Strongly \( J \)-clean skew triangular matrix rings

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Abstract Let \( R \) be an arbitrary ring with identity. An element \( a \in R \) is strongly \( J \)-clean if there exist an idempotent \( e \in R \) and element \( w \in J(R) \) such that \( a = e + w \) and \( ew = ew \). A ring \( R \) is strongly \( J \)-clean in case every element in \( R \) is strongly \( J \)-clean. In this note, we investigate the strong \( J \)-cleanness of the skew triangular matrix ring \( T_n(R, \sigma) \) over a local ring \( R \), where \( \sigma \) is an endomorphism of \( R \) and \( n = 2, 3, 4 \).

Keywords strongly \( J \)-clean ring · skew triangular matrix ring · local ring

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1 Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let \( R \) be a ring. \( J(R) \) and \( U(R) \) will denote, respectively, the Jacobson radical and the group of units in \( R \). An element \( a \in R \) is strongly clean if there exist an idempotent \( e \in R \) and a unit \( u \in R \) such that \( a = e + u \) and \( eu = ue \). A ring \( R \) is strongly clean if every element in \( R \) is strongly clean. Many authors have studied such rings from very different points of view (cf. [1-9]). An element \( a \in R \) is strongly \( J \)-clean provided that there exist an idempotent \( e \in R \) and element \( w \in J(R) \) such that \( a = e + w \) and \( ew = ew \). A ring \( R \) is strongly \( J \)-clean in case every element in \( R \) is strongly \( J \)-clean. Strong \( J \)-cleanness over commutative rings is studied in [1] and deduced the strong \( J \)-cleanness of \( T_n(R) \) for a large class of local rings \( R \), where \( T_n(R) \) denotes the ring of all upper triangular matrices over \( R \).

Let \( \sigma \) be an endomorphism of \( R \) preserving 1 and \( T_n(R, \sigma) \) be the set of all upper triangular matrices over the rings \( R \). For any \((a_{ij}), (b_{ij}) \in T_n(R, \sigma)\), we define \((a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})\), and \((a_{ij})(b_{ij}) = (c_{ij})\) where \( c_{ij} = \sum_{k=1}^{n} a_{ik}\sigma^{k-1}(b_{kj}) \). Then \( T_n(R, \sigma) \)

* This paper is dedicated to my mother Gönül Ünal.
is a ring under the preceding addition and multiplication. It is clear that \( T_n(R, \sigma) \) will be \( T_n(R) \) only when \( \sigma \) is the identity morphism. Let \( a \in R \) and the maps \( l_a : R \to R \) and \( r_a : R \to R \) denote, respectively, the abelian group endomorphisms given by \( l_a(r) = ar \) and \( r_a(r) = ra \) for all \( r \in R \). Thus, \( l_a - r_b \) is an abelian group endomorphism such that \( (l_a - r_b)(r) = ar - rb \) for any \( r \in R \).

Strong cleanness of \( T_n(R, \sigma) \) for several \( n \) was studied in [3]. In this article, we investigate the strong \( J \)-cleanness of \( T_n(R, \sigma) \) over a local ring \( R \) for \( n = 2, 3, 4 \) and then extend strong cleanness to such properties. In this direction we show that \( T_2(R, \sigma) \) is strongly \( J \)-clean if and only if for any \( a \in 1 + J(R) \), \( b \in J(R) \), \( l_a - r_{\sigma(b)} : R \to R \) is surjective and \( R/J(R) \cong \mathbb{Z}_2 \). Further if \( l_a - r_{\sigma(b)} \) and \( l_b - r_{\sigma(a)} \) are surjective for any \( a \in 1 + J(R) \), \( b \in J(R) \), then \( T_3(R, \sigma) \) is strongly \( J \)-clean if and only if \( R/J(R) \cong \mathbb{Z}_2 \). The necessary condition for \( T_3(R, \sigma) \) to be strongly \( J \)-clean is also discussed. In addition to these, if \( l_a - r_{\sigma(b)} \) and \( l_b - r_{\sigma(a)} \) are surjective for any \( a \in 1 + J(R) \), \( b \in J(R) \), then \( T_4(R, \sigma) \) is strongly \( J \)-clean if and only if \( R/J(R) \cong \mathbb{Z}_2 \).

2 The case \( n = 2 \)

By [Theorem 4.4, 2], the triangular matrix ring \( T_2(R) \) over a local ring \( R \) is strongly \( J \)-clean if and only if \( R \) is bleached and \( R/J(R) \cong \mathbb{Z}_2 \). We extend this result to the skew triangular matrix ring \( T_2(R, \sigma) \) over a local ring \( R \).

