On \((k,n)\)-closed submodules

Ece Yetkin Celikel

Abstract. Let \(R\) be a commutative ring with \(1 \neq 0\) and \(M\) an \(R\)-module. We will call a proper submodule \(N\) of \(M\) as a semi \(n\)-absorbing submodule of \(M\) if whenever \(r \in R\), \(m \in M\) with \(r^nm \in N\), then \(rm \in N\). We will say \(N\) to be a \((k,n)\)-closed submodule of \(M\) if whenever \(r \in R\), \(m \in M\) with \(r^nm \in N\), then \(r^nk \in N\). In this paper we introduce semi \(n\)-absorbing and \((k,n)\)-closed submodules of modules over commutative rings, and investigate their basic properties.

Keywords. \((m,n)\)-closed ideal · \(n\)-absorbing submodule · semi \(n\)-absorbing submodule · semi \(n\)-absorbing ideal · \((k,n)\)-closed submodule

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1 Introduction

Let \(R\) be a commutative ring with \(1 \neq 0\) and \(I\) be a proper ideal of \(R\). As stated in [3], \(I\) is called an \(n\)-absorbing (resp. strongly \(n\)-absorbing) ideal of \(R\) if whenever \(x_1 \cdots x_{n+1} \in I\) for \(x_1, \ldots, x_{n+1} \in R\) (resp. \(I_1 \cdots I_{n+1} \subseteq I\) for ideals \(I_1, \ldots, I_{n+1}\) of \(R\)), then there are \(n\) of the \(x_i\)'s (resp. \(n\) of the \(I_i\)'s) whose product is in \(I\). Recall that a proper ideal \(I\) of \(R\) is said to be semi-prime ideal if whenever \(r^2 \in I\) for some \(r \in R\), then \(r \in I\). For generalizations of semi-prime ideals the reader may consult [9]. In [4], D. F. Anderson and A. Badawi said \(I\) to be a semi \(n\)-absorbing ideal if \(x^{n+1} \in I\) for \(x \in R\) implies \(x^n \in I\). Also A. Badawi said that a proper ideal \(I\) of \(R\) is a \((m,n)\)-closed ideal if \(x^m \in I\) for \(x \in R\) implies that \(x^n \in I\) [4]. Let \(M\) be an \(R\)-module. A proper submodule \(N\) of \(M\) is called \(n\)-absorbing (resp. strongly \(n\)-absorbing) submodule of \(M\) if whenever \(a_1 \cdots a_m \in N\) for \(a_1, \ldots, a_m \in R\) and \(m \in M\) (resp. \(I_1 \cdots I_n \subseteq I\) for ideals \(I_1, \ldots, I_n\) of \(R\) and a submodule \(L\) of \(M\)), then either \(a_1 \cdots a_\alpha \in (N :_R M)\) (resp. \(I_1 \cdots I_\alpha \subseteq (N :_R M)\)) or there are \(n-1\) of \(a_i\)'s (\(I_i\)'s) whose product with \(m\) (resp. \(I\)) is in \(N\) [6]. A proper submodule \(N\) of an \(R\)-module \(M\) is called semi-prime if whenever \(r \in R\) and \(m \in M\) with \(r^2m \in N\), then \(rm \in N\).

A proper submodule \(N\) of \(M\) is called a quasi-prime submodule of \(M\) if whenever
$a, b \in R, m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$. More generally, we define $(k, n)$-closed submodules of an $R$-module $M$ as follows: let $R$ be a commutative ring with identity and $k, n$ be positive integers. We call a proper submodule $N$ of $M$ as a $(k, n)$-closed submodule of $M$ if whenever $r \in R, m \in M$ with $r^k m \in N$, then $r^k \in (N:_R M)$ or $r^{k-1} m \in N$. In particular, we call $N$ as a semi $n$-absorbing submodule of $M$ if whenever $r \in R, m \in M$ with $r^n m \in N$, then $r^n \in (N:_R M)$ or $r^{n-1} m \in N$. It is clear that a semi $n$-absorbing submodule is $(n, n)$-closed.

Throughout we assume that all rings are commutative with 1 and $k, n$ are positive integers. The radical of an ideal $I$ of $R$ is denoted by $\sqrt{I}$. We denote the set of invertible (unit) elements of $R$ by $U(R)$, i.e. $U(R) = \{u \in R : \text{there is a } v \in R \text{ such that } uv = vu = 1_R\}$. Let $N$ be a submodule of an $R$-module $M$. We will denote by $(N:_R M)$ the residual of $N$ by $M$, that is, the set of all $r \in R$ such that $rM \subseteq N$. An $R$-module $M$ is called a multiplication module if every submodule $N$ of $M$ has the form of $IM$ for some ideal $I$ of $R$. Note that, since $I \subseteq (N:_R M)$ then $N = IM \subseteq (N:_R M)M \subseteq N$. So that $N = (N:_R M)M$ [7]. For a submodule $N$ of $M$, if $N = IM$ for some ideal $I$ of $R$, then we say that $I$ is a presentation ideal of $N$. Clearly, every submodule of $M$ has a presentation ideal if and only if $M$ is a multiplication module. Let $N$ and $K$ be submodules of a multiplication $R$-module $M$ with $N = I_1M$ and $K = I_2M$ for some ideals $I_1$ and $I_2$ of $R$. The product of $N$ and $K$ denoted by $NK$ is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of $N$ and $K$ is independent of presentations of $N$ and $K$. Moreover, for $a, b \in M$, by $ab$, we mean the product of $Ra$ and $Rb$. Clearly, $NK$ is a submodule of $M$ and $NK \subseteq N \cap K$ (see [1]). It is well-known that if $R$ is a commutative ring and $M$ a non-zero multiplication $R$-module, then every proper submodule of $M$ is contained in a maximal submodule of $M$. [7, Theorem 2.5]. As a generalization of Jacobson radical of $R$, the radical of the module $M$ is defined by the intersection of all maximal submodules of $M$, that is $\text{Rad}(M) = \cap \{N : N$ is a maximal submodule of $M\}$. Let $N$ be a proper submodule of a non-zero $R$-module $M$. Then the $M$-radical of $N$ denoted by $M-\text{rad}(N)$ is defined to be the intersection of all prime submodules of $M$ containing $N$. If $M$ has no prime submodule containing $N$, then we say $M-\text{rad}(N) = M$.

In this study, we give many properties of $(k, n)$-closed submodules and also obtain relationships among semi $n$-absorbing submodules, $(k, n)$-closed submodules and the other concepts. For general background and terminology, the reader may consult [2] and [10].

2 Properties of $(k, n)$-closed submodules

In this section, we introduce and study basic properties of semi $n$-absorbing and $(k, n)$-closed submodules with many examples.

**Lemma 2.1** Let $N$ be a proper submodule of an $R$-module $M$. Then the following statements are equivalent:

1. $N$ is a $(k, n)$-closed submodule of $M$. 


2. If whenever $r \in R$ and $L$ is a submodule of $M$ with $r^k L \subseteq N$, then $r^{n-1} L \subseteq N$ or $r^n \in (N : R M)$.

