Lorentzian stationary surfaces and null curves in 4-dimensional space forms of index 2

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Abstract We discuss a property of the Gaussian curvature and the normal curvature of Lorentzian stationary surfaces in 4-dimensional semi-Riemannian space forms of index 2, which is related to a property of null curves via a characteristic initial value problem.

Keywords Lorentzian stationary surface · null curve · space form

Mathematics Subject Classification (2010) 53A10 · 53B30 · 53A04

1 Introduction

Recently, the geometry of Lorentzian surfaces in 4-dimensional semi-Riemannian space forms of index 2 has been studied actively (cf. [1], [2], [4], [5], [8], [12], [13], [16]). In this paper, we will study Lorentzian surfaces with zero mean curvature vector in 4-dimensional space forms of index 2.

Let $N^n_q(c)$ denote the $n$-dimensional semi-Riemannian space form of constant curvature $c$ and index $q$. Namely, it is either the $n$-dimensional semi-Euclidean space $R^n_q$ of index $q$, the $n$-dimensional pseudo-sphere $S^n_q(c)$ of constant curvature $c > 0$ and index $q$, or the $n$-dimensional pseudo-hyperbolic space $H^n_q(c)$ of constant curvature $c < 0$ and index $q$. We shall write $N^n_q(c) = N^n(c)$. A surface in $N^n_q(c)$ is called Lorentzian if the induced metric is Lorentzian. We say that a Lorentzian surface is stationary if it has zero mean curvature vector.

For a minimal surface in $N^4(c)$ and a spacelike maximal surface in $N^4_2(c)$, the Gaussian curvature $K$ and the normal curvature $K_\nu$ satisfy $(K - c)^2 - K_\nu^2 \geq 0$, where the equality holds at isotropic points. Those surfaces have similar geometric properties on the rigidity, isometric deformations and curvatures (cf. [14] and [10]). A related work can be seen also in [7].

For a Lorentzian stationary surface in $N^4_2(c)$, the value $(K - c)^2 - K_\nu^2$ also plays an important role, but its sign is not fixed. According to the
sign of \((K - c)^2 - K^2_\nu\), there are different situations on the rigidity and isometric deformability (see \([12]\)). So it should be important to investigate the geometric meaning of the sign of \((K - c)^2 - K^2_\nu\). In a previous paper [13], we consider the case where \(c = 0\), and discuss the sign of \(K^2 - K^2_\nu\) for Lorentzian stationary surfaces in \(R^4_2\), from the view point of the geometry of null curves in \(R^4_2\). Here we will consider the case where \(c \neq 0\).

Let \(\langle , \rangle\) denote the inner product of the semi-Euclidean space \(R^n_q\). When \(c > 0\), \(N^4_2(c)\) is the pseudo-sphere \(S^4_2(c)\) which is given as

\[
S^4_2(c) = \{ x \in R^5_2 | \langle x, x \rangle = 1/c \}.
\]

When \(c < 0\), \(N^4_2(c)\) is the pseudo-hyperbolic space \(H^4_2(c)\) which is given as

\[
H^4_2(c) = \{ x \in R^5_3 | \langle x, x \rangle = 1/c \}.
\]

Let \(M\) be a Lorentzian surface in \(N^4_2(c)\) \((c \neq 0)\), and \(f\) be the inclusion map. Then \(f\) is an \(R^2_2\)-valued map with \(\langle f, f \rangle = 1/c\) when \(c > 0\), and \(f\) is an \(R^3_2\)-valued map with \(\langle f, f \rangle = 1/c\) when \(c < 0\). There exists a local coordinate system \((u, v)\) on \(M\) such that the induced metric \(g\) satisfies

\[
g = a(du \otimes dv + dv \otimes du)
\]

for some function \(a \neq 0\) (cf. p.13 of [15], [6]). Then \(f_u\) and \(f_v\) are null, and \(\langle f_u, f_v \rangle = a \neq 0\). Let \(H^*\) be the mean curvature vector of \(M\) as a surface in the ambient semi-Euclidean space \(R^5_2(\supset S^4_2(c))\) or \(R^5_3(\supset H^4_2(c))\). Then we have

\[
H^* = \frac{f_{uv}}{\langle f_u, f_v \rangle}.
\]

