Abstract. In this paper we study the notion of semi-reduced prime subsemimodule and $NI$-semimodule related to a Morita Context $< R, S, P_{S R} S P_{R} Q_{R}, \theta, \phi >$. We characterize the $NI$-semimodule by semi-reduced prime subsemimodule.

Keywords. semi-reduced prime semimodule · $NI$-semimodule · Morita context

Mathematics Subject Classification (2010) 16D90 · 16Y60

1 Introduction

Though the notion of semiring was first introduced by Vandiver in 1934 [23], it has seen its tremendous growth in the last three decades of last century. This is evident from various monographs [6–8,10]. Recent work on semiring theory by Katsov [12,13], Maity [16,17], Bhuniya [1,2], Sardar and Gupta [19,21], Dey et al [20,3] (to name a few) indicates that this branch of mathematics has created a sustained research interest. One aspect of semiring theory is to investigate the validity of the ring theoretic analogues. One such generalization is the notion of Morita equivalence and Morita context of semirings. In this field Katsov et. al. [12–15] initiated the study and Sardar et. al. [19,21] made some contribution. The purpose of this paper is to continue the study of Morita context of semirings of the present authors [20,3]. Here we study $NI$—semimodule related to a Morita context of semirings.

In 2001 G.Marks [18] introduced the notion of an $NI$-ring as follows: A ring $R$ is called a $NI$-ring if $N^*(R) = N(R)$, where $N^*(R)$ is the unique maximal nil ideal of the ring $R$ and $N(R)$ is the set of all nilpotent elements of $R$. Kwak [11] characterized $NI$—ring in [11]. Following the trend of investigating the validity of various concepts of ring theory in the setting of semiring T.K. Dutta and M.L. Das [4] extended the notion of $NI$-rings to

$NI$-semimodule related to a Morita context

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In our venture of studying Morita context of semirings [20], [3], [21], [9] based on various concepts of semirings and semimodules, \( NI \)-semirings and \( NI \)-semimodules have been chosen as the central concept in this paper. Firstly we introduce the notion of a semireduced prime semimodule and subsemimodule (cf. Definitions 3.10, 3.11) related to a Morita context. Then we show that the intersection of all semireduced prime subsemimodules of \( R \) in a Morita context \( < R, S, R \rightarrow P, S \rightarrow Q, R, \theta, \phi > \) forms a unique maximal nil subsemimodule of \( P \) (cf. Proposition 3.16). Later we define some special subsets in \( P \) (cf. Definition 3.17, 3.22) and obtain some of their important properties (cf. Propositions 3.18, 3.19, 3.20, 3.21, 3.24, 3.25, 3.26) in order to characterize (cf. Theorem 3.29) the unique maximal nil subsemimodule \( N^*(P) \) of \( P \) for a Morita context \( < R, S, R \rightarrow P, S \rightarrow Q, R, \theta, \phi > \).

Lastly we have shown that the property of being \( NI \) is a Morita invariant property for a Morita context \( < R, S, R \rightarrow P, S \rightarrow Q, R, \theta, \phi > \) (cf. Theorem 3.38). It has also been observed (cf. Concluding Remark) that similar results also hold for the other component semimodule \( Q \) of the Morita context.

2 Preliminaries

In this section we recall some basic definitions and results that are needed for our work.

Definition 2.1 [13] Two semirings \( R \) and \( S \) are said to be Morita equivalent if there exists a progenerator \( R \rightarrow P \in \text{M}(R) \) for \( R \) such that \( S \cong \text{End}(R,P) \) as semirings; equivalently two semirings \( R \) and \( S \) are Morita equivalent if the categories \( R \text{-Mod} \) and \( S \text{-Mod} \) are equivalent categories.