In particular, a proper submodule $N$ of $M$ is a semi $n$-absorbing submodule of $M$ if and only if whenever $r \in R$, $L$ a submodule of $M$ with $r^n L \subseteq N$ implies either $r^n \in (N : R M)$ or $r^{n-1} L \subseteq N$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $N$ is a $(k, n)$-closed submodule of $M$. Let $r \in R$ and $L$ be a submodule of $M$ with $r^k L \subseteq N$. Assume that $r^{n-1} L \not\subseteq N$. Then $r^n m \not\in N$ for some $m \in L$. Since $r^k m \in N$ and $r^{n-1} m \not\in N$, we conclude $r^n \in (N : R M)$, as needed.

(2) $\Rightarrow$ (1) This part is clear. □

There are some relationships between $(k, n)$-closed submodules of $M$ and $(k, n)$-closed ideals of $R$.

**Theorem 2.2** Let $M$ be an $R$-module, and $N$ be a proper submodule of $M$. If $N$ is a $(k, n)$-closed submodule of $M$, then $(N : R M)$ is a $(k, n)$-closed ideal of $R$. If $M$ is a multiplication $R$-module, then the presentation ideal of a $(k, n)$-closed submodule of $M$ is a $(k, n)$-closed ideal of $R$.

**Proof.** Assume that $r \in R$ with $r^k \in (N : R M)$ but $r^n \not\in (N : R M)$. Then there is an element $m \in M$ with $r^m m \not\in N$ which means that $r^{n-1} m \not\in N$. Since $r^m m \in N$, $r^{n-1} m \not\in N$ and $r^n \not\in (N : R M)$, this situation contradicts with our hypothesis. Thus $(N : R M)$ is a $(k, n)$-closed ideal of $R$. □

However, the converse of Theorem 2.2 is not true in general. For example, consider $N = 6 \mathbb{Z}$ as a submodule of $\mathbb{Z}$-module $\mathbb{Z}$. While $(N : \mathbb{Z} \mathbb{Z}) = 6 \mathbb{Z}$ is clearly a $(2, 1)$-closed ideal of $\mathbb{Z}$, $N$ is not $(2, 1)$-closed submodule of $\mathbb{Z}$. In fact $2^2 \cdot 3^2 \in N$ but $2^1 \not\in (N : \mathbb{Z} \mathbb{Z}) = 6 \mathbb{Z}$ and $2^0 \cdot 3^2 \not\in N$.

**Theorem 2.3** Let $N$ be a proper submodule of the $R$-module $M$.

1. If $N$ is a $(k, n)$-closed submodule of $M$, then $(N : R m)$ is a $(k, n)$-closed ideal of $R$ for each $m \in M \setminus N$.
2. If $(N : R m)$ is a $(k, n)$-closed ideal of $R$ for each $m \in M \setminus N$, then $N$ is a $(k, n + 1)$-closed submodule of $M$.

**Proof.** (1) Suppose that $r^k \in (N : R m)$ and $r^n \not\in (N : R m)$ for some $m \in M \setminus N$. Hence $r^k m \in N$ but $r^m m \not\in N$ which means $r^{n-1} m \not\in N$. Since $N$ is a $(k, n)$-closed submodule of $M$, we have $r^n \in (N : R M) \subseteq (N : R m)$, a contradiction. Thus $(N : R m)$ is a $(k, n)$-closed ideal of $R$ for each $m \in M \setminus N$.

(2) Let $r^k m \in N$ for $r \in R$ and $m \in M$. Assume that $r^{n+1} \not\in (N : R M)$. Since $r^k \in (N : R m)$ and $(N : R m)$ is a $(k, n)$-closed ideal of $R$ for each $m \in M \setminus N$, we conclude that $r^n \in (N : R m)$. Therefore $r^m m \in N$. This means that $N$ is a $(k, n + 1)$-closed submodule of $M$. □

**Lemma 2.4** Let $M$ be a finitely generated $R$-module such that $M = R m_1 + \ldots + R m_t$, $N$ be a proper submodule of $M$ and $k > n$. Then
1. If \((N : R m_i)\) is a \((k,n)\)-closed ideal of \(R\) for all \(i = 1,\ldots,t\), then \((N : R M)\) is a \((k,n)\)-closed ideal of \(R\). In particular, if \(M = Rm\) be a cyclic \(R\)-module and \(N\) is a proper submodule of \(M\), then \((N : R m)\) is a \((k,n)\)-closed ideal of \(R\) if and only if \((N : R M)\) is a \((k,n)\)-closed ideal of \(R\).

2. Let \(R\) be a division ring and \(M = Rm\) be a cyclic \(R\)-module. Then \((N : R m)\) is a \((k,n)\)-closed ideal of \(R\) if and only if \((N : R m')\) is a \((k,n)\)-closed ideal of \(R\) for all elements \(m' \in M\).

**Proof.** (1) Assume that \((N : R m_i)\) is a \((k,n)\)-closed ideal of \(R\) for all \(i = 1,\ldots,t\).

Suppose that \(r^k \in (N : R M)\) and \(r^0 \notin (N : R M)\) for some \(r \in R\). Then \(r^j \notin (N : R m_i)\) for some \(j = 1,\ldots,t\). Hence \(r^k \notin (N : R m_1)\), and so \(r^k \notin (N : R M)\) which contradicts with our assumption. Thus \((N : R M)\) is a \((k,n)\)-closed ideal of \(R\). The “in particular” part is clear.

(2) Suppose that \(R\) is a division ring and \(M = Rm\) is a cyclic \(R\)-module. Then one can easily obtain that \((N : R m) = (N : R m')\), so we are done. \(\square\)

**Theorem 2.5** Let \(R\) be a division ring and \(N\) be a proper submodule of a cyclic \(R\)-module \(M = Rm\).

1. If \((N : R m)\) is a \((k,n)\)-closed ideal of \(R\), then \(N\) is a \((k,n+1)\)-closed submodule of \(M\).

2. If \((N : R m)\) is a semi \(n\)-absorbing ideal of \(R\), then \(N\) is a semi \((n+1)\)-absorbing submodule of \(M\).

**Proof.** (1) From Theorem 2.3 and Lemma 2.4 (2), we are done.

(2) Since a semi \(n\)-absorbing ideal of \(R\) is a \((n+1,n)\)-closed ideal, \(N\) is a \((n+1,n+1)\)-closed submodule of \(M\) by (1), so it is clear. \(\square\)

In Theorem 2.5, the condition ”division ring” on \(R\) is necessary. Otherwise, if \((N : R m)\) is a \((k,n)\)-closed ideal of \(R\), then \(N\) is not need to be \((k,n+1)\)-closed submodule of \(M\) as in the following example.

**Example 2.1** Consider \(N = 8\mathbb{Z}\) as a submodule of \(\mathbb{Z}\)-module \(\mathbb{Z}\). Then \((N : 2\mathbb{Z}) = 8\mathbb{Z}\) is \((2,1)\)-closed ideal but \(N\) is not \((2,2)\)-closed submodule of \(M\). In fact \(2^2 \cdot 2 \in N\) but neither \(2 \cdot 2 \in N\) nor \(2^2 \in (N : 2\mathbb{Z})\).