So \(M\) is stationary as a surface in \(N^4_2(c)\) if and only if

\[
f_{uv} = -c\langle f_u, f_v \rangle f.
\]

Now we state the result as follows:

**Theorem 1.1.** Let \(M\) be a Lorentzian stationary surface in \(N^4_2(c)\) \((c \neq 0)\), and let \(f(u, v)\) be the inclusion map such that \(f_u\) and \(f_v\) are null. Assume that \(\{f_u, f_v, f_{uu}, f_{uv} \}\) are linearly independent. Then the Gaussian curvature \(K\) and the normal curvature \(K_\nu\) satisfy

\[
(K - c)^2 - K^2_\nu = \frac{\langle f_{uu}, f_{uu} \rangle \langle f_{uv}, f_{uv} \rangle}{\langle f_u, f_v \rangle^4}.
\]

So, under the condition of Theorem 1.1, the sign of \((K - c)^2 - K^2_\nu\) is the same as that of \(\langle f_{uu}, f_{uu} \rangle \langle f_{uv}, f_{uv} \rangle\). If \(\{f_u, f_v, f_{uu}, f_{uv}\}\) are linearly dependent at a point \(p\), then \(K_\nu = 0\) and \((K - c)^2 - K^2_\nu = (K - c)^2 \geq 0\) at \(p\).

By [11] we can see the existence of Lorentzian stationary surfaces in \(N^4_2(c)\) satisfying \((K - c)^2 - K^2_\nu = 0\). Next we show the existence of Lorentzian stationary surfaces in \(N^4_2(c)\) such that \((K - c)^2 - K^2_\nu\) is positive, or negative.
Theorem 1.2. Let \( P(u) \) and \( Q(v) \) be null curves in \( \mathbb{N}^4_2(c) \) \((c \neq 0)\) which satisfy \( P(0) = Q(0) \) and \( \langle P'(0), Q'(0) \rangle \neq 0 \). Suppose that \( \{ P'(0), Q'(0), P''(0), Q''(0) \} \) are linearly independent, and \( \langle P''(0), P''(0) \rangle \langle Q''(0), Q''(0) \rangle > 0 \) (resp. \(< 0\)). Then there exists a Lorentzian stationary surface in \( \mathbb{N}^4_2(c) \) whose inclusion map \( f(u,v) \) satisfies \( f_{u} \) and \( f_{v} \) are null, \( f(u,0) = P(u) \), \( f(0,v) = Q(v) \), and \( (K - c)^2 - K'_{\nu} > 0 \) (resp. \(< 0 \)) near \((u,v) = (0,0)\).

In Section 2, we recall the method of moving frames and basic facts for Lorentzian surfaces in \( \mathbb{N}^4_2(c) \). In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, using Theorem 1.2, we give examples. In Section 5, we give a remark on the right hand side of the equation in Theorem 1.1.

2 Preliminaries

We use the following convention on the ranges of indices:

\[ 1 \leq A, B, \cdots \leq 4, \quad 1 \leq i, j, \cdots \leq 2, \quad 3 \leq \alpha, \beta, \cdots \leq 4. \]

Let \( \{ e_A \} \) be a local orthonormal frame field in \( \mathbb{N}^4_2(c) \), and \( \{ \omega^A \} \) be the dual coframe field, so that the metric \( g \) of \( \mathbb{N}^4_2(c) \) is given by

\[ g = \omega^1 \otimes \omega^1 - \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 - \omega^4 \otimes \omega^4. \]

The connection forms \( \omega^A_B \) are given by

\[ de_B = \sum_A \omega^A_B e_A. \]

Then \( \omega^A_B = -\varepsilon_A \varepsilon_B \omega^B_A \), where \( \varepsilon_1 = \varepsilon_3 = 1 \) and \( \varepsilon_2 = \varepsilon_4 = -1 \). In other words, \( \omega^A_B = -\omega_B^A \) if \(|A - B|\) is even, and \( \omega^A_B = \omega_B^A \) if \(|A - B|\) is odd. The structure equations are given by

\[ d\omega^A = -\sum_B \omega^A_B \wedge \omega^B, \quad (2.1) \]

\[ d\omega^A_B = -\sum_C \omega^A_C \wedge \omega^C_B + \frac{1}{2} \sum_{C,D} R^A_{BCD} \omega^C \wedge \omega^D, \quad (2.2) \]

\[ R^A_{BCD} = \varepsilon_B (\delta^A_C \delta_{RD} - \delta^A_D \delta_{BC}). \quad (2.3) \]