Remark 2.1 Note that if for any ring \( R \), \( R/J(R) \cong \mathbb{Z}_2 \), then \( 2 \in J(R) \), \( 1 + J(R) = U(R) \) and \( 1 + U(R) = J(R) \). For if, \( f \) is the isomorphism \( R/J(R) \cong \mathbb{Z}_2 \) then \( f(1 + J(R)) = 1 + 2\mathbb{Z} \). Hence \( f(2 + J(R)) = 2 + 2\mathbb{Z} = 0 + 2\mathbb{Z} \). So \( 2 + J(R) = 0 + J(R) \), that is \( 2 \in J(R) \). \( 1 + J(R) \subseteq U(R) \). Let \( u \in U(R) \). Then \( f(1 + J(R)) = 1 + 2\mathbb{Z} = f(1 + J(R)) \). Hence \( u - 1 \in J(R) \) and so \( u \in 1 + J(R) \). Thus, \( U(R) \subseteq 1 + J(R) \) and \( U(R) = 1 + J(R) \).

Lemma 2.1 Let \( R \) be a ring and let \( \sigma \) be an endomorphism of \( R \). If \( T_n(R, \sigma) \) is strongly \( J \)-clean for some \( n \in \mathbb{N} \), then so is \( R \).

Proof. Let \( e = \text{diag}(1, 0, \ldots, 0) \in T_n(R, \sigma) \). Then \( R \cong eT_n(R, \sigma)e \). From Corollary 3.5 in [2], \( R \) is strongly \( J \)-clean. \( \square \)

Theorem 2.2 Let \( R \) be a local ring, and let \( \sigma \) be an endomorphism of \( R \). Then the following are equivalent:

1. \( T_2(R, \sigma) \) is strongly \( J \)-clean.
2. If \( a \in 1 + J(R), b \in J(R) \), then \( l_a - r_{\sigma(b)} : R \to R \) is surjective and \( R/J(R) \cong \mathbb{Z}_2 \)

Proof. (1) \( \Rightarrow \) (2) From Lemma 2.2, \( R \) is strongly \( J \)-clean and by Lemma 4.2 in [2], \( R/J(R) \cong \mathbb{Z}_2 \). By Remark 2.1, \( 1 + J(R) = U(R) \). Let \( a \in 1 + J(R), b \in J(R), v \in R \). Then \( A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma) \). By hypothesis, there exists an idempotent \( E = \begin{pmatrix} e & f \\ 0 & f \end{pmatrix} \in T_2(R, \sigma) \) such that \( A - E \in J(T_2(R, \sigma)) \) and \( AE = EA \). Since \( R \) is local, all idempotents in \( R \) are 0 and 1. Thus, we see that \( e = 1, f = 0 \); otherwise, \( A - \)
E \notin J(T_2(R, \sigma)). \text{ So } E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}. \text{ As } AE = EA, \text{ we get } -v + x\sigma(b) = ax. \text{ Hence, } \\
ax - x\sigma(b) = -v \text{ for some } x \in R. \text{ As a result, } l_a - r_{\sigma(b)} : R \to R \text{ is surjective.}

(2) \Rightarrow (1) \text{ Let } A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma).

\textbf{Case 1.} \text{ If } a, b \in J(R), \text{ then } A \in J(T_2(R, \sigma)) \text{ is strongly } J\text{-clean.}

\textbf{Case 2.} \text{ If } a, b \in 1 + J(R), \text{ then } A - I_2 \in J(T_2(R, \sigma)); \text{ hence, } A = I_2 + (A - I_2) \in T_2(R, \sigma) \text{ is strongly } J\text{-clean.}

\textbf{Case 3.} \text{ If } a \in 1 + J(R), b \in J(R), \text{ by hypothesis, } l_a - r_{\sigma(b)} : R \to R \text{ is surjective.}

Thus, \text{ ax } - x\sigma(b) = v \text{ for some } x \in R. \text{ Choose } E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \in T_2(R, \sigma). \text{ Then } E^2 = E \in T_2(R, \sigma), AE = EA \text{ and } A - E \in J(T_2(R, \sigma)). \text{ That is, } A \in T_2(R, \sigma) \text{ is strongly } J\text{-clean.}

\textbf{Case 4.} \text{ If } a \in J(R), b \in 1 + J(R), \text{ then } a + 1 \in 1 + J(R), b + 1 \in J(R) \text{ and by hypothesis, } l_{a+1} - r_{\sigma(b+1)} : R \to R \text{ is surjective. Thus } \text{ ax } - x\sigma(b) = -v \text{ for some } x \in R. \text{ Choose } E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \in T_2(R, \sigma). \text{ Then } E^2 = E \in T_2(R, \sigma), AE = EA \text{ and } A - E \in J(T_2(R, \sigma)). \text{ Hence, } A \in T_2(R, \sigma) \text{ is strongly } J\text{-clean. Therefore } A \in T_2(R, \sigma) \text{ is strongly } J\text{-clean.} \quad \square