Definition 2.2 If \( R \) and \( S \) are two semirings \( R \rightarrow P, S \rightarrow Q \) are \( R \)-\( S \)-bicomodule and \( S \rightarrow R \)-bicomodule respectively, and \( \theta : P \otimes Q \rightarrow R \) and \( \phi : Q \otimes P \rightarrow S \) are respectively \( R \rightarrow R \)-bicomodule homomorphism and \( S \rightarrow S \)-bicomodule homomorphism such that \( \theta(p \otimes q)p' = p\phi(q \otimes p') \) and \( \phi(q \otimes p)q' = q\theta(p \otimes q') \) for all \( p, p' \in P \) and \( q, q' \in Q \) then the quintuple \( < R, S, R \rightarrow P, S \rightarrow Q, \theta, \phi > \) is called a Morita context for semirings. Two semirings \( R \), \( S \) are Morita equivalent if and only if there exists a Morita context \( < R, S, R \rightarrow P, S \rightarrow Q, \theta, \phi > \) with \( \theta \) and \( \phi \) surjective. Readers are referred to [13, 12, 8] for more notions of semirings, semimodules and Morita equivalence of semirings.

Definition 2.3 [21] Let \( R \) and \( S \) be Morita equivalent semirings via Morita context \( < R, S, R \rightarrow P, S \rightarrow Q, \theta, \phi > \). Then we see that the lattices of ideals of \( R \) and the lattices of subsemimodules of \( P \) are isomorphic. Moreover this
isomorphism takes finitely generated ideals to finitely generated subsemimodules and vice-versa. Similar isomorphism can be defined for other pairs of the Morita context as follows.

\[ f_1 : \text{Id}(R) \to \text{Sub}(P) \quad \text{and} \quad g_1 : \text{Sub}(P) \to \text{Id}(R) \]

\[ f_1(I) := \{ \sum_{k=1}^{n} i_k p_k \mid p_k \in P, i_k \in I \text{ for all } k, n \in \mathbb{Z}^+ \}, \]

and 
\[ g_1(N) := \{ \sum_{k=1}^{n} \theta(p_k \otimes q_k) \mid p_k \in N, q_k \in Q \text{ for all } k, n \in \mathbb{Z}^+ \} \]

**Definition 2.4** [3] Let \( R, S \) be two Morita equivalent semirings via Morita context \( \langle R, S, R_P S, S_Q R, \theta, \phi \rangle \). An element \( a \) of \( P \) is said to be nilpotent if for any \( q \in Q \) there exists a positive integer \( n = n(q, a) \), depending on \( q \) and \( a \), such that \( \theta(a \otimes q)^{n-1} a = 0 \).

**Definition 2.5** [3] Let \( R, S \) be two Morita equivalent semirings via Morita context \( \langle R, S, R_P S, S_Q R, \theta, \phi \rangle \). An element \( a \) of \( P \) is said to be strongly nilpotent if there exists a positive integer \( n \) such that \( \theta(a \otimes Q)^{n-1} a = 0 \).

**Definition 2.6** [3] Let \( R, S \) be two Morita equivalent semirings via Morita context \( \langle R, S, R_P S, S_Q R, \theta, \phi \rangle \). A subsemimodule \( K \) of \( P \) is said to be prime if for subsemimodules \( M, N \) of \( P \), \( g_1(M)N \subseteq K \) implies either \( M \subseteq K \) or \( N \subseteq K \) (Since \( g_1(M) \) is an ideal of \( R \) (cf. Definition 2.3) and \( N \) is a left \( R \)-subsemimodule of \( P \) so \( g_1(M)N \) is meaningful).

**Definition 2.7** [3] Let \( R, S \) be two Morita equivalent semirings via Morita context \( \langle R, S, R_P S, S_Q R, \theta, \phi \rangle \). \( P \) is said to be a prime semimodule if for subsemimodules \( M, N \) of \( P \), \( g_1(M)N = 0 \) implies either \( M = 0 \) or \( N = 0 \).

### 3 Main results

We first formulate few definitions in order to obtain the main results.

**Definition 3.1** A subsemimodule \( M \) of a semimodule \( P \) is called a \( k \)-subsemimodule (Similar as the definition of a \( k \)-ideal of a semiring [22]) if for \( x, y \in P \), \( x + y \in M \) and \( y \in M \) implies that \( x \in M \).