Let \( M \) be a Lorentzian surface in \( \mathbb{N}^4_2(c) \). We choose the frame \( \{ e_A \} \) so that \( \{ \varepsilon_i \} \) are tangent to \( M \). In the following the argument will be restricted to \( M \). Then \( \omega^a = 0 \) on \( M \), and by (2.1), we have

\[ 0 = -\sum_i \omega^a_i \wedge \omega^i. \]
So there exists a symmetric tensor \( h_{ij}^\alpha \) so that

\[
\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega^j,
\]  

(2.4)

where \( h_{ij}^\alpha \) are the components of the second fundamental form \( h \) of \( M \).

The Gaussian curvature \( K \) and the normal curvature \( K_\nu \) of \( M \) are given by

\[
d\omega_2 = -K \omega^1 \wedge \omega^2, \quad d\omega_4 = -K_\nu \omega^1 \wedge \omega^2.
\]  

(2.5)

Then by (2.2), (2.3), (2.4) and (2.5), we have

\[
K = c - h_{11}^3 h_{22}^3 + (h_{12}^3)^2 + h_{11}^4 h_{22}^4 - (h_{12}^4)^2,
\]  

(2.6)

and

\[
K_\nu = h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4 - h_{12}^3 h_{22}^4 + h_{22}^3 h_{12}^4.
\]  

(2.7)

The mean curvature vector \( H \) of \( M \) is given by

\[
H = \frac{1}{2} \sum_\alpha (h_{11}^\alpha - h_{22}^\alpha) e_\alpha.
\]

We say that \( M \) is stationary if \( H = 0 \) on \( M \). When \( M \) is stationary, by (2.6) and (2.7), we have

\[
K = c - (h_{11}^3)^2 + (h_{12}^3)^2 + (h_{11}^4)^2 - (h_{12}^4)^2,
\]  

(2.8)

and

\[
K_\nu = 2(h_{11}^3 h_{12}^4 - h_{12}^3 h_{11}^4).
\]  

(2.9)

3 Proof of Theorems

Proof of Theorem 1.1. Let \( M \) be a Lorentzian stationary surface in \( N_2^4(c) \) \((c \neq 0)\), and let \( f(u, v) \) be the inclusion map such that \( f_u \) and \( f_v \) are null. Then the equation (1.1) is satisfied. We assume that \( \{f_u, f_v, f_{uu}, f_{vv}\} \) are linearly independent.

Set \( a = \langle f_u, f_v \rangle \neq 0 \). Changing the parameter, we assume that \( a > 0 \). We note that the Gaussian curvature \( K \) is intrinsically given by

\[
K = -\frac{1}{a} (\log a)_{uv} = -\frac{1}{a^2} \left( a_{uv} - \frac{a_u a_v}{a} \right).
\]  

(3.1)

Since \( f_u \) and \( f_v \) are null, and noting (1.1), we can see that

\[
a_u = \langle f_{uu}, f_v \rangle, \quad a_v = \langle f_u, f_{vv} \rangle,
\]  

(3.2)
\[a_{uv} = \langle f_{uu}, f_v \rangle + \langle f_{uu}, f_{vv} \rangle = \langle (-caf)_u, f_v \rangle + \langle f_{uu}, f_{vv} \rangle = -ca^2 + \langle f_{uu}, f_{vv} \rangle. \quad (3.3)\]

Set
\[e_1 = \frac{1}{\sqrt{2a}} (f_u + f_v), \quad e_2 = \frac{1}{\sqrt{2a}} (f_u - f_v). \quad (3.4)\]

Then \(\{e_1, e_2\}\) is an orthonormal frame field on \(M\) of signature \((+, -)\).