**Proposition 2.6** Let \(N\) be a proper submodule of an \(R\)-module \(M\) and \(k > t\). Then the following statements are equivalent:

1. \(N\) is a \((k,n)\)-closed submodule of \(M\).

2. \((N : R r^k m) = (N : R r^{n-1} m)\) or \(r^n \notin (N : R M)\) for \(r \in R\) and \(m \in M\).

**Proof.** (1)\(\Rightarrow\) (2) Suppose that \(N\) is a \((k,n)\)-closed submodule of \(M\) and \(r^n \notin (N : R M)\). Let \(s \in (N : R r^k m)\). Hence \(r^k (sm) \in N\). Since \(N\) is \((k,n)\)-closed and \(r^n \notin (N : R M)\), we get \(r^{k-1} sm \in N\). It follows \(s \in (N : R r^{n-1} m)\), that is \((N : R r^k m) \subseteq (N : R r^{n-1} m)\). Since the inverse inclusion is always hold, this completes the proof.

(2)\(\Rightarrow\) (1) Suppose that \(r \in R\), \(m \in M\) with \(r^k m \in N\). If \(r^n \in (N : R M)\), then we are done. So assume that \((N : R r^k m) = (N : R r^{n-1} m)\). Thus \(r^{n-1} m \in N\), as needed. \(\square\)
The relations among the concepts of semi-prime, semi-\( n \)-absorbing, quasi-prime, \( n \)-absorbing submodules and \((k,n)\)-closed submodules are provided in the following theorem.

**Theorem 2.7** Let \( M \) be an \( R \)-module and \( N \) be a proper submodule of \( M \). Then the following statements hold:

1. Let \( N \) be a semi-prime submodule of \( M \). Then \( N \) is a \((k,n)\)-closed submodule of \( M \) for all positive integers \( k \) and \( n \). Moreover \( N \) is a semi-\( n \)-absorbing submodule of \( M \) for all positive integer \( n \).
2. If \( N \) is an \( n \)-absorbing submodule of \( M \), then \( N \) is a semi-\( n \)-absorbing submodule of \( M \).
3. If \( N \) is an \( n \)-absorbing submodule of \( M \), then \( N \) is a \((k,n)\)-closed submodule of \( M \) for every positive integer \( k \).
4. If \( N \) is a \((k,n)\)-closed submodule of \( M \), then \( N \) is a \((k_1,n_1)\)-closed submodule of \( M \) for all \( k_1 \leq k \) and \( n_1 \geq n \).
5. If \( N \) is a semi-\( n \)-absorbing submodule of \( M \), then \( N \) is a semi-\( n_1 \)-absorbing submodule of \( M \) for all \( n_1 \geq n \).
6. If \( N \) is a quasi-prime submodule of \( M \), then \( N \) is a \((k,n)\)-closed submodule of \( M \) for all positive integers \( k \geq n \geq 2 \).

**Proof.** (1), (2), (3) and (4) are clear from the definitions.

(5) Induction method on \( n \). For \( n = 1 \), it is clear. So suppose that \( n \geq 2 \) and \( N \) is a semi \((n-1)\)-absorbing submodule of \( M \). We show that \( N \) is semi-\( n \)-absorbing. Let \( r \in R \) and \( m \in M \) with \( r^nm \in N \). Assume that \( r^n \in (N:RM) \). Hence \( r^{n-1}(rm) \in N \) which implies that \( r^{n-2}(rm) = r^{n-1}m \in N \) by induction hypothesis. Thus \( N \) is a semi-\( n \)-absorbing of \( M \) for all \( n \geq 2 \).

(6) We show that \( N \) is a \((k,2)\)-closed submodule of \( M \) for all \( k \geq 2 \) by using mathematical induction on \( k \). Suppose that \( N \) is a quasi-prime submodule of \( M \). Then \( N \) is a \((k,2)\)-closed submodule of \( M \) for \( k = 2 \) directly from their definitions. Now suppose that \( N \) is a \((t,2)\)-closed submodule of \( M \) for all \( 2 \leq t < k \) and our aim is to show that \( N \) is \((k,2)\)-closed. Let \( rk \in N \) for \( r \in R \) and \( m \in M \). Assume that \( r^2 \notin (N:RM) \). Since \( r^{k-1}(rm) \in N \), and \( N \) is \((k-1,2)\)-closed by induction hypothesis, we conclude that \( r^{k}(rm) = r^k m \in N \). Since \( N \) is \((2,2)\)-closed and \( r^2 \notin (N:RM) \), we get \( rm \in N \). Thus \( N \) is a \((k,2)\)-closed submodule of \( M \) for all \( k \geq 2 \). Consequently, \( N \) is a \((k,n)\)-closed submodule of \( M \) for all positive integers \( n \) with \( k \geq n \geq 2 \) by (4).

**Example 2.2** The converses of (1)-(6) in Theorem 2.7 are not true in general as these situations are shown in the following examples.

1. Let \( N = 30\mathbb{Z} \) as a submodule of the \( \mathbb{Z} \)-module \( \mathbb{Z} \). Since \( N = 2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} \) is intersection of semi-prime submodules of \( \mathbb{Z} \), it is semi-2-absorbing \(((2,2)\)-closed) submodule of \( \mathbb{Z} \) from Theorem 2.10. Also it is \((3,2)\)-closed submodule of \( \mathbb{Z} \) from Theorem 2.7 (4). However \( N \) is not 2-absorbing submodule of \( \mathbb{Z} \). In fact \( 2 \cdot 3 \cdot 5 \in N \) but \( 2 \cdot 3 \notin (N:2\mathbb{Z}) \) and \( 2 \cdot 5 \notin N \) and \( 3 \cdot 5 \notin N \). So the converses of (2) and (3) are not true.
2. Consider the submodule \( N = (0) \) of \( \mathbb{Z} \)-module \( \mathbb{Z}_{p^n} \) where \( p \) is a prime and \( n \) is positive integer. Then \( N \) is a \((n,n)\)-closed submodule of \( \mathbb{Z}_{p^n} \), but \( N \) is not \((n,n-1)\)-closed as \( p^n \mathbb{T} = 0 \in N \) but neither \( p^{n-2} \mathbb{T} \in N \) nor \( p^{n-1} \in (N : \mathbb{Z}_{p^n}) = (p^n) \). Note that \( N \) in \( \mathbb{Z} \)-module \( \mathbb{Z}_{p^n} \) is a semi \( n \)-absorbing submodule of \( \mathbb{Z}_{p^n} \), but it is not quasi-prime as \( p^n \mathbb{T} \in N \) but \( p \mathbb{T} \notin N \). Also it is not semi \((n-1)\) absorbing (it is also not semi-prime clearly) submodule as \( p^{n-1} \in p \mathbb{T} \notin N \) but neither \( p^{n-1} \in (N : \mathbb{Z}_{p^n}) \) nor \( p^{n-2} p = p^{n-1} \in N \). Thus the coverses of (1), (4), (5) and (6) are not true.

**Theorem 2.8** Let \( N \) be a proper submodule of \( M \). If \( N \) is a semi \( n \)-absorbing submodule of \( M \), then \( N \) is a \((k,n)\)-closed submodule of \( M \) for all positive integer \( k \).