\textbf{Corollary 2.3} \text{ Let } R \text{ be a local ring, and let } \sigma \text{ be an endomorphism of } R. \text{ Then the following are equivalent:}

(1) \text{ } T_2(R, \sigma) \text{ is strongly } J\text{-clean.}

(2) \text{ } R/J(R) \cong \mathbb{Z}_2 \text{ and } T_2(R, \sigma) \text{ is strongly clean.}

\textbf{Proof.} \text{ (1) } \Rightarrow (2) \text{ It is clear.}

(2) \Rightarrow (1) \text{ Let } a \in 1 + J(R), b \in J(R), v \in R. \text{ Then } A = \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma). \text{ By hypothesis, there exists an idempotent } E = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \in T_2(R, \sigma) \text{ such that } A - E \in J(T_2(R, \sigma)) \text{ and } AE = EA. \text{ Since } R \text{ is local, we see that } e = 0, f = 1; \text{ otherwise, } A - E \notin J(T_2(R, \sigma)). \text{ So } E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}. \text{ It follows from } AE = EA \text{ that } v + x\sigma(b) = ax, \text{ and so } ax - v = x\sigma(b). \text{ Therefore } l_a - r_{\sigma(b)} : R \to R \text{ is surjective. By Theorem 2.3, } T_2(R, \sigma) \text{ is strongly } J\text{-clean as } R/J(R) \cong \mathbb{Z}_2. \quad \square

\textbf{Corollary 2.4} \text{ Let } R \text{ be a ring, and } R/J(R) \cong \mathbb{Z}_2. \text{ If } J(R) \text{ is nil, then } T_2(R, \sigma) \text{ is strongly } J\text{-clean.}

\textbf{Proof.} \text{ Clearly } R \text{ is local. Let } a \in 1 + J(R), b \in J(R). \text{ Then we can find some } n \in \mathbb{N} \text{ such that } b^n = 0. \text{ For any } v \in R, \text{ we choose } x = (l_{a-1} + l_{a-2}r_b + \cdots + l_{a-n}r_{b^{n-1}})(v). \text{ It can be easily checked that } (l_a - r_b)(x) = (l_{a-1} + l_{a-2}r_b + \cdots + l_{a-n}r_{b^{n-1}})(v) = (v + a^{-1}vb + \cdots + a^{-n+1}vb^{n-1} - (a^{-1}vb + \cdots + a^{-n}vb^{n})) = v. \text{ Hence, } l_a - r_b : R \to R \text{ is surjective. Similarly, } l_a - r_{\sigma(b)} \text{ is surjective since } \sigma(b) \in J(R). \text{ This completes the proof by Theorem 2.3.} \quad \square
Example 2.1 Let \( \mathbb{Z}_{2^n} = \mathbb{Z}/2^n\mathbb{Z}, \ n \in \mathbb{N} \), and let \( \sigma \) be an endomorphism of \( \mathbb{Z}_{2^n} \). Then, \( T_2(\mathbb{Z}_{2^n}, \sigma) \) is strongly J-clean. As \( \mathbb{Z}_{2^n} \) is a local ring with the Jacobson radical \( 2\mathbb{Z}_{2^n} \), obviously, \( J(\mathbb{Z}_{2^n}) \) is nil, and we are through by Corollary 2.4.

Example 2.2 Let \( \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} \), let

\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\},
\]

and let \( \sigma : R \to R, \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) \mapsto \left( \begin{array}{cc} a & -b \\ 0 & a \end{array} \right) \). Then \( T_2(R, \sigma) \) is strongly J-clean. Obviously, \( \sigma \) is an endomorphism of \( R \). It is easy to check that \( J(R) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_2, b \in \mathbb{Z}_4 \right\} \), and then \( R/J(R) \cong \mathbb{Z}_2 \) is a field. Thus, \( R \) is a local ring. In addition, \( (J(R))^4 = 0 \), thus \( J(R) \) is nil. Therefore we obtain the result by Corollary 2.4.

3 The case \( n = 3 \)

We now extend Theorem 2.2 to the case of \( 3 \times 3 \) skew triangular matrix rings over a local ring.

Theorem 3.1 Let \( R \) be a local ring. If \( l_a - r_{\sigma(b)} \) and \( l_b - r_{\sigma(a)} \) are surjective for any \( a \in 1 + J(R), b \in J(R) \), then \( T_3(R, \sigma) \) is strongly J-clean if and only if \( R/J(R) \cong \mathbb{Z}_2 \).

Proof. \((\Leftarrow)\) We noted in Remark 2.1, in this case we have \( \sigma(J(R)) \subseteq J(R), \sigma(U(R)) \subseteq U(R), 1 + J(R) = U(R) \) and \( 1 + U(R) = J(R) \) and we use them in the sequel intrinsically. Let \( A = (a_{ij}) \in T_3(R, \sigma) \). We divide the proof into six cases.