**Definition 3.2** Let \( R, S \) be two Morita equivalent semirings via Morita context \( \langle R, S, R_P S, S_Q R, \theta, \phi \rangle \). A subsemimodule \( M \) of the bisemimodule \( P \) is said to be completely prime if \( \theta(a \otimes Q)b \subseteq M \) implies that \( a \in M \) or \( b \in M \) for \( a, b \in P \).
Definition 3.3 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. A subsemimodule $M$ of the bisemimodule $P$ is said to be completely semiprime if $\theta(a \otimes Q)a \subseteq M$ implies that $a \in M$ for $a \in P$.

Definition 3.4 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. Then a nonempty subset $H$ of $P$ is said to be an $m$-system of $P$ if $c, d \in H$ implies there exist $p \in P$ and $q, q' \in Q$ such that $\theta(c \otimes q) \theta(p \otimes q')d \in H$.

Definition 3.5 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. A subsemimodule $M$ of $P$ is nil if each of its elements are nilpotent (cf. Definition 2.4).

Definition 3.6 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. A subsemimodule $M$ of $P$ is said to be strongly nil subsemimodule if $N(P) = N_Q(P)$, where $N(P)$ is the set of all nilpotent elements of $P$ and $N_Q(P)$ is the set of all strongly nilpotent (cf. Definition 2.5) elements of $P$.

Definition 3.7 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. For a subsemimodule $M$ of $P$ the congruence on $P$, denoted by $\rho_M$ and defined as $a \rho_M b$ if and only if $a + m_1 = b + m_2$ for some $m_1, m_2 \in M$, is called the Bourne-congruence on $P$ defined by the subsemimodule $M$.

Now we denote the Bourne-congruence on $(\rho_M)$ class of an element $p$ of $P$ by $p/\rho_M$ or simply by $p/M$ and denote the set of all such congruence classes of the elements of the semimodule $P$ by $P/\rho_M$ or simply by $P/M$.

Definition 3.8 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. For an subsemimodule $M$ of $P$ if the Bourne-congruence $\rho_M$, defined by $M$, is proper i.e. $0/M \neq P$ then we define on $P/M$ the following operations: $a/M + b/M = (a + b)/M$ and $r.a/M = ra/M$ also $a/M.s = as/M$ for all $a, b \in P$ and $r \in R$, $s \in S$. Now $P/M$ is an $R - S$ bisemimodule with these operations. We call this semimodule the Bourne factor semimodule or simply the factor semimodule of $P$ by $M$.

Definition 3.9 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. $P$ is said to be right symmetric if for $a, b, c \in P$, $\theta(a \otimes Q)\theta(b \otimes Q)c = 0$ implies $\theta(a \otimes Q)\theta(c \otimes Q)b = 0$. A subsemimodule $M$ of $P$ is said to be right symmetric if $\theta(a \otimes Q)\theta(b \otimes Q)c \subseteq M$ implies $\theta(a \otimes Q)\theta(c \otimes Q)b \subseteq M$ for $a, b, c \in P$.

Similar are the definitions of left symmetric semimodule and left symmetric subsemimodule.

Definition 3.10 Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, \rho >$. Then $P$ is said to be a semireduced prime semimodule if $P$ is prime semimodule and it has no nonzero nil subsemimodules.
Definition 3.11 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S Q_R, \theta, \phi >$. A proper subsemimodule $M$ of $P$ is said to be a semi-reduced prime subsemimodule if $M$ is a prime subsemimodule of $P$ and $P/M$ contains no nonzero nil subsemimodules.

Definition 3.12 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S Q_R, \theta, \phi >$. For $q \in Q$, a nonempty subset $M$ of $P$ called a $q$-semigroup if $\theta(x \otimes q)y \in M$ for all $x, y \in M$.

