Translation surfaces of linear Weingarten type

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Abstract We give a relatively simple proof that if a translation surface in Euclidean space satisfies a relation of type $aH + bK = c$, for some real numbers $a, b, c$, where $H$ and $K$ are the mean curvature and the Gauss curvature of the surface, respectively, then $a = 0$ or $b = 0$. In particular, $K$ is constant or $H$ is constant. Our method of proof extends to the Lorentzian ambient space.

Keywords translation surface · linear Weingarten surface · mean curvature · Gauss curvature

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1 Introduction and results.

A Weingarten surface in Euclidean space $\mathbb{R}^3$ is a surface $S$ whose mean curvature $H$ and Gauss curvature $K$ satisfies a non-trivial relation $\Psi(H, K) = 0$. These surfaces were introduced by the very Weingarten in the context of the problem of finding all surfaces isometric to a given surface of revolution and have been extensively studied in the literature [13]. In order to simplify the study of Weingarten surfaces, it is natural to impose some additional geometric condition on the surface, as for example, that $S$ is ruled or rotational [1,3,4,7,12].

Following this strategy, Dillen, Goemans and Van de Woestyne considered Weingarten surfaces that are graphs of type $z = f(x) + g(y)$, where $f$ and $g$ are smooth functions defined in some intervals $I, J \subset \mathbb{R}$, respectively [2]. A surface $S$ in $\mathbb{R}^3$ is called a translation surface if it can locally parametrize as $X(x, y) = (x, y, f(x) + g(y))$. In particular, a translation surface has the property that the translations of a parametric curve $x = \text{const}$ by the parametric curves $y = \text{const}$ remain in $S$ (similarly for the parametric curves $x = \text{const}$). In the cited paper, the authors classify all translation surfaces of Weingarten type:
Theorem 1.1 ([2]) A translation surface in $\mathbb{R}^3$ of Weingarten type is a plane, a generalized cylinder, a Scherk’s minimal surface or an elliptic paraboloid.

The proof given in [2] (see also [6]) discusses many cases, involving the solvability of a large number of ODE systems and it requires the use of a computer algebra program (as Maple) to manipulate the algebraic operations. In fact, in [2] it is described the procedure and only some cases are mentioned. Previously some authors obtained partial results assuming simpler functions $f$ and $g$, as for example, that they are polynomial in its variables, simplifying and doing easier the computations ([11,15]).

In this paper we provide a significantly simpler proof of Theorem 1.1 when the Weingarten relation is linear in its variables. A linear Weingarten surface in Euclidean space $\mathbb{R}^3$ is a surface where there exists a relation

$$a H + b K = c, \quad (1.1)$$

for some real numbers $a, b, c$, not all zero. In the class of linear Weingarten surfaces, we mention two families of surfaces that correspond with trivial choices of the constants $a$ and $b$: surfaces with constant Gauss curvature ($a = 0$) and surfaces with constant mean curvature ($b = 0$). In Theorem 1.1, only the first three surfaces are linear Weingarten surfaces, which have constant $H$ or constant $K$: a plane ($H = K = 0$), a generalized cylinder ($K = 0$) and the Scherk’s minimal surface parametrized as $z = \log(\cos(\lambda y)) - \log(\cos(\lambda x))$, $\lambda > 0$ ($H = 0$). We observe that for an elliptic paraboloid, the Weingarten relation $\Psi(H,K) = 0$ is not linear. Indeed, if this surface writes as $z = a(x^2 + y^2)$, $a \neq 0$, then we have $\Psi(\alpha, \beta) = 2\sqrt{2|a| \text{ sign}(a)} \alpha - \beta^{3/4} - 2|a| \beta^{1/4}$. Finally, we point out that the classification of linear Weingarten surfaces in the general case is almost completely open today; see for example [5,9,12].

The result that we prove is:

Theorem 1.2 A translation surface in Euclidean space $\mathbb{R}^3$ of linear Weingarten type is a surface with constant Gauss curvature $K$ or constant mean curvature $H$. In particular, the surface is congruent with a plane, a generalized cylinder or a Scherk’s minimal surface.