Case 1. If \( a_{11}, a_{22}, a_{33} \in 1 + J(R) \), then \( A = I_3 + (A - I_3) \), and so \( A - I_3 \in J(T_3(R, \sigma)) \). Then \( A \in T_3(R, \sigma) \) is strongly J-clean.

Case 2. If \( a_{11} \in J(R), a_{22}, a_{33} \in 1 + J(R) \), then we have \( e_{12} \in R \) such that \( a_{11} e_{12} - e_{12} \sigma(a_{22}) = -a_{12} \). Further, we have some \( e_{13} \in R \) such that \( a_{11} e_{13} - e_{13} \sigma^2(a_{33}) = e_{12} \sigma^2(a_{23}) - a_{13} \). Choose

\[
E = \begin{pmatrix} 0 & e_{12} & e_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R, \sigma).
\]

Then \( E^2 = E \), and \( A = E + (A - E) \), where \( A - E \in J(T_3(R, \sigma)) \). Furthermore,

\[
EA = \begin{pmatrix} 0 & e_{12} \sigma(a_{22}) & e_{12} \sigma(a_{23}) + e_{13} \sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},
\]

\[
AE = \begin{pmatrix} 0 & a_{11} e_{12} + a_{12} a_{11} e_{13} + a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},
\]

and so \( EA = AE \). That is, \( A \in T_3(R, \sigma) \) is strongly J-clean.

Case 3. If \( a_{11} \in 1 + J(R), a_{22} \in J(R), a_{33} \in 1 + J(R) \), then we have an \( e_{12} \in R \) such that \( a_{11} e_{12} - e_{12} \sigma(a_{22}) = a_{12} \). Further, we have some \( e_{23} \in R \) such that
$a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$. Thus $-a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) = -e_{12}\sigma(a_{22})\sigma(e_{23}) = e_{12}\sigma(a_{23}) - e_{12}\sigma(e_{23})\sigma^2(a_{33})$. Choose

$$E = \begin{pmatrix} 1 & e_{12} & -e_{12}\sigma(e_{23}) \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R, \sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R, \sigma))$. Furthermore,

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) & a_{13} + e_{12}\sigma(a_{23}) - e_{12}\sigma(e_{23})\sigma^2(a_{33}) \\ 0 & 0 & e_{23}\sigma(a_{33}) \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} - a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) + a_{13} \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. Thus, $A \in T_3(R, \sigma)$ is strongly $J$-clean.

**Case 4.** If $a_{11}, a_{22} \in 1 + J(R), a_{33} \in J(R)$, then we find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$. Thus, there exists $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} - a_{12}\sigma(e_{23})$. Choose

$$E = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, \sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R, \sigma))$. Furthermore,

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) + e_{13}\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} + e_{23}\sigma(a_{33}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} + a_{12}\sigma(e_{23}) \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $EA = AE$. Therefore $A \in T_3(R, \sigma)$ is strongly $J$-clean.

**Case 5.** If $a_{11} \in 1 + J(R), a_{22}, a_{33} \in J(R)$, then we have some $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$. Further, there exists $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(e_{23})$. Choose

$$E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, \sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R, \sigma))$. Hence

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) + e_{13}\sigma^2(a_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} & a_{11}e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
and so $EA = AE$. Thus $A \in T_3(R, \sigma)$ is strongly $J$-clean.

**Case 6.** If $a_{11} \in J(R), a_{22} \in 1 + J(R), a_{33} \in J(R)$, then we find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$. Hence there is $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$. It is easy to verify that

$$e_{12}\sigma(a_{23}) + e_{12}\sigma(e_{23})\sigma^2(a_{33}) = e_{12}\sigma(a_{22}e_{23}) = a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}).$$

Choose

$$E = \begin{pmatrix} 0 & e_{12} & e_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R, \sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R, \sigma))$. In addition,

$$EA = \begin{pmatrix} 0 & e_{12}\sigma(a_{22}) & e_{12}\sigma(a_{23}) + e_{12}\sigma(e_{23})\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} + e_{23}\sigma(a_{33}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & a_{11}e_{12} + a_{12}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

and so $EA = AE$. Consequently, $A \in T_3(R, \sigma)$ is strongly $J$-clean.

**Case 7.** If $a_{11}, a_{22}, a_{33} \in J(R)$, then we find $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$. Further, we have an $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = -a_{13} - a_{12}\sigma(e_{23})$. Choose

$$E = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R, \sigma).$$

Then $E^2 = E$, and $A = E + (A - E)$, where $A - E \in J(T_3(R, \sigma))$. Furthermore,

$$EA = \begin{pmatrix} 0 & e_{13}\sigma(a_{33}) \\ 0 & e_{23}\sigma(a_{33}) \\ 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & 0 & a_{11}e_{13} + a_{12}\sigma(e_{23}) + a_{13} \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. As a result, $A \in T_3(R, \sigma)$ is strongly $J$-clean.