Set \(V = f_{uu} - \frac{a_u}{a} f_u\), which is nonzero, because \(\{f_u, f_v, f_{uu}, f_{vv}\}\) are linearly independent. Using (3.2), we see that
\[\langle V, f \rangle = \langle V, f_u \rangle = \langle V, f_v \rangle = 0, \quad \langle V, V \rangle = \langle f_{uu}, f_{uu} \rangle.\]

So \(V\) is a nonzero normal vector to \(M\).

(i) We consider the case where \(\langle f_{uu}, f_{uu} \rangle > 0\). We can write \(\langle f_{uu}, f_u \rangle = r_1^2\) for some positive function \(r_1\). Then the vector
\[e_3 = \frac{1}{r_1} V = \frac{1}{r_1} \left( f_{uu} - \frac{a_u}{a} f_u \right)\]
is a spacelike unit normal vector to \(M\). Set
\[W = f_{vv} - \frac{a_v}{a} f_v - \frac{1}{r_1} \left( ca^2 + a_{uv} - \frac{a_u a_v}{a} \right) e_3,\]
which is nonzero, because \(\{f_u, f_v, f_{uu}, f_{vv}\}\) are linearly independent. Using (3.2) and (3.3), we can see that
\[\langle W, f \rangle = \langle W, f_u \rangle = \langle W, f_v \rangle = \langle W, e_3 \rangle = 0.\]

So \(W\) is a nonzero normal vector which is orthogonal to \(e_3\). Since the normal plane is 2-dimensional Minkowskian, \(W\) is a timelike vector.

By (3.1) we can rewrite \(W\) as follows:
\[W = f_{vv} - \frac{a_v}{a} f_v + \frac{a^2}{r_1} (K - c)e_3.\]

Then, by (3.1), (3.2) and (3.3),
\[\langle W, W \rangle = \langle f_{vv}, f_{vv} \rangle - \frac{a^4}{r_1^2} (K - c)^2 =: -r_2^2 < 0, \quad (3.5)\]
for some positive function \(r_2\), and the vector
\[e_4 = \frac{1}{r_2} W = \frac{1}{r_2} \left( f_{vv} - \frac{a_v}{a} f_v + \frac{a^2}{r_1} (K - c)e_3 \right)\]

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is a timelike unit normal vector which is orthogonal to $e_3$. So $\{e_3, e_4\}$ is a normal orthonormal frame field to $M$ of signature $(+, -)$.

Using (3.1), (3.2), (3.3) and (3.5), we can compute that

$$\langle f_{uu}, e_3 \rangle = r_1, \quad \langle f_{vv}, e_3 \rangle = -\frac{a^2}{r_1} (K - c), \quad (3.6)$$

$$\langle f_{uu}, e_4 \rangle = 0, \quad \langle f_{vv}, e_4 \rangle = -r_2. \quad (3.7)$$

We denote by $D$ the flat connection of $R^{5}_5$ when $c > 0$, or $R^{5}_3$ when $c < 0$. Using (3.4), (1.1), (3.6) and (3.7), we have the components of the second fundamental form $h$ as follows:

$$h_{11}^3 = \langle D_{e_1} e_1, e_3 \rangle = \frac{1}{2a} \langle f_{uu} + 2f_{uv} + f_{vv}, e_3 \rangle = \frac{1}{2a} \left( r_1 - \frac{a^2}{r_1} (K - c) \right) = h_{22}^3,$$

$$h_{12}^3 = \langle D_{e_1} e_2, e_3 \rangle = \frac{1}{2a} \langle f_{uu} - f_{vv}, e_3 \rangle = \frac{1}{2a} \left( r_1 + \frac{a^2}{r_1} (K - c) \right),$$

$$h_{11}^4 = -\langle D_{e_1} e_1, e_4 \rangle = \frac{r_2}{2a} = h_{22}^4, \quad h_{12}^4 = -\langle D_{e_1} e_2, e_4 \rangle = -\frac{r_2}{2a}.$$

By (2.9), the normal curvature $K_\nu$ is given by

$$K_\nu = -\frac{r_1 r_2}{a^2}. \quad (3.8)$$

By (3.5) and (3.8), we get

$$(K - c)^2 - K_\nu^2 = \frac{r_1^2}{a^4} \langle f_{vv}, f_{vv} \rangle = \frac{\langle f_{uu}, f_{uu} \rangle \langle f_{vv}, f_{vv} \rangle}{\langle f_u, f_v \rangle^4}.$$