**Proof.** If \( k \leq n \), the claim is clear. So suppose that \( k > n \) and say \( t := k - n \). Let \( r^t m \in N \) for some \( r \in R \) and \( m \in M \). Assume that \( r^t \notin (N : R M) \). Hence \( r^t (r^{-1} m) \in N \). Since \( N \) is semi \( n \)-absorbing and \( r^t \notin (N : R M) \), we get \( r^{t-1} (r^{-1} m) = r^t (r^{-1} m) \in N \). This follows \( r^{t-1} (r^{-1} m) = r^n (r^{t-2}) \in N \) as again \( N \) is a semi \( n \)-absorbing submodule of \( M \). It implies that \( r^n (r^{t-3} m) \in N \). So we continue with this argument and obtain that \( r^n m \in N \) at the \( r^n \) step. Finally we conclude \( r^n m \in N \) which means that \( N \) is a \((k,n)\)-closed submodule of \( M \). \( \square \)

**Corollary 2.9** Let \( N \) be a proper submodule of \( M \) and \( k > n \). Then \( N \) is a \((k,n)\)-closed submodule of \( M \) if and only if \( N \) is a semi \( n \)-absorbing submodule of \( M \).

**Proof.** Suppose that \( N \) is \((k,n)\)-closed and \( r^n m \in N \) for \( r \in R \) and \( m \in M \). So \( r^n m \in N \), and this implies that either \( r^k \notin (N : R M) \) or \( r^{n-1} m \in N \). Thus \( N \) is a semi \( n \)-absorbing submodule of \( M \). The converse part follows from Theorem 2.8. \( \square \)

**Theorem 2.10** Let \( \{N_\lambda\}_{\lambda \in \Lambda} \) be a family of semi-prime submodules of \( M \). Then \( \cap_{\lambda \in \Lambda} N_\lambda \) is a \((k,n)\)-closed submodule of \( M \) for all positive integers \( k \) and \( n \).

**Proof.** Suppose that \( r^k m \in \cap_{\lambda \in \Lambda} N_\lambda \) for \( r \in R \) and \( m \in M \). Then \( r^k m \in N_\lambda \) for all \( \lambda \in \Lambda \). Since each \( N_\lambda \) is semi-prime, we conclude that \( rm \in N_\lambda \) for all \( \lambda \in \Lambda \). Thus \( rm \in \cap_{\lambda \in \Lambda} N_\lambda \) for all \( \lambda \in \Lambda \). This implies that \( r^{n-1} m \in \cap_{\lambda \in \Lambda} N_\lambda \) for all \( n \). From Theorem 2.7 (4), \( \cap_{\lambda \in \Lambda} N_\lambda \) is \((k,n)\)-closed for all integers \( k \) and \( n \). \( \square \)

**Corollary 2.11** Let \( N \) be a proper submodule of an \( R \)-module \( M \). Then \( M - \text{rad}(N) \) and \( \text{Rad}(M) \) are \((k,n)\)-closed submodules of \( M \) for all integers \( k \) and \( n \).

**Proof.** The result is follows from Theorem 2.10. \( \square \)

**Lemma 2.12** [5] Let \( R \) be a commutative ring, \( M \) a finitely generated multiplication \( R \)-module and \( N_1, \ldots, N_i \) are pairwise comaximal \( R \)-submodules of \( M \). Then the following statements hold:

1. \( N_1N_2 = N_1 \cap N_2 \).
2. \( N_1 \cap \cdots \cap N_i = 1 \) and \( N_i \) are comaximal.
3. \( N_1 \cdots N_i = N_1 \cap \cdots \cap N_i \).
Theorem 2.13 Let $M$ be finitely generated multiplication $R$-module and $N_1, \ldots, N_t$ be semi-prime submodules of $M$. If $N_1, \ldots, N_t$ are pairwise comaximal, then $N_1 \cap \cdots \cap N_t$ is a $(k, n)$-closed submodule of $M$ for all positive integers $k$ and $n$. In particular, if $N$ is semi-prime, then $N^n$ is a $(k, n)$-closed submodule of $M$.

Proof. It follows from Theorem 2.10 and Lemma 2.12. □

D.F. Anderson and A. Badawi proved in Theorem 2.3 [4] that the intersection of two semi $n$-absorbing ideals is also a semi $n$-absorbing ideal of $R$. However this situation is not true for submodules of any module. The intersection of two semi $n$-absorbing submodules may not to be semi $n$-absorbing as the following:

Example 2.3 Consider $\mathbb{Z}$ as $\mathbb{Z}$-module and two submodules $N = p^n\mathbb{Z}$ and $K = q^n\mathbb{Z}$ of $\mathbb{Z}$ where $p$ and $q$ are prime integers. Clearly both of them are semi $n$-absorbing submodules of $\mathbb{Z}$. However $N \cap K = p^n\mathbb{Z}$ is not semi $n$-absorbing since $p^n(q^n) \notin N \cap K$ but $p^n(q^n) \notin N \cap K$ and $p^n \notin (N \cap K : \mathbb{Z} \mathbb{Z})$.

Theorem 2.14 Let $N_1, \ldots, N_t$ be semi $n_1$-absorbing submodules of $M$. Then $\bigcap_{\lambda \in \Lambda} N_\lambda$ is a $(k, n)$-closed submodule of $M$.

Proof. Let $r^m m \in N$ for $r \in R$ and $m \in M$. If $r^{n_1} \in (N_{\lambda_1} ) M$ for all $\lambda_1 \in \Lambda$, then $r^m \in (\bigcap (N_{\lambda_1} ) M = (\bigcap N_{\lambda_1} ) M)$, we are done. Suppose that $r^m \notin (N_{\lambda_0} ) M$ for some $\lambda_0 \in \Lambda$. Then $r^m \notin (N_{\lambda_0} ) M$ for all $N_{\lambda_1} \subseteq N_{\lambda_0}$. Hence $r^m m \in N_{\lambda_1}$ for all $N_{\lambda_1} \subseteq N_{\lambda_0}$ as each $N_{\lambda_1}$ is $(k, n)$-closed. Therefore $r^m m \in (\bigcap N_{\lambda_1}$ which means that $\bigcap_{\lambda \in \Lambda} N_\lambda$ is a $(k, n)$-closed submodule of $M$. □

Theorem 2.15 Let $N_1$ and $N_2$ be proper submodules of an $R$-module $M$.

1. If $N_1$ is a semi $n_1$-absorbing and $N_2$ is a semi $n_2$-absorbing submodule of $M$, then $N_1 \cap N_2$ is a semi $(n_1+1)$-closed submodule of $M$ where $n = \max\{n_1, n_2\}$.
2. If $N_1, \ldots, N_t$ be semi $n_1$-absorbing submodules of $M$. Then $N_1 \cap \cdots \cap N_t$ is a semi $(n_1+1)$-absorbing submodule of $M$.
3. If $N_1, \ldots, N_t$ be semi $n_1$-absorbing submodules of $M$. Then $N_1 \cap \cdots \cap N_t$ is a semi $(n_1+2)$-absorbing submodule of $M$ where $n = \max\{n_1, \ldots, n_t\}$.