**Case 8.** If $a_{11}, a_{22}, a_{33} \in J(R)$, then $A = 0 + A$, where $A \in J(T_3(R, \sigma))$. Hence, $A \in T_3(R, \sigma)$ is strongly $J$-clean.

Thus, $T_3(R, \sigma)$ is strongly $J$-clean.

$(\Rightarrow)$ Similar to Theorem 2.2, we easily complete the proof. $\Box$

**Corollary 3.2** Let $R$ be a ring, and $R/J(R) \cong \mathbb{Z}_2$. If $J(R)$ is nil, then $T_3(R, \sigma)$ is strongly $J$-clean.

**Proof.** Obviously $R$ is local. Let $a \in U(R), b \in J(R)$. Then we can find some $n \in \mathbb{N}$ such that $b^n = 0$; hence, $(\sigma(b))^n = 0$. For any $v \in R$, we choose $x = (l_{a-1} + l_{a-2}r_{\sigma(b)} + \cdots + l_{a-n}r_{\sigma(b)^{n-1}})(v)$. It can be easily checked that $(l_{a-1} - r_{\sigma(b)})(x) = (l_{a-1} + l_{a-2}r_{\sigma(b)} + \cdots + l_{a-n}r_{\sigma(b)^{n-1}})(v) = (v + a^{-1}r_{\sigma(b)} + \cdots + a^{-n}r_{\sigma(b)^{n-1}}) - (a^{-1}r_{\sigma(b)} + \cdots + a^{-n}r_{\sigma(b)^{n-1}}) = v$. Thus, $l_a - r_{\sigma(b)} : R \rightarrow R$ is surjective. Likewise, $l_b - r_{\sigma(a)} : R \rightarrow R$ is surjective. Consequently, $T_3(R, \sigma)$ is strongly $J$-clean by Theorem 3.1. $\Box$
4 A characterization

We will consider the necessary and sufficient conditions under which the skew triangular matrix ring $T_3(R, \sigma)$ is strongly $J$-clean.

**Lemma 4.1** Let $R$ be a local ring. If $T_3(R, \sigma)$ is strongly $J$-clean, then $l_a - r_{\sigma^2(b)}$, $l_b - r_{\sigma(a)}$ and $l_b - r_{\sigma^2(a)}$ are surjective for any $a \in 1 + J(R), b \in J(R)$.

**Proof.** Let $a \in 1 + J(R), b \in J(R)$. Clearly, $T_2(R, \sigma)$ is strongly $J$-clean. By Theorem 2.2, $l_a - r_{\sigma(b)}$ is surjective. As $1 - b \in 1 + J(R)$ and $1 - a \in J(R)$, we get $l_{1 - b} - r_{\sigma(1 - a)} : R \to R$ is surjective. For any $v \in R$, we have $(1 - b)x - x\sigma(1 - a) = -v$. Thus, $bx - x\sigma(a) = v$ and so $l_b - r_{\sigma(a)} : R \to R$ is surjective.

Let $v \in R$ and let

$$A = \begin{pmatrix} b & 0 & v \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} \in T_3(R, \sigma).$$

We have an idempotent $E = (e_{ij}) \in T_3(R, \sigma)$ such that $A - E \in J(T_3(R, \sigma))$ and $EA = \sigma E$. This implies that $e_{11}, e_{22}, e_{33}$ are all idempotents. As $a \in 1 + J(R), b \in J(R)$, we have $e_{11} = 0, e_{22} = 0$ and $e_{33} = 1$; otherwise, $A - E \notin J(T_3(R, \sigma))$. As $E = 1 - v$, we have

$$E = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

for some $e_{13}, e_{23} \in R$. Observing that

$$\begin{pmatrix} 0 & 0 & be_{13} + v \\ 0 & 0 & be_{23} \\ 0 & 0 & a \end{pmatrix} = AE = \sigma E = \begin{pmatrix} 0 & 0 & e_{13}\sigma^2(a) \\ 0 & 0 & e_{23}\sigma(a) \\ 0 & 0 & a \end{pmatrix},$$

we have $be_{13} - e_{13}\sigma^2(a) = -v$. Thus, $l_b - r_{\sigma^2(a)} : R \to R$ is surjective. Since $1 - a \in J(R)$ and $1 - b \in 1 + J(R)$, we have $l_{1 - a} - r_{\sigma^2(1 - b)} : R \to R$ is surjective. Thus, we can find some $x \in R$ such that $(1 - a)x - x\sigma^2(1 - b) = -v$. This implies that $ax - x\sigma^2(b) = v$, hence $l_a - r_{\sigma^2(b)}$ is surjective. \qed

**Theorem 4.2** Let $R$ be a local ring and let $\sigma$ be an endomorphism of $R$. Then the following are equivalent:

1. $T_3(R, \sigma)$ is strongly $J$-clean.
2. $R/J(R) \cong \mathbb{Z}_2$, and $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R), b \in J(R)$.