(ii) Next we consider the case where $\langle f_{uu}, f_{uu} \rangle < 0$. We can write $\langle f_{uu}, f_{uu} \rangle = -r_1^2$ for some positive function $r_1$. Then the vector

$$e_4 = \frac{1}{r_1} V = \frac{1}{r_1} \left( f_{uu} - \frac{a_u}{a} f_u \right)$$

is a timelike unit normal vector to $M$. Set

$$W = f_{vv} - \frac{a_v}{a} f_v + \frac{1}{r_1} \left( c a_v^2 + a_{uv} - \frac{a_a a_v}{a} \right) e_4$$

$$= f_{vv} - \frac{a_v}{a} f_v - \frac{a^2}{r_1} (K - c) e_4.$$
Then we can see that $W$ is a nonzero normal vector which is orthogonal to $e_4$. So $W$ is spacelike, and

$$\langle W, W \rangle = \langle f_{vv}, f_{vv} \rangle + \frac{a^4}{r_1^2} (K - c)^2 =: r_2^2 > 0,$$

(3.9)

for some positive function $r_2$. Then the vector

$$e_3 = \frac{1}{r_2} W = \frac{1}{r_2} \left( f_{vv} - \frac{a_v}{a} f_v - \frac{a^2}{r_1} (K - c) e_4 \right)$$

is a spacelike unit normal vector which is orthogonal to $e_4$. So $\{e_3, e_4\}$ is a normal orthonormal frame field to $M$ of signature $(+, -)$.

Using (3.1), (3.2), (3.3) and (3.9), we can get

$$\langle f_{uu}, e_4 \rangle = -r_1, \quad \langle f_{vv}, e_4 \rangle = -\frac{a^2}{r_1} (K - c),$$

$$\langle f_{uu}, e_3 \rangle = 0, \quad \langle f_{vv}, e_3 \rangle = r_2.$$

Then, as in the case (i),

$$h_{11}^3 = h_{22}^3 = \frac{r_2}{2a}, \quad h_{12}^3 = -\frac{r_2}{2a},$$

$$h_{11}^4 = h_{22}^4 = \frac{1}{2a} \left( r_1 + \frac{a^2}{r_1} (K - c) \right), \quad h_{12}^4 = \frac{1}{2a} \left( r_1 - \frac{a^2}{r_1} (K - c) \right),$$

and

$$K_\nu = \frac{r_1 r_2}{a^2}.$$

(3.10)

By (3.9) and (3.10), we have

$$(K - c)^2 - K_\nu^2 = -\frac{r_1^2}{a^4} \langle f_{vv}, f_{vv} \rangle = \frac{\langle f_{uu}, f_{uu} \rangle \langle f_{vv}, f_{vv} \rangle}{\langle f_u, f_v \rangle^4}.$$

Remark. Changing the roles of $u$ and $v$, we can show the theorem also in the case where $\langle f_{vv}, f_{vv} \rangle > 0$ or $\langle f_{vv}, f_{vv} \rangle < 0$.

(iii) We consider the case where both $f_{uu}$ and $f_{vv}$ are null at a point $p \in M$. Set

$$N_1 = f_{uu} - \frac{a_u}{a} f_u, \quad N_2 = f_{vv} - \frac{a_v}{a} f_v.$$

Then $N_1$ and $N_2$ are null and linearly independent normal vector at $p$. So $\langle N_1, N_2 \rangle \neq 0$ at $p$, because the normal plane is 2-dimensional Minkowskian.

By (3.1), (3.2) and (3.3), we have

$$\langle N_1, N_2 \rangle = \langle f_{uu}, f_{vv} \rangle - \frac{a_u a_v}{a} = -a^2 (K - c).$$
So $K \neq c$ at $p$. When $K < c$ at $p$, set
\[ e_3 = \frac{1}{a\sqrt{2(c - K)}} (N_1 + N_2), \quad e_4 = \frac{1}{a\sqrt{2(c - K)}} (N_1 - N_2). \]
When $K > c$ at $p$, set
\[ e_3 = \frac{1}{a\sqrt{2(K - c)}} (N_1 - N_2), \quad e_4 = \frac{1}{a\sqrt{2(K - c)}} (N_1 + N_2). \]
In each case, $\{e_3, e_4\}$ is a normal orthonormal basis at $p$ of signature $(+, -)$.