The following proposition and the subsequent corollary and the proposition ensure the existence of semireduced prime constituent bisemimodule in the Morita Context $< R, S, R_P S, S Q_R, \theta, \phi >$.

Proposition 3.13 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S Q_R, \theta, \phi >$ and $M'$ be a $q$-semigroup in $P - \{0\}$, where $q \in Q$. Suppose that $M$ be an subsemimodule of $P$ maximal with respect to the property $M \cap M' = \emptyset$. Then $M$ is a semi-reduced prime subsemimodule of $P$.

Proof. Let $g_1(A)B \subseteq M$, for two subsemimodules $A$ and $B$ of $P$. If possible, let $A \nsubseteq M$ and $B \nsubseteq M$. Then there exists $a \in A$ and $b \in B$ such that $a, b \notin M$. Therefore $(M+ < a >) \cap M' \neq \emptyset$ and $(M+ < b >) \cap M' \neq \emptyset$. Let $x_1 \in (M+ < a >) \cap M'$ and $x_2 \in (M+ < b >) \cap M'$. So $\theta(x_1 \otimes q)x_2 \in g_1(M+ < a >)(M+ < b >) \subseteq M + g_1(< a >) < b > \subseteq M + M \subseteq M$. Also since $M'$ is an $q$-semigroup, $\theta(x_1 \otimes q)x_2 \in M'$. Therefore $M \cap M' \neq \emptyset$, a contradiction to the hypothesis. Hence $g_1(A)B \subseteq M$ implies that either $A \subseteq M$ or $B \subseteq M$. Thus $M$ is a prime subsemimodule of $P$.

Suppose $I/M$ be a nonzero nil subsemimodule of $P/M$. Then $M \subseteq I$ and $I$ is a subsemimodule of $P$. Therefore $I \cap M' \neq \emptyset$. Let $x \in I \cap M'$. So $x \in I$ and $x \in M'$. Now $x/M \in I/M$ implies that, for any $q \in Q$ there exists a positive integer $n$ depending on $x$ and $q$ such that $\theta(x \otimes q)^{n-1}x/M = 0/M$. Thus for all $q \in Q$ there exists a positive integer $n$ such that $\theta(x \otimes q)^{n-1}x \in M$ as $M$ is an $k$-subsemimodule of $P$. Also since $M'$ is a $q$-semigroup in $P - \{0\}$, $x \in M$ implies $\theta(x \otimes q)^{n-1}x \in M'$ for positive integer $n$. Hence, $M \cap M' \neq \emptyset$, a contradiction. This shows that $M$ is a semi-reduced prime subsemimodule of $P$.

Now we deduce the following result from the above proposition.

Corollary 3.14 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S Q_R, \theta, \phi >$ and $x$ a nonzero element of $P$. Also assume $M' = \{ x, \theta(x \otimes q)x, \theta(x \otimes q^2)x, ... \}$ where $q \in Q$ such that $\theta(x \otimes q)^{n-1}x \neq 0$ for all natural number $n$. Then there exists a semi-reduced prime subsemimodule $M$ of $P$ such that $M \cap M' = \emptyset$.

Proof. Since $x$ is a nonzero element of $P$ and $\theta(x \otimes q)^{n-1}x \neq 0$ for all natural number $n$, $M'$ is a $q$-semigroup in $P - \{0\}$. Let $\mathfrak{F} = \{ I : I$ is a
subsemimodule of $P$ such that $I \cap M' = \phi$. Then it is clear that $\{0\} \in \mathcal{F}$. Thus $\mathcal{F}$ is a nonempty poset with respect to set inclusion. Suppose that $\mathcal{C} = \{K_i : i \in A\}$ be a chain in $\mathcal{F}$. Then $K = \bigcup_{i \in A} K_i$ is an upper bound of $\mathcal{C}$ in $\mathcal{F}$. Now by Zorn’s lemma, $\mathcal{F}$ has a maximal element, say $M$. This leads to, $M$ is a subsemimodule of $P$ maximal with respect to the property $M \cap M' = \phi$. Thus by proposition 3.13, $M$ is a semi-reduced prime subsemimodule of $P$.

\[ \square \]

**Proposition 3.15** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, R_{P_{S,S}} Q_{R, \theta, \phi} >$ and $M'$ an $q$-semigroup in $P - \{0\}$, where $q \in Q$. Assume $I$ is an subsemimodule of $P$ such that $I \subseteq M$ and $M \cap M' = \phi$.