**Proof.** (1) $\Rightarrow$ (2) is obvious from Lemma 4.1.

(2) $\Rightarrow$ (1) Clear from Theorem 4.1. \qed

**Corollary 4.3** Let $R$ be a local ring and let $\sigma$ be an endomorphism of $R$. Then the following are equivalent:

1. $T_2(R, \sigma)$ is strongly $J$-clean.
2. $T_3(R, \sigma)$ is strongly $J$-clean.
3. $R/J(R) \cong \mathbb{Z}_2$ and $l_a - r_{\sigma(b)}$ is surjective for any $a \in 1 + J(R), b \in J(R)$.

**Proof.** (1) $\iff$ (3) is proved by Theorem 2.2.

(2) $\iff$ (3) is obvious from Theorem 4.2. \qed
5 The case $n = 4$

We now extend the preceding discussion to the case of $4 \times 4$ skew triangular matrix rings over a local ring.

**Theorem 5.1** Let $R$ be a local ring. If $l_a - r_{\sigma(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R)$, $b \in J(R)$, then $T_4(R, \sigma)$ is strongly $J$-clean if and only if $R/J(R) \cong \mathbb{Z}_2$.

**Proof.** ($\Leftarrow$) As $R/J(R) \cong \mathbb{Z}_2$, $\sigma(J(R)) \subseteq J(R)$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \in T_4(R, \sigma).$$

We show the existence of

$$E = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ 0 & e_{22} & e_{23} & e_{24} \\ 0 & 0 & e_{33} & e_{34} \\ 0 & 0 & 0 & e_{44} \end{pmatrix} \in T_4(R, \sigma),$$

such that $E^2 = E$, $AE = EA$ and $A - E \in J(T_4(R, \sigma))$. One can easily derive from $E^2 = E$ that

(a) $e_{12} = e_{11}e_{12} + e_{12}\sigma(e_{22})$
(b) $e_{13} = e_{11}e_{13} + e_{12}\sigma(e_{23}) + e_{13}\sigma(e_{33})$
(c) $e_{23} = e_{22}e_{23} + e_{23}\sigma(e_{33})$

and from $AE = EA$ that

(d) $a_{11}e_{12} - e_{12}\sigma(a_{22}) = e_{11}a_{12} - a_{12}\sigma(e_{22})$
(e) $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = e_{11}a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23}) - a_{13}\sigma^2(e_{33})$
(f) $a_{22}e_{23} - e_{23}\sigma(a_{33}) = e_{22}a_{23} - a_{23}\sigma(e_{33})$

**Case 1.** If $a_{22} \in J(R)$, $a_{11} \in 1 + J(R)$ then $e_{22} = 0$, $e_{11} = 1$. Hence, (d) implies that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$ and by assumption there exists $e_{12} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{22})})(e_{12}) = a_{12}$.

(A) If $a_{33} \in 1 + J(R)$, then $e_{33} = 1$. From (f), $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$ and (b) implies that $e_{13} = -e_{12}\sigma(e_{23})$.

(B) If $a_{33} \in J(R)$, then $e_{33} = 0$. By (c), $e_{23} = 0$. From (e), we have $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$ and by assumption there exists $e_{13} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{33})})(e_{13}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$.

**Case 2.** If $a_{22} \in 1 + J(R)$, $a_{11} \in 1 + J(R)$, then $e_{22} = 1$, $e_{11} = 1$. By (a) implies that $e_{12} = 0$.

(C) If $a_{33} \in 1 + J(R)$, then $e_{33} = 1$. From (b), we have $e_{13} = 0$ and (c) implies that $e_{23} = 0$.

(D) If $a_{33} \in J(R)$, then $e_{33} = 0$. By (f), we have $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$, and (e) gives rise to $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$ and by assumption there exists $e_{13} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{33})})(e_{13}) = a_{13} + e_{12}\sigma(a_{23}) - a_{12}\sigma(e_{23})$.

**Case 3.** If $a_{22} \in 1 + J(R)$, $a_{11} \in J(R)$, then $e_{22} = 1$, $e_{11} = 0$. By (d), $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$ and there exists $e_{12} \in R$ such that $(l_{a_{11}} - r_{\sigma(a_{22})})(e_{12}) = -a_{12}$.
(E) If \(a_{33} \in 1 + J(R)\), then \(e_{33} = 1\). From (c), we have \(e_{23} = 0\). Then from (e), we have
\[a_{11}e_{13} - e_{13}a_{33} = e_{13}a_{23} - a_{13}\]
If \(a_{33} \in J(R)\), then \(e_{33} = 0\). From (f), we have \(a_{22}e_{23} - e_{23}a_{33} = a_{23}\) and there exists \(e_{23} \in R\) such that \((l_{a_{22}} - r_{a_{33}})(e_{23}) = a_{23}\). Then (b) implies that \(e_{13} = e_{12}a(23)\).