This shows that in the family of translation surfaces, linear Weingarten surfaces correspond with trivial choices of $a, b$ in (1.1), that is, $a = 0$ or $b = 0$. We point out that an early work of Liu proved that the only translations surfaces with constant $K$ or constant $H$ are the first three surfaces of Theorem 1.1 ([8]). Finally, and with minor modifications, we extend in Theorem 3.1 our results to the Lorentzian ambient space (see also [2]).
2 Proof of Theorem 1.2

The mean curvature \( H \) and the Gauss curvature \( K \) are expressed in a local parametrization \( X \) as

\[
H = \frac{lN - 2mM + nL}{2(LN - M^2)}, \quad K = \frac{ln - m^2}{LN - M^2},
\]

where \( \{L, M, N\} \) and \( \{l, m, n\} \) are the coefficients of the first fundamental form and the second fundamental form, respectively. Assume that \( S \) is a translation surface expressed locally as \( X(x, y) = (x, y, f(x) + g(y)) \) for some smooth functions \( f \) and \( g \). Then \( H \) and \( K \) are

\[
H = f''(1 + g'^2) + g''(1 + f'^2) \quad \sqrt{W} + b \frac{f''g''}{(1 + f'^2 + g'^2)^2} = 0.
\]

We multiply (2.3) by \( W^2 \) and divide by \( (1 + g'^2)(1 + f'^2) \) obtaining

\[
a \left( \frac{f''}{1 + f'^2} + \frac{g''}{1 + g'^2} \right) \sqrt{W} + b \frac{f''g''}{1 + f'^2 + g'^2} = 0.
\]

Introduce the next notation:

\[
F = \frac{f''}{1 + f'^2}, \quad G = \frac{g''}{1 + g'^2}.
\]

In particular, since \( f'' \neq 0 \) and \( g'' \neq 0 \), then \( F \neq 0 \) and \( G \neq 0 \). Then (2.4) writes as

\[
a(F + G)\sqrt{W} + bFG = 0.
\]
Let us observe that this identity implies $F + G \neq 0$, since on the contrary, $b = 0$. From (2.6), we have

$$1 + f'^2 + g'^2 = W = \frac{b^2}{a^2} \left( \frac{FG}{F + G} \right)^2.$$

We differentiate this equation with respect to $x$ and then with respect to $y$. Because the left hand side is a sum of a function of $x$ and a function of $y$, this calculation yields 0. On the other hand, the right hand side concludes

$$6 \frac{b^2 F^2 G^2 F' G'}{(F + G)^4} = 0.$$  \hfill (2.7)

This implies $F'G' = 0$. We discuss the two possibilities:

1. Suppose that there exists $x_0 \in I$ such that $F'(x_0) \neq 0$. Then $G' = 0$ in some subinterval $J' \subset J$ and this implies that $G$ is a constant function in $J'$. By the definition of $G$ in (2.5), and since $g'' \neq 0$, then $G \neq 0$. Differentiating (2.6) with respect to $y$, we obtain

$$a(F + G)g'g'' \sqrt{W} = 0.$$  \hfill (2.9)

As $a, F + G, g'' \neq 0$, then $g' = 0$. This implies $g'' = 0$, a contradiction.

2. Therefore $F' = 0$ in the interval $I$. This implies that $F$ is a constant function. Differentiating now (2.6) with respect to $x$ together a similar argument as in the previous case, we obtain a contradiction.

2.2 Case $c \neq 0$.

Dividing in (1.1) by $c$, and after a change of notation, the relation (1.1) writes as

$$a \frac{f''(1 + g'^2) + g''(1 + f'^2)}{(1 + f'^2 + g'^2)^{3/2}} + b \frac{f''g''}{(1 + f'^2 + g'^2)^2} = 1,$$  \hfill (2.8)

or equivalently

$$a(F + G)\sqrt{W} + bFG = \frac{W^2}{(1 + f'^2)(1 + g'^2)},$$  \hfill (2.9)

where $F$ and $G$ are defined as in (2.5). We differentiate (2.9) separately with respect to $x$ and with respect to $y$:

$$a \left( F'\sqrt{W} + (F + G) \frac{f'f''}{\sqrt{W}} \right) + bF'G = \frac{4W f'f''}{(1 + f'^2)(1 + g'^2)} - \frac{2f'f''W^2}{(1 + f'^2)^2(1 + g'^2)}.$$
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\[ a \left( G' \sqrt{W} + (F + G) \frac{g'g''}{\sqrt{W}} \right) + bFG' = \frac{4Wg'g''}{(1 + f'^2)(1 + g'^2)} - \frac{2g'g''W^2}{(1 + f'^2)(1 + g'^2)\sqrt{W}}. \]