Proof. (1) Let $r \in R$ and $m \in M$ such that $r^{n_1+1} m \in N_1 \cap N_2$. First observe from Corollary 2.9 that $N_1$ and $N_2$ are $(n_1, n_2)$-closed and $(n_2, n_1)$-closed submodules of $M$, respectively. Hence we have $r^{n_1} \in (N_1 : R M)$ or $r^{n_1+1} m \in N_1$ and $r^{n_2} \in (N_2 : R M)$ or $r^{n_2+1} m \in N_2$. If $r^{n_1} \in (N_1 : R M)$ and $r^{n_2} \in (N_2 : R M)$, then $r^m \in (N_1 : R M) \cap (N_2 : R M) = (N_1 \cap N_2 : R M)$. If $r^m \in (N_1 : R M)$ and $r^{n_2+1} m \in N_2$, then $r^m m \in N_1 \cap N_2$. If symmetrically $r^{n_1+1} m \in N_1$, and $r^{n_2} \in (N_2 : R M)$, then again we have $r^m m \in N_1 \cap N_2$. For the last, if $r^{n_1+1} m \in N_1$ and $r^{n_2+1} m \in N_2$, then $r^{n_1+1} m \in N_1 \cap N_2$ and $N_1 \cap N_2$. Thus we conclude either $r^{n_1+1} \in (N_1 \cap N_2 : R M)$ or $r^m m \in (N_1 \cap N_2 : R M)$, as needed.

(2) One can easily obtain the proof by using induction method on $t$.

(3) We use induction method on $t$. If $t = 3$, then the claim is clear from (1) and (2). So assume that $t > 3$ and the claim is satisfied for $t - 1$. Then $N_1 \cap \cdots \cap N_{t-1}$
is semi \((n_{t-1} + 2)\)-absorbing. If \(n_{t-1} + 2 < n_t\), then \(N_1 \cap \cdots \cap N_t\) is semi \((n_{t-1} + 1)\)-absorbing submodule of \(M\) by part (1). Thus \(N_1 \cap \cdots \cap N_t\) is semi \((n_{t-1} + 2)\)-absorbing submodule of \(M\) by Theorem 2.7 (5). If \(n_{t-1} + 2 = n_t\), then \(N_1 \cap \cdots \cap N_t\) is semi \((n_{t-1} + 2)\)-absorbing submodule of \(M\) by part (2). If \(n_{t-1} + 2 > n_t\), then \(N_1 \cap \cdots \cap N_t\) is \((n_{t-1} + 3)\)-absorbing by part (1). Here observe that \(n_{t-1} + 3 = n_t + 2\) as \(n_{t-1} + 2 > n_t\) and \(n_{t-1} < n_t\). Therefore \(N_1 \cap \cdots \cap N_t\) is semi \((n_t + 2)\)-absorbing submodule of \(M\). □

**Theorem 2.16** Let \(R\) be a division ring, \(M\) a cyclic \(R\)-module, and \(N_1, \ldots, N_t\) be \((k_j, n_j)\)-closed submodules of \(M\). Then \(N_1 \cap \cdots \cap N_t\) is a \((k, n+1)\)-closed submodule of \(M\) for all integers \(k \leq \min\{k_1, \ldots, k_t\}\) and \(n \geq \min\{k, \max\{n_1, \ldots, n_t\}\}\).

**Proof.** Suppose that \(N_1, \ldots, N_t\) are \((k_j, n_j)\)-closed submodules of \(M\). Hence \((N_i :_RM)\), \(\ldots, (N_i :_RM)\) are \((k, n)\)-closed ideals of \(R\) by Theorem 2.2. Then \(\bigcap_{j=1}^{t} (N_j :_RM) = (\bigcap_{j=1}^{t} N_j :_RM)\) is a \((k, n)\)-closed ideal of \(R\) for \(k \leq \min\{k_1, \ldots, k_t\}\) and \(n \geq \min\{k, \max\{n_1, \ldots, n_t\}\}\) by Theorem 2.3 in [4]. Thus we conclude that \(\bigcap_{j=1}^{t} N_j\) is a \((k, n+1)\)-closed submodule of \(M\) by Theorem 2.5. □

A non-zero submodule \(N\) of an \(R\)-module \(M\) is called a secondary submodule of \(M\) if for each \(r \in R\) the homothety \(N \rightarrow N\) is surjective or nilpotent (resp. surjective or zero). In this case \(P = \sqrt{(0 :_R N)}\) is a prime ideal, and we call \(N\) a \(P\)-secondary submodule of \(M\). For more details concerning secondary submodule of a module refer to [8].

**Theorem 2.17** Let \(N\) be a secondary submodule of an \(R\)-module \(M\). If \(K\) is a semi \(n\)-absorbing submodule of \(M\), then \(N \cap K\) is a secondary submodule of \(M\).

**Proof.** Suppose that \(N\) is a \(P\)-secondary submodule of \(M\) and \(r \in R\). If \(r \in P = \sqrt{(0 :_R N)}\), then clearly \(r \in \sqrt{(0 :_R N \cap K)}\). So assume that \(r \notin P\). Since \(r^n \notin P\) for all \(n \geq 0\), this implies that \(r^n N = N\). It is needed to show that \(r(N \cap K) = (N \cap K)\). Let \(m \in N \cap K\). Since \(N = r^n N\), there is an element \(m_1\) of \(N\) such that \(m = r^n m_1 \in N \cap K \subseteq K\). Since \(K\) is semi \(n\)-absorbing, we conclude either \(r^n \in (K :_RM)\) or \(r^{n-1} m \in K\). If \(r^n \in (K :_RM)\), then \(N = r^n N \subseteq K\), and so \(r(N \cap K) = rN = N \cap K\). If \(r^{n-1} m \in K\), then \(m = r^m m_1 \in r(N \cap K)\), we are done. □

**Corollary 2.18** Let \(N\) and \(K\) be proper submodules of an \(R\)-module \(M\) with \(K \subseteq N\). If \(N\) is a secondary semi \(n\)-absorbing submodule of \(M\), then \(K\) is a semi \(n\)-absorbing submodule of \(M\).

**Proof.** This is a direct consequence of Theorem 2.17. □

Let \(N\) and \(K\) be submodules of \(M\) with \(K \subseteq N\). If \(N\) is a semi \(n\)-absorbing submodule of \(M\), then \(K\) is not need to be a semi \(n\)-absorbing submodule of \(M\) as the following example verifying this case. So Example 2.4 shows that the condition "secondary" in Corollary 2.18 is necessary.
Example 2.4 Consider a submodule \( N = 4\mathbb{Z} \) of \( \mathbb{Z} \)-module \( \mathbb{Z} \) and \( K = 12\mathbb{Z} \). Then \( K \) is clearly a semi 2-absorbing submodule and \( K \subseteq N \), but \( N \) is not semi 2-absorbing submodule of \( M \) as \( 2^2 \cdot 3 \in K \) but \( 2^2 \notin (K :_R M) \) and \( 2 \cdot 3 \notin K \).