**Case 4.** If \(a_{22} \in J(R)\), \(a_{11} \in J(R)\), then \(e_{22} = 0\), \(e_{11} = 0\). Hence, (a) implies that \(e_{12} = 0\).

(G) If \(a_{33} \in 1 + J(R)\), then \(e_{33} = 1\). From (f), \(a_{22}e_{23} - e_{23}a_{33} = -a_{23}\) and there exists \(e_{23} \in R\) such that \((l_{a_{22}} - r_{a_{33}})(e_{23}) = a_{23}\). So (e) gives us \(a_{11}e_{13} - e_{13}a_{33} = -a_{12}a(23) - a_{13}\). Hence, there exists \(e_{13} \in R\) such that \((l_{a_{11}} - r_{a_{33}})(e_{13}) = -a_{12}a(23) - a_{13}\).

(H) If \(a_{33} \in J(R)\), then \(e_{33} = 0\). From (c), we have \(e_{23} = 0\) and by (b) we obtain \(e_{13} = 0\).

Similar to preceding calculations from \(E^2 = E\) we have

1. \(e_{14} = e_{11}e_{14} + e_{12}a(24) + e_{13}a(34) + e_{14}a(44)\)
2. \(e_{24} = e_{22}e_{24} + e_{23}a(34) + e_{24}a(44)\)
3. \(e_{34} = e_{33}e_{34} + e_{34}a(44)\)

and from \(AE = EA\) we have

4. \(a_{11}e_{14} - e_{14}a(44) = -a_{12}a(24) - a_{13}a(34) - a_{14}a(44) + e_{11}a_{14} + e_{12}a_{24} + e_{13}a_{34}\)
5. \(a_{22}e_{24} - e_{24}a(44) = -a_{23}a(34) - a_{24}a(44) + e_{22}a_{24} + e_{23}a_{34}\)
6. \(a_{33}e_{34} - e_{34}a(44) = -a_{34}a(44) + e_{33}a_{34} + e_{34}a(44)\)

To complete the proof, we only need to show the existence of \(e_{14}, e_{24}\) and \(e_{34}\) in \(R\) satisfying preceding conditions (1)-(6).

**Case 1.** If \(a_{44} \in J(R)\), \(a_{33} \in 1 + J(R)\), then \(e_{44} = 0\) and \(e_{33} = 1\), otherwise \(A - E \notin J(T_4(R, \sigma))\).\( \) By (6), \(a_{33}e_{34} - e_{34}a(44) = a_{34}\). Then by (5), \(a_{22}e_{24} - e_{24}a(44) = -a_{23}a(34) + e_{22}a_{24} + e_{23}a_{34}\). Then \(e_{13} = 1\), otherwise \(A - E \notin J(T_4(R, \sigma))\). Hence, there exists \(e_{14} \in R\) and \(e_{14} \in R\) such that \((l_{a_{11}} - r_{a(44)})(e_{14}) = -a_{12}a(24) - a_{13}a(34) + a_{14} + e_{12}a(24) + e_{13}a(34)\). If \(a_{11} \in J(R)\), then \(e_{11} = 0\) and by (1), \(e_{14} = e_{12}a(24) + e_{13}a(34)\).

**Case 2.** If \(a_{44} \in 1 + J(R)\), \(a_{33} \in 1 + J(R)\), then \(e_{44} = e_{33} = 1\). Then by (3), \(e_{34} = 0\). Again there are two possibilities:

(C) If \(a_{22} \in U(R)\), then \(e_{22} = 1\) and by (2), \(e_{24} = 0\). If \(a_{11} \in U(R)\), then \(e_{11} = 1\) and by (1), \(e_{14} = 0\). If \(a_{11} \in J(R)\), then \(e_{11} = 0\). Then by equation (4), \(a_{11}e_{14} - e_{14}a(44) = e_{12}a(24) + e_{13}a(34)\). Hence, there exists \(e_{14} \in J(R)\) such that \((l_{a_{11}} - r_{a(44)})(e_{14}) = e_{12}a(24) + e_{13}a(34)\).
If $a_{22} \in J(R)$, then \(e_{22} = 0\) and by (5), \(a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = -a_{24} + e_{23}\sigma(a_{34})\).

So, there exists \(e_{24} \in R\) such that \((l_{a_{22}} - r_{\sigma(a_{44})})(e_{24}) = -a_{24} + e_{23}\sigma(a_{34})\).