Dividing the first equation by \( f'f'' \) and the second one by \( g'g'' \), we have, respectively,

\[ \frac{4W}{(1 + f'^2)(1 + g'^2)} - a\frac{F + G}{\sqrt{W}} = a\frac{F'\sqrt{W}}{f'f''} + b\frac{F'G'}{f'f''} + \frac{2W^2}{(1 + f'^2)(1 + g'^2)} \]

\[ \frac{4W}{(1 + f'^2)(1 + g'^2)} - a\frac{F + G}{\sqrt{W}} = a\frac{G'\sqrt{W}}{g'g''} + b\frac{FG'}{g'g''} + \frac{2W^2}{(1 + f'^2)(1 + g'^2)}. \]

Thus

\[ a\frac{F'\sqrt{W}}{f'f''} + b\frac{F'G'}{f'f''} + \frac{2W^2}{(1 + f'^2)(1 + g'^2)} = a\frac{G'\sqrt{W}}{g'g''} + b\frac{FG'}{g'g''} + \frac{2W^2}{(1 + f'^2)(1 + g'^2)}. \]

From (2.9), we replace the value of \( W \) in the above expression, obtaining

\[
\begin{align*}
& a \left( \frac{F'}{f'f''} + 2\frac{(F + G)}{1 + f'^2} - \frac{G'}{g'g''} - \frac{2(F + G)}{1 + g'^2} \right) \sqrt{W} \\
& = -b \left( \frac{F'G'}{f'f''} + \frac{2FG}{1 + f'^2} - \frac{FG'}{g'g''} - \frac{2FG}{1 + g'^2} \right). \quad (2.10)
\end{align*}
\]

Now we write (2.8) as

\[ a \left( f''(1 + g'^2) + g''(1 + f'^2) \right) \sqrt{W} + b f''g'' = W^2. \]

We differentiate this expression first with respect to \( x \) and with respect to \( y \):

\[
\begin{align*}
& a \left( f'''(1 + g'^2) + 2f'f''g'' \right) \sqrt{W} + a \left( f''(1 + g'^2) + g''(1 + f'^2) \right) \frac{f'f''}{\sqrt{W}} \\
& + b f'''g'' = 4f'f''W.
\end{align*}
\]

\[
\begin{align*}
& a \left( 2f'g'g'' + g''(1 + f'^2) \right) \sqrt{W} + a \left( f''(1 + g'^2) + g''(1 + f'^2) \right) \frac{g'g''}{\sqrt{W}} \\
& + b f'''g'' = 4g'g''W.
\end{align*}
\]

From both equations, we deduce the value of \( W \) on the right hand sides and we equal both expressions, deducing

\[
\begin{align*}
& a \left( \frac{f''}{f'f''}(1 + g'^2) + 2g'' - \frac{g''}{g'g''}(1 + f'^2) \right) \sqrt{W} = b \left( f'' g'' - \frac{g''}{f'f''}(1 + f'^2) \right). \quad (2.11)
\end{align*}
\]
If we write (2.10) and (2.11) as \(aP_1\sqrt{W} = bQ_1\) and \(aP_2\sqrt{W} = bQ_2\), respectively, we obtain \(P_1Q_2 - P_2Q_1 = 0\). After some manipulations, this identity writes as

\[
(f'f''g'' - f'''g'g'') \left(2f'f''g''g'' + f'f''g'''(1 + f'^2) - f'''g'g''(1 + g'^2)\right) = 0,
\]

that is, \(P_2Q_2 = 0\). We discuss by cases:

1. There exists \((x_0, y_0) \in I \times J\) such that \(P_2(x_0, y_0) \neq 0\). Then \(Q_2 = 0\) in some sub-rectangle of \(I \times J\). By (2.11), we obtain \(aP_2 = 0\), a contradiction.