Using (3.1), (3.2) and (3.3), we have at $p$,
\[ \langle f_{uu}, N_1 \rangle = 0, \quad \langle f_{uu}, N_2 \rangle = -a^2(K - c), \]
\[ \langle f_{uv}, N_1 \rangle = -a^2(K - c), \quad \langle f_{vv}, N_2 \rangle = 0. \]
So, when $K < c$ at $p$, we get
\[ h_{11}^3 = h_{22}^3 = \sqrt{\frac{c - K}{2}}, \quad h_{12}^3 = 0, \]
\[ h_{11}^4 = h_{22}^4 = 0, \quad h_{12}^4 = \sqrt{\frac{c - K}{2}}, \]
and
\[ K_u = c - K, \quad (K - c)^2 - K_u^2 = 0. \]
When $K > c$ at $p$, we get
\[ h_{11}^3 = h_{22}^3 = 0, \quad h_{12}^3 = \sqrt{\frac{K - c}{2}}, \]
\[ h_{11}^4 = h_{22}^4 = \sqrt{\frac{K - c}{2}}, \quad h_{12}^4 = 0, \]
and
\[ K_v = -(K - c), \quad (K - c)^2 - K_v^2 = 0. \]
Now, from (i), (ii) and (iii), we have proved Theorem 1.1.

Proof of Theorem 1.2. Let $P(u)$ and $Q(v)$ be null curves in $N^4_2(c)$ ($c \neq 0$) which satisfy $P(0) = Q(0)$ and $\langle P'(0), Q'(0) \rangle \neq 0$. By Theorem 2 of [3], there exists a Lorentzian conformal stationary immersion $F(s, t)$ from a domain containing $(0, 0)$ of $R^3$ into $N^4_2(c)$, which satisfies
\[ \langle F_s, F_s \rangle = -\langle F_t, F_t \rangle \neq 0, \quad \langle F_s, F_t \rangle = 0, \]
\[ F(s, s) = P(s), \quad F(s, -s) = Q(s). \]
It is a kind of characteristic initial value problem. Set $f(u, v) = F(u + v, u - v)$, which gives a Lorentzian stationary surface $M$ in $N^4_2(c)$ such that $f_u$ and $f_v$ are null, and $f(u, 0) = P(u), f(0, v) = Q(v)$.

Assume that $\{P'(0), Q'(0), P''(0), Q''(0)\}$ are linearly independent, and $\langle P'', P'' \rangle \langle Q'', Q'' \rangle > 0$ (resp. $< 0$). Then, using Theorem 1.1 for the surface $M$, we have $(K - c)^2 - K^2_v > 0$ (resp. $< 0$) near $(u, v) = (0, 0)$. Thus we have proved Theorem 1.2.
4 Examples

Let $R^5_2$ be the 5-dimensional semi-Euclidean space of index 2 with standard coordinate system $(x_1, x_2, x_3, x_4, x_5)$ and signature $(+, +, +, -, -)$. Then

$$S^2_4(1) = \{ (x_1, x_2, x_3, x_4, x_5) \in R^5_2 | x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = 1 \}.$$ 

**Example 4.1.** Let $P(u)$ be a null curve in $S^2_4(1)$ given by

$$P(u) = \left( \cos \left( \frac{u^2}{2} \right), \sin \left( \frac{u^2}{2} \right), u \cosh u, u, u \sinh u \right).$$

Then

$$P(0) = (1, 0, 0, 0, 0), \quad P'(0) = (0, 0, 1, 1, 0), \quad P''(0) = (0, 1, 0, 0, 2),$$
and $\langle P'''(0), P''(0) \rangle = -3$. Next, let $Q(v)$ be a null curve in $S^2_4(1)$ given by

$$Q(v) = (2 \cos v - \cos(\sqrt{2}v), \sqrt{2} \sin v, 0, \sin(\sqrt{2}v), -\sqrt{2} \cos v + \sqrt{2} \cos(\sqrt{2}v)).$$

Then

$$Q(0) = (1, 0, 0, 0, 0), \quad Q'(0) = (0, \sqrt{2}, 0, \sqrt{2}, 0), \quad Q''(0) = (0, 0, 0, 0, -\sqrt{2}),$$
and $\langle Q''(0), Q'(0) \rangle = -2$.