Let \( R \) be an integral domain. Recall that if for every element \( r \) of its field of fractions \( F \), at least one of \( r \) or \( r^{-1} \) belongs to \( R \), then \( R \) is called valuation domain.

Proposition 2.19 Let \( R \) be a valuation domain with quotient field \( K \). Let \( M \) be an \( R \)-module and \( N \) a proper submodule of \( M \). Then \( N \) is a semi \( n \)-absorbing submodule of \( M \) if and only if whenever \( r \in K \), \( m \in M \) with \( r^{n+1} \in N \) implies that \( r^n m \in N \).

Proof. Suppose that \( N \) is a semi \( n \)-absorbing submodule of \( M \). Assume that \( r^{n+1} m \in N \) but \( r^{n+1} \notin (N :_R M) \) for some \( r \in K \) and \( m \in M \). If \( r \in R \), then we are done. So assume that \( r \notin R \). Since \( R \) is a valuation domain, \( r^{-1} \in R \). Hence we get \( r^{-1}(r^{n+1}m) = r^n m \notin N \). The converse part is clear. \( \square \)

Definition 2.20 Let \( N \) be a proper submodule of \( M \).

1. \( N \) is said to be strongly semi \( n \)-absorbing submodule if whenever \( I \) is an ideal and \( L \) is a submodule of \( M \) with \( I^n L \subseteq N \) implies that \( I^n \subseteq (N :_R M) \) or \( I^{n-1} L \subseteq N \).
2. \( N \) is said to be strongly \((k,n)\)-closed submodule if whenever \( I \) is an ideal and \( L \) is a submodule of \( M \) with \( I^k L \subseteq N \) implies that \( I^n \subseteq (N :_R M) \) or \( I^{n-1} L \subseteq N \).

Note that every strongly \((k,n)\)-closed submodule is a \((k,n)\)-closed submodule of \( M \). Clearly a \((1,1)\)-closed submodule is also a strongly \((1,1)\)-closed submodule of \( M \). Also observe that a strongly semi \( n \)-absorbing submodule is a \( n \)-absorbing submodule of \( M \).

Lemma 2.21 Let \( N \) be a proper submodule of \( M \). Then the following statements are equivalent:

1. \( N \) is a strongly \((k,n)\)-closed submodule of \( M \).
2. If \( I \) is an ideal of \( R \) and \( m \in M \) with \( I^k m \subseteq N \), then \( I^n \subseteq (N :_R M) \) or \( I^{n-1} m \subseteq N \).

Proof. (1) \( \implies \) (2) It is obvious.

(2) \( \implies \) (1) Suppose that \( I^k L \subseteq N \) for an ideal \( I \) of \( R \) and a submodule \( L \) of \( M \). Assume that \( I^{n-1} L \notin N \). Then there is an element \( m \) of \( L \) such that \( I^{n-1} m \notin N \) for some \( m \in L \). Since \( I^k m \subseteq N \), we have \( I^n \subseteq (N :_R M) \) by (2). Thus \( N \) is a strongly \((k,n)\)-closed submodule of \( M \). \( \square \)

Theorem 2.22 Let \( R \) be a principal ideal domain and \( N \) be a proper submodule of an \( R \)-module \( M \). Then the following are equivalent:

1. \( N \) is a \((k,n)\)-closed submodule of \( M \).
2. \( N \) is a strongly \((k,n)\)-closed submodule of \( M \).

Proof. (1) \( \implies \) (2) Since \( I \) is principal, \( I = (a) \) for some \( a \in R \). So we are done by Lemma 2.21.

(2) \( \implies \) (1) It is clear. \( \square \)
Proposition 2.23 Let $N$ be a proper submodule of an $R$-module $M$. If $N$ is a $(k,n)$-closed submodule of $M$, then $(N :_M I) = \{ m \in M : Im \subseteq N \}$ is a $(k,n)$-closed submodule of $M$ for all ideal $I$ of $R$. Moreover if $N$ is a strongly $(k,n)$-closed submodule of $M$, then $(N :_M I^k) = (N :_M I^{n-1})$.

Proof: Suppose that $r^k m \in (N :_M I)$ for $r \in R$ and $m \in M$. Hence $r^k Im \subseteq N$, which implies that either $r^k \in (N :_R M)$ or $r^{n-1} Im \subseteq N$ by Lemma 2.1. This means $r^n \in ((N :_R M) :_R I) = ((N :_M I) :_R M)$ or $r^{n-1} m \in (N :_M I)$. Thus $(N :_M I)$ is a $(k,n)$-closed submodule of $M$ for all ideal $I$ of $R$. Now suppose that $N$ is a strongly $(k,n)$-closed submodule of $M$. Since $(N :_M I^{n-1}) \subseteq (N :_M I^k)$ is always true, it is sufficient to show the inverse inclusion. Let $m \in (N :_M I^k)$. Then $I^k m \in N$, and we have $I^n \subseteq (N :_R M)$ or $I^{n-1} m \in N$ from Lemma 2.21. If $I^{n-1} m \in N$, then $m \in (N :_M I^{n-1})$, so we are done. So suppose that $I^n \subseteq (N :_R M)$. Thus $I^k \subseteq (N :_R M)$, as needed. □

Theorem 2.24 Let $N$ be a proper submodule of $M$. Then the following statements are equivalent:

1. $N$ is a strongly $(k,n)$-closed submodule of $M$.
2. For any ideal $I$ of $R$ and $N \subseteq L$ a submodule of $M$ with $I^k L \subseteq N$ implies that $I^n \subseteq (N :_R M)$ or $I^{n-1} L \subseteq N$.

Proof. (1) $\implies$ (2) It is clear.

(2) $\implies$ (1) Let $K$ be a submodule of $M$ and $I$ an ideal of $R$ such that $I^k K \subseteq N$. Hence $I^k(K + N) = I^k K + I^k N \subseteq N$. Put $L = K + N$. Since $N$ is strongly $(k,n)$-closed, we conclude that either $I^n \subseteq (N :_R M)$ or $I^{n-1} L \subseteq N$ by hypothesis (2). Thus $I^n \subseteq (N :_R M)$ or $I^{n-1} K \subseteq N$. □

Theorem 2.25 Let $N$ be a $(k,2)$-closed submodule of $M$, and $L$ a submodule of $M$. Then:

1. If $L^2 M \subseteq N$, then $2L^2 \subseteq (N :_R M)$.
2. If $2 \in U(R)$, then $N$ is a strongly $(k,2)$-closed submodule of $M$.

Proof. (1) Suppose that $L^2 M \subseteq N$. Then $l_1^2 m, l_2 m, (l_1 + l_2)^2 m \in N$ for all $m \in M$, for all $l_1, l_2$. Since $N$ is $(k,2)$-closed, we conclude that (either $l_1^2 \in (N :_R M)$ or $l_1 m \in N$) and (either $l_2^2 \in (N :_R M)$ or $l_2 m \in N$) and (either $(l_1 + l_2)^2 \in (N :_R M)$ or $(l_1 + l_2)m \in N$ which means $l_1^2 m, l_2^2 m, (l_1 + l_2)^2 m \in N$. Then $2l_1 l_2 m = ((l_1 + l_2)^2 - l_1^2 - l_2^2) m \in N$. Thus $2L^2 M \subseteq N$, and so $2L^2 \subseteq (N :_R M)$.