If \(a_{11} \in J(R)\), then \(e_{11} = 0\). From equation (4), \(a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) - a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma(a_{34})\).

By assumption, there exists \(e_{14} \in R\) such that \((l_{a_{11}} - r_{\sigma(a_{44})})(e_{14}) = -a_{12}\sigma(e_{24}) - a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma(a_{34})\).

If \(a_{11} \in U(R)\), then \(e_{11} = 1\). By equation (1), \(e_{14} = -e_{12}\sigma(e_{24})\).

**Case 3.** If \(a_{34} \in 1 + J(R), a_{33} \in J(R)\). In this case \(e_{33} = 0\) and \(e_{44} = 1\).

By (6), \(a_{33}e_{34} - e_{34}\sigma(a_{44}) = -a_{34}\). Hence, there exists \(e_{34} \in R\) such that \((l_{a_{33}} - r_{\sigma(a_{44})})(e_{34}) = -a_{34}\).

Using (5), \(a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = e_{22}e_{24} + e_{23}\sigma(a_{34}) - a_{23}\sigma(a_{34}) - a_{24}\).

Then there are two possibilities:

- **(E)** If \(a_{22} \in 1 + J(R)\), then \(e_{22} = 1\) and from (2), \(e_{24} = -e_{23}\sigma(e_{34})\).

  By (4), \(a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = e_{11}(1 + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}\).

  If \(a_{11} \in J(R)\), then \(e_{11} = 0\). So there exists \(e_{14} \in R\) such that \((l_{a_{11}} - r_{\sigma(a_{44})})(e_{14}) = e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}\).

  If \(a_{11} \in U(R)\), then \(e_{11} = 1\) and by (1), \(e_{14} = -e_{12}\sigma(e_{24}) - e_{13}\sigma^2(e_{34})\).

- **(F)** If \(a_{22} \in J(R)\), then \(e_{22} = 0\) and by hypothesis there exists \(e_{24} \in R\) such that \((l_{a_{22}} - r_{\sigma(a_{44})})(e_{24}) = -a_{24} + e_{23}\sigma(a_{34}) - a_{23}\sigma(e_{34})\).

  From equation (4), \(a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = e_{11} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}\).

  If \(a_{11} \in J(R)\), then \(e_{11} = 0\). From (4) and by hypothesis, there exists \(e_{14} \in R\) such that \((l_{a_{11}} - r_{\sigma(a_{44})})(e_{14}) = e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34}) - a_{12}\sigma(e_{24}) - a_{13}\sigma^2(e_{34}) - a_{14}\).

  If \(a_{11} \in U(R)\), then \(e_{11} = 1\) and by (1), \(e_{14} = -e_{12}\sigma(e_{24}) - e_{13}\sigma^2(e_{34})\).

**Case 4.** If \(a_{44} \in J(R), a_{33} \in J(R)\). In this case \(e_{33} = e_{44} = 0\).

- **(G)** If \(a_{22} \in J(R)\), then \(e_{22} = 0\). By (2), \(e_{24} = 0\). If \(a_{11} \in J(R)\), then \(e_{11} = 0\) and from (1), \(e_{14} = 0\). If \(a_{11} \in U(R)\), then \(e_{11} = 1\). Hence, equation (4) becomes \(a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})\).

  By hypothesis there exists \(e_{14} \in R\) such that \((l_{a_{11}} - r_{\sigma(a_{44})})(e_{14}) = a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})\).

- **(H)** If \(a_{22} \in 1 + J(R)\), then \(e_{22} = 1\) and from (5), \(a_{22}e_{24} - e_{24}\sigma^2(a_{44}) = a_{24} + e_{23}\sigma(a_{34})\).

  By assumption, there exists \(e_{24} \in R\) such that \((l_{a_{22}} - r_{\sigma(a_{44})})(e_{24}) = a_{24} + e_{23}\sigma(a_{34})\).

  If \(a_{11} \in U(R)\), then \(e_{11} = 1\) and by (4), \(a_{11}e_{14} - e_{14}\sigma^3(a_{44}) = -a_{12}\sigma(e_{24}) - a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})\).

  Hence, there exists \(e_{14} \in R\) such that \((l_{a_{11}} - r_{\sigma(a_{44})})(e_{14}) = -a_{12}\sigma(e_{24}) + a_{14} + e_{12}\sigma(a_{24}) + e_{13}\sigma^2(a_{34})\).

  If \(a_{11} \in J(R)\), then \(e_{11} = 0\) and from (1), \(e_{14} = e_{12}\sigma(e_{24})\).

  Thus, we always find \(e_{14}, e_{24}\) and \(e_{34}\) in \(R\).

\(\Rightarrow\) Analogous to Theorem 2.2 we easily obtain the result. \(\square\)

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**References**