2. Therefore \(P_2 = 0\) in \(I \times J\). Using (2.11) again, we have \(Q_2 = 0\) in \(I \times J\).

These two equations write as

\[
\frac{f'''}{f'f''} = \frac{g'''}{g'g''} \quad (2.12)
\]

and

\[
2(f'' - g'') + \frac{g'''}{g'g''}(1 + f'^2) - \frac{f'''}{f'f''}(1 + g'^2) = 0. \quad (2.13)
\]

Because the left hand side of Equation (2.12) depends only on the variable \(x\), and the right hand side depends only on the variable \(y\), then there exists a constant \(\lambda \in \mathbb{R}\) such that

\[
\frac{f'''}{f'f''} = \frac{g'''}{g'g''} = 2\lambda \quad (2.14)
\]

and thus

\[
\frac{f'''}{f'} = 2\lambda f'', \quad \frac{g'''}{g'} = 2\lambda g''.
\]

Substituting the above information in (2.13), we get

\[
2(f'' - g'') + 2\lambda(1 + f'^2)g'' - 2\lambda(1 + g'^2)f'' = 0,
\]

or equivalently

\[
f'' - g'' + \lambda g'' - \lambda f'' = \lambda f''g'^2 - \lambda g''f'^2. \quad (2.15)
\]

If \(\lambda \neq 0\), we differentiate this equation with respect to \(x\) and then with respect to \(y\) and we deduce

\[
f'f''g''' = g'g''f'''.
\]

As we are assuming that \(f'', g'' \neq 0\), we conclude that

\[
\frac{f'''}{f'f''} = \frac{g'''}{g'g''} = \mu
\]

for some constant \(\mu \in \mathbb{R}\). Substituting in (2.14) we deduce that \(\mu \neq 0\) and that \(f''', g'''\) are both constant functions. Then \(f''' = g''' = 0\), so (2.14) yields \(\lambda = 0\), a contradiction.
Therefore, \( \lambda = 0 \) in (2.14). Equation (2.15) says now that \( f'' = g'' = \rho \), for some real number \( \rho \neq 0 \). Then (2.8) writes as
\[
ap(2 + f'^2 + g'^2) = W + b\rho^2 W^{-\frac{1}{2}}.
\]
Differentiating with respect to \( x \) and simplifying by \( f'f'' \), we get
\[
2a\rho = 3W^\frac{1}{2} + b\rho^2 W^{-\frac{3}{2}},
\]
which implies that \( W \) is a constant function. As \( W = 1 + f'^2 + g'^2 \), this would say that \( f'' = g'' = 0 \), a contradiction.

3 The Lorentzian case

We consider the Lorentz-Minkowski space \( L^3 \), that is, the real vector space \( \mathbb{R}^3 \) endowed with the metric \((dx)^2 + (dy)^2 - (dz)^2\) where \( (x, y, z) \) are the canonical coordinates. A surface \( S \) immersed in \( L^3 \) is said non degenerate if the induced metric on \( S \) is non degenerated. The induced metric on \( S \) can only be of two types: a Riemannian metric and in such a case \( S \) is called a spacelike surface, or a Lorentzian metric, and \( S \) is called a timelike surface. For both types of surfaces, it is defined the mean curvature \( H \) and the Gauss curvature \( K \) and we say again that the surface is of linear Weingarten type if there exists a linear relation between \( H \) and \( K \) as in (1.1).

Similarly, in Lorentzian setting we can extend the concept of translation surface. A surface \( S \) in \( L^3 \) is again locally a graph on one of the coordinate planes, since this property is not metric but because \( S \) is immersed. Thus a translation surface in \( L^3 \) is a surface that writes locally as the graph of a function which is the sum of two real functions. However, in \( L^3 \) we can say a bit more. If \( S \) is spacelike, then \( S \) is a graph on the \( xy \)-plane and if \( S \) is a timelike surface, then \( S \) is a graph on the \( xz \)-plane or on the \( yz \)-plane [14]. Therefore, if \( S \) is a translation surface in \( L^3 \), we may suppose that:

1. If \( S \) is spacelike, then \( S \) writes locally as \( z = f(x) + g(y) \).
2. If \( S \) is timelike, then \( S \) writes locally as \( y = f(x) + g(z) \) or as \( x = f(y) + g(z) \).