(2) Let $2 \in U(R)$. Since $2L^2 M \subseteq N$ from (1), we conclude that $L^2 \subseteq (N :_R M)$. □

Now we extend well-known results about prime submodules, $n$-absorbing submodules and $(m,n)$-closed ideals to $(k,n)$-closed submodules.

Theorem 2.26 Let $N$ be a proper submodule of $M$, and $S$ be a multiplicatively closed subset of $R$ such that $(N :_R M) \cap S = \emptyset$. If $N$ is a $(k,n)$-closed submodule
of $M$, then $S^{-1}N$ is a $(k,n)$-closed submodule of $S^{-1}M$. In particular, if $N$ is a semi $n$-absorbing submodule of $M$, then $S^{-1}N$ is a semi $n$-absorbing submodule of $S^{-1}M$.

**Proof.** Let $\left( \frac{r}{s_1} \right)^k \left( \frac{m}{s_2} \right) \in S^{-1}N$. Hence $ur^km \in N$ for some $u \in S$. Hence $(ur)^k m \in N$. Since $N$ is $(k,n)$-closed, $(ur)^{n-1} m \in N$ or $(ur)^n \in (N:M)$ which follows either $\left( \frac{r}{s_1} \right)^{n-1} \left( \frac{m}{s_2} \right) \in S^{-1}N$ or $\left( \frac{r}{s_1} \right)^n = u^s \frac{r^s m}{u^s s_1} \in S^{-1}(N:M) \subseteq (S^{-1}N:S^{-1}R)$. "In particular" part is clear as a semi $n$-absorbing submodule is a $(n,n)$-closed submodule of $M$. □

**Corollary 2.27** Let $S$ be a multiplicatively closed subset of $R$ such that $S \cap (N:M) = \emptyset$ with $2 \in S$. If $N$ is a strongly $(k,2)$-closed submodule of $M$, then $S^{-1}N$ is a strongly $(k,2)$-closed submodule of $S^{-1}M$.

**Proof.** Let $S^{-1}K$ be a submodule of $S^{-1}M$ such that $(S^{-1}K)^k(S^{-1}M) \subseteq S^{-1}N$. Since $2 \in S$, $2 \notin U(S^{-1}R)$, we are done by Theorem 2.25 (2). □

**Corollary 2.28** Let $N$ be a proper submodule of $M$, and $P$ a prime submodule of $M$ containing $N$. Then $N$ is a $(k,n)$-closed submodule of $M$ if and only if $NP$ is a $(k,n)$-closed submodule of $M_P$.

**Proof.** If $N$ is a $(k,n)$-closed submodule of $M$, then $NP$ is a $(k,n)$-closed submodule of $M_P$ by Theorem 2.26. Conversely suppose that $r \in R$, $m \in M$ with $r^k m \in N$. Let $\Omega = \{ u \in R : ur^k m \in N \}$. Then $\left( \frac{r}{s_1} \right)^n \frac{m}{s_2} \in NP$ implies that $\left( \frac{r}{s_1} \right)^{n-1} \frac{m}{s_2} \in NP$ or $\left( \frac{r}{s_1} \right)^n \in (NP:R_P M_P)$ as $NP$ is $(k,n)$-closed. Therefore $ur^k m \in NP$ for some $u \in R \setminus P$. Hence $\Omega \subseteq P$. Also $\Omega \nsubseteq P'$ where $P'$ is any prime submodule of $M$ with $I \subseteq P'$. Therefore $\Omega = R$, which means that $r^k m \in N$. Thus $N$ is a $(k,n)$-closed submodule of $M$. □

**Theorem 2.29** Let $M, M'$ be $R$-modules, and $f : M \to M'$ an $R$-module homomorphism.

1. If $N'$ is a $(k,n)$-closed (resp. semi $n$-absorbing) submodule of $M'$, then $f^{-1}(N')$ is a $(k,n)$-closed (resp. semi $n$-absorbing) submodule of $M$.
2. If $f$ is onto and $N$ is a $(k,n)$-closed (resp. semi $n$-absorbing) submodule of $M$ containing $Ker f$, then $f(N)$ is a $(k,n)$-closed (resp. semi $n$-absorbing) submodule of $M'$.

**Proof.** The reader can easily obtain the proof, so it is omitted. □

**Corollary 2.30** Let $M, M'$ be $R$-modules and $N, K$ be proper submodules of $M$. Then the following statements hold:

1. If $M \subseteq M'$ and $N$ is a $(k,n)$-closed (resp. semi $n$-absorbing) submodule of $M'$, then $N \cap M$ is a $(k,n)$-closed (resp. semi $n$-absorbing) submodule of $M$. 183
2. If \( K \subseteq N \), then \( N/K \) is a \((k,n)\)-closed (resp. semi \( n \)-absorbing) submodule of \( M/K \) if and only if \( K \) is a \((k,n)\)-closed (resp. semi \( n \)-absorbing) submodule of \( M \).

**Theorem 2.31** Let \( M_1, M_2 \) be \( R \)-modules with \( M = M_1 \oplus M_2 \), and let \( N_1, N_2 \) be proper submodules of \( M_1, M_2 \), respectively.

1. \( N_1 \) is a \((k_1,n_1)\)-closed submodule of \( M_1 \) if and only if \( N_1 \oplus N_2 \) is a \((k,n)\)-closed submodule of \( M_1 \oplus M_2 \) for all positive integers \( k_1 \leq k \) and \( n \geq n_1 \).
2. \( N_2 \) is a \((k_2,n_2)\)-closed submodule of \( M_2 \) if and only if \( M_1 \oplus N_2 \) is a \((k,n)\)-closed submodule of \( M_1 \oplus M_2 \) for all positive integers \( k_2 \leq k \) and \( n \geq n_2 \).

**Proof.** (1) Suppose that \( N_1 \) is a \((k_1,n_1)\)-closed submodule of \( M_1 \). Assume that \( r^{k_1} (m_1, m_2) \in N_1 \oplus M_2 \) but \( r^{k_1-1} (m_1, m_2) \notin N_1 \oplus M_2 \). Then \( r^{k_1-1} m_1 \notin N_1 \), which implies \( r^{k_1} m_1 \in (N_1 :_R M_1) \). Thus \( r^{k_1} \in (N_1 :_R M_1) \). Consequently, \( N_1 \oplus M_2 \) is a \((k,n)\)-closed submodule of \( M_1 \oplus M_2 \) for all positive integers \( k_1 \leq k \) and \( n \geq n_1 \) by Theorem 2.7 (4). The converse part can be obtained easily by using the similar argument.

(2) It can be easily verified similar to (1). \( \Box \)

**Theorem 2.32** Let \( M_1, M_2 \) be \( R \)-modules, \( N_1 \) be a \((k_1,n_1)\)-closed submodule of \( M_1 \), and \( N_2 \) be a \((k_2,n_2)\)-closed submodule of \( M_2 \). Then \( N_1 \oplus N_2 \) is a \((k,n)\)-closed submodule of \( M_1 \oplus M_2 \) for all positive integers \( k \leq \min \{k_1,k_2\} \) and \( n \geq \max \{n_1,n_2\} + 1 \).

