–ideal of $R$ because $R$ is a right WPZI ring. If $Ra+r(a^nR) \neq R$, then there exists a maximal left ideal $M$ of $R$ containing $Ra+r(a^nR)$. Since $R$ is a left $SF$ ring, $R/M$ is a flat left $R$–module, so there exists $b \in M$ such that $a = ab$ because $a \in M$. Hence $1 - b \in r(a^n)$. If $1 - b = 0$, then $1 = b \in M$, a contradiction. Therefore $1 - b \neq 0$. Since $r(a^n)$ is a $W$–ideal of $R$, there exists $m \geq 1$ such that $(1 - b)^m \neq 0$ and $R(1 - b)^m \subseteq r(a^n)$. Hence $(1 - b)^m \in r(a^nR)$, which implies $(1 - b)^m \in M$. Since $1 - (1 - b)^m = (1 - (1 - b) + (1 - b)^2 + \cdots + (1 - b)^{m-1})b \in M$, $1 \in M$, which is a contradiction. Hence $Ra+r(a^nR) = R$, which implies $Ra+r(a^n) = R$. Let $1 = ca + x$, where $c \in R$ and $x \in r(a^n)$. So $a^n = a^nca$. Write $d = a^{n-1} - a^{n-1}ca$. Then $d^2 = 0$. If $d \neq 0$, then similar to the proof mentioned above, we have $Rd + r(d) = R$, so there exists $u \in R$ such that $d = dud$. Hence there exists $y \in R$ such that $a^{n-1} = a^{n-1}ya$. If $d = 0$, then $a^{n-1} = a^{n-1}ca$. Repeating the process above, we obtain that $a = ava$ for some $w \in R$. So $R$ is a regular ring. By Proposition 1.1(1), $R$ is an Abelian ring, so $R$ is a strongly regular ring. \[\Box\]

A ring $R$ is called right weakly semicommutative, if for any $a, b \in R$, $ab = 0$ implies $aRb^n = 0$ for some $n \geq 1$. Clearly, semicommutative rings are right weakly semicommutative and right weakly semicommutative rings are Abelian.

Theorem 2.2 If $R$ is a right weakly semicommutative left $SF$ ring, then $R$ is a strongly regular ring.

Proof. Assume that $a \in R$. If $Ra+r(aR) \neq R$, then there exists a maximal left ideal $M$ of $R$ containing $Ra+r(aR)$. Since $R$ is a left $SF$ ring and $R/M$ is a simple left $R$–module, $R/M$ is flat. Hence $a = ab$ for some $b \in M$. Since $R$ is a right weakly semicommutative ring, there exists $n \geq 1$ such that $aR(1-b)^n = 0$, hence $(1-b)^n \subseteq M$. \[\Box\]
Thus $1 \in M$, a contradiction. Therefore $Ra + r(aR) = R$, which implies $a \in aRa$. Therefore $R$ is a regular ring. Since $R$ is an Abelian ring, $R$ is a strongly regular ring. □

Lemma 2.3 If $R$ is a left SF ring and left WPZI ring, then $Z_l(R) = 0$.

Proof. If $Z_l(R) \neq 0$, then there exists $0 \neq a \in Z_l(R)$ such that $a^2 = 0$. If $Z_l(R) + r(aR) \neq R$, then there exists a maximal left ideal $M$ of $R$ containing $Z_l(R) + r(aR)$. Since $R$ is a left SF ring, $R/M$ is a flat left $R$-module. Since $a \in Z_l(R) \subseteq M$, $a = ab$ for some $b \in M$. So $b \neq 1$ and $a \in l(1 - b)$. Since $R$ is a left WPZI ring, there exists $n \geq 1$ such that $(1-b)^n \neq 0$ and $l((1-b)^n)$ is a $W$-ideal of $R$. Hence $aR \subseteq l((1-b)^n)$, which shows that $(1-b)^n \subseteq r(aR) \subseteq M$, which implies $1 \in M$. This contradiction leads to $Z_l(R) + r(aR) = R$. Let $1 = x + y$ where $x \in Z_l(R)$ and $y \in r(aR)$. Thus $ay = 0$ and $a = ax$. Since $x \in Z_l(R)$, $l(1-x) = 0$, which implies $a = 0$, a contradiction. Therefore $Z_l(R) = 0$. □

Lemma 2.4 Let $R$ be a left SF ring. If $R$ is a left WPZI ring, then $R$ is a right weakly semicommutative ring.

Proof. Assume that $a, b \in R$ with $ab = 0$. If $b = 0$, then $aRb = 0$, we are done. If $b \neq 0$, then there exists $n \geq 1$ such that $b^n \neq 0$ and $l(b^n)$ is a $W$-ideal of $R$. If $aRb^n \neq 0$, then $acb^n \neq 0$ for some $c \in R$. By Lemma 2.3, $Z_l(R) = 0$, so there exists a nonzero left ideal $L$ of $R$ such that $L \cap l(acb^n) = 0$. Let $0 \neq x \in L$. Then $xab = 0$ because $ab = 0$. If $xa = 0$, then $xacb^n = 0$, which implies $x \in L \cap l(acb^n)$. Thus $x = 0$, which is a contradiction. Hence $xa \neq 0$. Since $l(b^n)$ is a $W$-ideal of $R$, there exists $m \geq 1$ such that $(xa)^m \neq 0$ and $(xa)^mRb^n = 0$. Hence $(xa)^m-1x \in l(acb^n) \cap L$, so $(xa)^m-1x = 0$, which implies $(xa)^m = 0$, a contradiction. Hence $aRb^n = 0$, we are done. □

Using Theorem 2.2 and Lemma 2.4, we have the following theorem:

Theorem 2.5 If $R$ is a left SF ring and left WPZI ring, then $R$ is a strongly regular ring.

By Theorem 2.5, we obtain the following two corollaries which are generalization of Rege ([6], Remark 3.13).

Corollary 2.6 If $R$ is a left SF ring and semicommutative ring, then $R$ is a strongly regular ring.

Corollary 2.7 If $R$ is a left SF ring and reversible ring, then $R$ is a strongly regular ring.

Acknowledgements Project supported by the Foundation of Natural Science of China (11471282, 11171291) and Natural Science Fund for Colleges and Universities in Jiangsu Province (11KJB110019). I would like to thank the referee for his/her helpful suggestions and comments.

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