Then there exists a semi-reduced prime subsemimodule $M$ of $P$ such that $I \subseteq M$.

**Proof.** Let $\mathcal{F} = \{J : J$ is an subsemimodule of $P$ containing $I$ and $J \cap M' = \phi\}$. Since $I \in \mathcal{F}$, $\mathcal{F}$ is a nonempty set. Now $\mathcal{F}$ is a poset with respect to the set inclusion. Let $\mathcal{C} = \{K_i : i \in A\}$ be a chain in $\mathcal{F}$. Then $K = \bigcup_{i \in A} K_i$ is an upper bound of $\mathcal{C}$ in $\mathcal{F}$. Therefore by Zorn’s lemma, $\mathcal{F}$ has a maximal element, say $M$. Then $I \subseteq M$. Hence $M$ is maximal with respect to the property $M \cap M' = \phi$. The rest of the proof follows in analogous manner as the proof of Proposition 3.13.

\[ \square \]

In the following proposition, we describe the properties of semi-reduced prime subsemimodules of a constituent bisemimodule $P$ in a Morita Context $< R, S, R_{P_{S,S}} Q_{R, \theta, \phi} >$.

**Proposition 3.16** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, R_{P_{S,S}} Q_{R, \theta, \phi} >$. Then $N^*(P) = \bigcap\{I : I$ is a semi-reduced prime subsemimodule of $P\}$, where $N^*(P)$ denotes the unique \(^1\) maximal nil subsemimodule of $P$.

**Proof.** If possible, we assume $N^*(P) \notin I$ for some semi-reduced prime subsemimodule $I$ of $P$. Then $(N^*(P) + I)/I$ is a nonzero nil subsemimodule of $P/I$, a contradiction. Hence, $N^*(P) \subseteq I$ for each semi-reduced prime subsemimodule of $P$. Thus $N^*(P) \subseteq \bigcap\{I : I$ is a semi-reduced prime subsemimodule of $P\}$.

To prove the converse, let $a \in \bigcap\{I : I$ is a semi-reduced prime subsemimodule of $P\}$. If possible, assume $a \notin N^*(P)$. Then $a$ is not a nilpotent element of $P$. Therefore by corollary 3.14, there exists a semi-reduced prime subsemimodule $I$ of $P$ such that $a \notin I$, a contradiction. Therefore $a \in N^*(P)$. So $\bigcap\{I : I$ is a semi-reduced prime subsemimodule of $P\}$.

\(^1\) The existence of the unique maximal nil subsemimodule of $P$ is guaranteed by Zorn’s lemma.

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\( P \subseteq N^*(P) \). Hence \( N^*(P) = \bigcap \{ I : I \) is a semi-reduced prime subsemimodule of \( P \} \).

\[ \square \]

We now define some special subsets in \( P \) of the Morita context \(< R, S, P, S, Q, R, \theta, \phi >\) and obtain some of their important properties (cf. propositions 3.18, 3.19, 3.20, 3.21) in order to characterize the unique maximal nil subsemimodule \( N^*(P) \) of \( P \) (cf. Theorem 3.29).

**Definition 3.17** Let \( R, S \) be two Morita equivalent semirings via Morita context \(< R, S, P, S, Q, R, \theta, \phi >\). For a semi-reduced prime subsemimodule \( M \) of \( P \), we define the following subsets of \( P \):

(i) \( N^*(M) = \{ x \in P : \theta(x \otimes Q)y \subseteq N^*(P) \) for some \( y \in P - M \}; \)

(ii) \( \overline{N^*} = \{ x \in P : \theta(x \otimes Q)^{n-1}x \subseteq N^*(M) \) for some positive integer \( n \}

Some properties of the subsets \( N^*(M) \) and \( \overline{N^*} \) are obtained in the following propositions.