In [2], Theorem 1.1 was extended to non-degenerate surfaces of \( L^3 \), obtaining a similar result. If we restrict to the case that the surface is a linear Weingarten surface, then the only translation surfaces appear with trivial choices of \( a \) and \( b \), that is, \( a = 0 \) or \( b = 0 \), and the surfaces have constant mean curvature \( H \) or constant Gauss curvature \( K \). Similarly, we extend the proof of Theorem 1.2 as follows:

Theorem 3.1 A non-degenerate translation surface in Lorentz-Minkowski space \( L^3 \) of linear Weingarten type is a surface with constant Gauss curvature \( K \) or constant mean curvature \( H \).
Translations surfaces in $\mathbb{L}^3$ with constant mean curvature or constant Gauss curvature were classified in [8] and they are a plane, a Scherk’s minimal surface or a generalized cylinder.

**Proof.** The proof of Theorem 3.1 is similar as in Theorem 1.2 and we only sketch the differences. Moreover, we will carry jointly the cases that the surface $S$ is spacelike or timelike. Again, we suppose by contradiction that $H$ and $K$ are not constant functions, in particular, $a,b \neq 0$. The expressions of $H$ and $K$ in local coordinates are

$$H = \varepsilon \frac{1}{2} \frac{LN - 2mM + nL}{LN - M^2}, \quad K = \varepsilon \frac{ln - m^2}{LN - M^2},$$

where $\varepsilon = -1$ is $S$ is spacelike and $\varepsilon = 1$ if $S$ is timelike ([10,14]).

Suppose that $S$ writes as $z = f(x) + g(y)$ if $S$ is spacelike or $y = f(x) + g(z)$ if $S$ is timelike. Then

$$H = -\varepsilon f''(1 + \varepsilon g'') + \varepsilon g''(1 + \varepsilon f'') \quad \text{and} \quad K = -\varepsilon f''g''(1 + \varepsilon f'')^2,$$

with $W = 1 + \varepsilon f'' - g'' > 0$. As we are assuming that $K$ is not constant, the expression of $K$ implies $f''$, $g'' \neq 0$. Let

$$F = \frac{f''}{1 + \varepsilon f'}, \quad G = \frac{g''}{1 - g'}.$$

Suppose $c = 0$ in (1.1). Then Equation (2.6) is the same, obtaining (2.7). Then the contradiction arrives similarly as in the Euclidean case.

Suppose $c \neq 0$ in (1.1). After a change of constants $a$ and $b$, we assume that $c = 1$. Now the linear Weingarten condition (1.1) expresses as

$$a(F + G)\sqrt{W} + bFG = \varepsilon \frac{W^2}{(1 + \varepsilon f'')(1 + g'')(1 + g'^2)}, \quad (3.1)$$

Now (2.10) and (2.11) write, respectively, as

$$a \left( \frac{F'}{f'} + \frac{2(F+G)}{\varepsilon + f'^2} + \varepsilon \frac{g'}{g'g''} + \varepsilon \frac{2(F+G)}{-1 + g'^2} \right) \sqrt{W}$$

$$+ b \left( \frac{F'G}{f'} + \frac{2FG}{\varepsilon + f'^2} + \varepsilon \frac{FG'}{g'g''} + \varepsilon \frac{2FG}{-1 + g'^2} \right) = 0$$

$$a \left( \frac{f''}{f'}(-1 + g'^2) + 2g'' + 2\varepsilon f'' + \varepsilon(\varepsilon + f'^2) \frac{g'''}{g'g''} \right) \sqrt{W}$$

$$+ \varepsilon b \left( \frac{f''}{g'g''} + \varepsilon g'' \frac{f'''}{f'} \right) = 0.$$
We deduce

\[
\left( f' f'' g'' + \varepsilon f''' g' g'' \right) \left( 2 f' f'' g'' (f'' + \varepsilon g') + f' f'' g'' (f^2 + \varepsilon) \right) + \varepsilon f''' g' g'' (g^2 - 1) = 0.
\]

Now the discussion by cases is similar as it was done in the Euclidean case, obtaining that \( W \) is a constant function. Hence \( f'' = g'' = 0 \), obtaining a contradiction.

\[ \square \]

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