**Proof.** Suppose that \( r \in R \) and \( (m_1,m_2) \in M \) such that \( r^{k_1} (m_1,m_2) \in N_1 \oplus N_2 \). Hence \( r^{k_1} m_1 \in N_1 \). Since \( r^{k_1} m_1 \in N_1 \) and \( n_1 \) is a \((k_1,n_1)\)-closed submodule of \( M_1 \), we have \( r^{k_1} m_1 \in N_1 \). Similarly, since \( r^{k_2} m_2 \in N_2 \) and \( n_2 \) is a \((k_2,n_2)\)-closed submodule of \( M_2 \), we get \( r^{k_2} m_2 \in N_2 \). Thus \( r^n m_1 \in N_1 \) and \( r^n m_2 \in N_2 \) for all \( n \geq \max \{n_1,n_2\} \). Therefore \( r^n (m_1,m_2) \in N_1 \oplus N_2 \), as needed. \( \Box \)

D.F. Anderson and A. Badawi determined in [4] when the powers of principal prime ideal or the ideals of the form \( p_1^{t_1} \cdots p_t^{t_t} \) where \( p_1, \ldots, p_t \) are non associate prime elements of \( R \) and \( t_1, \ldots, t_t \) are positive integers are \((m,n)\)-closed ideal of an integral domain \( R \). Analogous to them, we may conclude many results for submodules of multiplication modules over commutative rings. Some of them are presented as the following.

**Theorem 2.33** Let \( R \) be an integral domain and \( M \) a multiplication \( R \)-module. Let \( (N:_R M) = p^n R \) where \( p \) is prime element of \( R \) and \( k > 0 \). If \( N \) is a \((k,n)\)-closed submodule of \( M \), then the following statements are hold:

1. \( t = ka + r \), where \( a \) and \( r \) are integers such that \( a \geq 0 \), \( 1 \leq r \leq n \), \( a(k \mod n) + r \leq n \), and if \( a \neq 0 \), then \( k = n + c \) for an integer \( c \) with \( 1 \leq c \leq n - 1 \).
2. If \( k = bn + c \) for integers \( b \) and \( c \) with \( b \geq 2 \) and \( 0 \leq c \leq n - 1 \), then \( t \in \{1, \ldots, n\} \).
   If \( k = n + c \) for an integer \( c \) with \( 0 \leq c \leq n - 1 \), then \( t \in \bigcup_{h=1}^{n} \{ki + h : i \in \mathbb{Z} \} \) and \( 0 \leq ic \leq n - h \).
Proof. Suppose that $N$ is a $(k,n)$-closed submodule of $M$. Then $(N : R M)$ is a $(k,n)$-closed ideal of $M$ by Theorem 2.2. So we are done from Theorem 3.1 in [4]. □

**Corollary 2.34** Let $M$ be a multiplication $R$-module where $R$ is an integral domain, and $(N : R M) = p^i R$ where $p$ is prime element of $R$, $t > 0$. If $N$ is a semi $n$-absorbing submodule of $M$, then $t = na + r$, where $a$ and $r$ are integers such that $a \geq 0$, $1 \leq r < n$; that is $t \in \bigcup_{i=1}^{n} \{ni + h : i \in \mathbb{Z} \text{ and } 0 \leq i < n - h\}$. Proof. From Theorem 2.2 and Theorem 3.8 in [4], the result is clear. ⊓ ⊔

2. There is a positive integer such that $t = na + r$, where $a$ and $r$ are integers such that $a \geq 0$, $1 \leq r < n$; that is $t \in \bigcup_{i=1}^{n} \{ni + h : i \in \mathbb{Z} \text{ and } 0 \leq i < n - h\}$. Proof. Since a semi $n$-absorbing submodule is a $(n,n)$-closed submodule of $M$, the result is clear by Theorem 2.33. □

**Corollary 2.35** Let $R$ be an integral domain and $(N : R M) = p^i R$ where $p$ is a prime element of $R$ and $t$ is a positive integer. Then $N$ is a semi $2$-absorbing submodule of $M$, then $t \in \{1, 2\}$.

Consider a $\mathbb{Z}$-module $M = \mathbb{Z}$ and a submodule $N = 2^t \mathbb{Z}$ of $M$. It is shown in Example 2.1 that $N$ is not a semi 2-absorbing submodule of $M$ for $t = 3$.

**Theorem 2.36** Let $R$ be a principal ideal domain, $N$ a proper submodule of a multiplication $R$-module $M$ and $k, n$ be integers with $1 \leq n \leq k$. If $N$ is a (strongly) $(k,n)$-closed submodule of $M$, then $N = P_1^{t_1} \cdots P_i^{t_i}$ where $P_1, \ldots, P_i$ are nonassociate prime submodules of $M$, $t_1, \ldots, t_i$ are positive integers, and one of the following two conditions holds:

1. If $k = bn + c$ for integers $b$ and $c$ with $b \geq 2$ and $0 \leq c \leq n - 1$, then $t_j \in \{1, \ldots, n\}$ for every $1 \leq j \leq i$.
2. If $k = n + c$ for an integer $c$ with $0 \leq c \leq n - 1$, then $t_j \in \bigcup_{h=1}^{n} \{hv + h : v \in \mathbb{Z} \text{ and } 0 \leq hv < n - h\}$ for every $1 \leq j \leq i$.

Proof. Suppose that $N$ is $(k,n)$-closed submodule of $M$. Then $(N : R M)$ is a $(k,n)$-closed ideal of $R$ by Theorem 2.2. Hence $(N : R M) = P_1^{t_1} \cdots P_i^{t_i} R$ for some nonassociate prime elements of $R$, $t_1, \ldots, t_i$ are positive integers, and the conditions (1) or (2) is satisfied for $k$ and $n$ by [4]. Thus $N = P_1^{t_1} \cdots P_i^{t_i} M$. Put $P_i^{t_i} = p_i^{t_i} M$ for all $i = 1, \ldots, t$, so $N = P_1^{t_1} \cdots P_i^{t_i}$, we are done. □

**Theorem 2.37** Let $N$ be a proper submodule of a multiplication $R$-module $M$ where $R$ is an integral domain and $k, n$ be integers with $1 \leq n \leq k$. Suppose that $N = P^t$, where $P$ is a prime submodule of $M$ and $t$ is a positive integer. If $N$ is a $(k,n)$-closed submodule of $M$, then one of the following statements holds:

1. $1 \leq t \leq n$.
2. There is a positive integer $a$ such that $t = ka + r = na + d$ for an integer $r$ and $d$ with $1 \leq r, d \leq n - 1$.
3. There is a positive integer $a$ such that $t = ka + r = n(a + 1)$ for an integer $r$ with $1 \leq r \leq n - 1$.

Proof. From Theorem 2.2 and Theorem 3.8 in [4], the result is clear. □
References


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Author

Ece Yetkin Celikel,
Department of Mathematics,
Faculty of Art and Science,
Gaziantep University,
Gaziantep, Turkey,
E-mail: yetkinece@gmail.com