We can see that $P(0) = Q(0)$, $\langle P'(0), Q'(0) \rangle = -\sqrt{2} \neq 0$, $\langle P''(0), Q'(0), P''(0), Q''(0) \rangle$ are linearly independent, and

$$\langle P''(0), P''(0) \rangle \langle Q''(0), Q''(0) \rangle = 6 > 0.$$ 

So by Theorem 1.2, there exists a Lorentzian stationary surface in $S^2_4(1)$ whose inclusion map $f(u, v)$ satisfies $f_u$ and $f_v$ are null, $f(u, 0) = P(u)$, $f(0, v) = Q(v)$, and $(K - 1)^2 - K^2_{\nu} > 0$ near $(u, v) = (0, 0)$.

**Example 4.2.** Let $P(u)$ be a null curve in $S^2_4(1)$ given by

$$P(u) = \left( \cosh \left( \frac{u^2}{2} \right), u \cos u, u \sin u, u, \sinh \left( \frac{u^2}{2} \right) \right),$$

which satisfies

$$P(0) = (1, 0, 0, 0, 0), \quad P'(0) = (0, 1, 0, 1, 0), \quad P''(0) = (0, 0, 2, 0, 1),$$
and $\langle P'''(0), P''(0) \rangle = 3$. Let $Q(v)$ be a null curve in $S^2_4(1)$ given by

$$Q(v) = \left( \cos \left( \frac{v^2}{2} \right), \sin \left( \frac{v^2}{2} \right), v, v \cos v, v \sin v \right),$$

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which satisfies
\[ Q(0) = (1, 0, 0, 0, 0), \quad Q'(0) = (0, 0, 1, 1, 0), \quad Q''(0) = (0, 1, 0, 0, 2), \]
and \( \langle Q''(0), Q''(0) \rangle = -3. \)

We can see that \( P(0) = Q(0), \langle P'(0), Q'(0) \rangle = -1 \neq 0, \)
\( \{ P'(0), Q'(0), P''(0), Q''(0) \} \) are linearly independent, and
\[ \langle P''(0), P''(0) \rangle \langle Q''(0), Q''(0) \rangle = -9 < 0. \]

By Theorem 1.2, there exists a Lorentzian stationary surface in \( S^4_2(1) \) whose
inclusion map \( f(u, v) \) satisfies \( f_u \) and \( f_v \) are null, \( f(u, 0) = P(u), f(0, v) = Q(v) \), and \( (K + 1)^2 - K^2 < 0 \) near \( (u, v) = (0, 0) \).

Let \( \sigma : R^5_2 \to R^5_3 \) be the canonical anti-isometry from \( R^5_2 \) to \( R^5_3 \), which
naturally induces the anti-isometry from \( S^4_2(1) \) to \( H^4_2(-1) \) (cf. p.110 of [9]).
For the null curves \( P(u) \) and \( Q(v) \) in \( S^4_2(1) \) in Example 4.1 or 4.2, considering null curves \( \sigma(P(u)) \) and \( \sigma(Q(v)) \) in \( H^4_2(-1) \), we can get Lorentzian stationary surfaces in \( H^4_2(-1) \) such that \( (K + 1)^2 - K^2 \) is positive, or negative.

5 A remark

Let \( f(u, v) \) be a Lorentzian surface in a semi-Euclidean space \( R^n_q \) with co-
ordinate system \( (u, v) \) such that \( f_u \) and \( f_v \) are null. Then the value
\[ \frac{\langle f_{uu}, f_{uu} \rangle \langle f_{vv}, f_{vv} \rangle}{\langle f_u, f_v \rangle^4} \]
is an invariant, which is independent of the choice of such coordinate sys-
tems. So, some more general treatment will be possible. Furthermore, it
should be interesting to find and study higher order invariants for Lorentzian
surfaces in \( R^n_q \).

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Received: 6.VII.2016 / Accepted: 8.II.2017

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