**Proposition 3.18** Let \( R, S \) be two Morita equivalent semirings via Morita context \(< R, S, P, S, Q, R, \theta, \phi >\). Then for any semi-reduced prime subsemimodule \( M \) of \( P \), \( N^*(M) \subseteq M \) and \( N^*(M) \subseteq \overline{N^*} \).

**Proof.** Let \( x \in N^*(M) \). Then \( \theta(x \otimes Q)y \subseteq N^*(P) \) for some \( y \in P - M \). By proposition 3.16, \( \theta(x \otimes Q)y \subseteq M \) for any semi-reduced prime subsemimodule \( M \) of \( P \). Since \( M \) is prime and \( y \in P - M \), \( x \in M \) [3]. Hence \( N^*(M) \subseteq M \).

Next we prove \( N^*(M) \subseteq \overline{N^*} \).

\[ \square \]

**Proposition 3.19** Let \( R, S \) be two Morita equivalent semirings via Morita context \(< R, S, P, S, Q, R, \theta, \phi >\) and \( M \) be a semi-reduced prime subsemimodule of \( P \). Then \( N^*(M) = \{ x \in P : \theta(x \otimes Q)y \subseteq N^*(P) \) for some \( y \in P - M \} \), where \( \langle y \rangle \) denotes the subsemimodule of \( P \) generated by \( y \).

**Proof.** Let \( A = \{ x \in P : \theta(x \otimes Q)y \subseteq N^*(P) \) for some \( y \in P - M \} \). Since \( y \in \langle y \rangle \), \( A \subseteq N^*(M) \). Let \( x \in N^*(M) \). Then \( \theta(x \otimes Q)y \subseteq N^*(P) \) for some \( y \in P - M \). Now any element of \( \langle y \rangle \) is of the form

\[ ny + \sum_{i=1}^{m} \phi(\alpha_i \otimes x_i) + \sum_{j=1}^{t} \theta(z_j \otimes \beta_j)y + \sum_{k=1}^{s} \theta(\mu_k \otimes v_k), \]

where \( x_i, z_j, \mu_k \in P \) and \( \alpha_i, \beta_j, \lambda_k, \mu_k \in Q \) and \( n, m, t, s \) are nonnegative integers. Hence \( \theta(x \otimes Q)y \subseteq N^*(P) \) as \( N^*(P) \) is a subsemimodule of \( P \). Therefore \( x \in A \). Consequently \( N^*(M) = A \).

\[ \square \]

**Proposition 3.20** Let \( R, S \) be two Morita equivalent semirings via Morita context \(< R, S, P, S, Q, R, \theta, \phi >\) and \( M \) be any semi-reduced prime subsemimodule of \( P \). Then \( N^*(M) \) is a bisubsemimodule of \( P \).
Proof. Since 0 ∈ \(N^*(M)\), \(N^*(M)\) is a nonempty subset of \(P\). Let \(x_1, x_2 \in N^*(M)\). Then there exist \(y_1, y_2 \in P - M\) such that \(\theta(x_1 \otimes Q) < y_1 \leq N^*(P)\) and \(\theta(x_2 \otimes Q) < y_2 \leq N^*(P)\). Since \(M\) is a prime subsemimodule of \(P\), \(P - M\) is an \(m - \) system. So there exist \(p, \alpha, \beta \in Q\) such that \(\theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 \in P - M\). Now \(\theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 \leq \theta(x_1 \otimes Q) < y_1 > \leq y_2 >\). Therefore, \(\theta((x_1 + x_2) \otimes Q) < \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 \leq \theta(x_1 \otimes Q) < \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 + \theta(x_2 \otimes Q) < \theta(y_1 \otimes \alpha)\theta(p \otimes \beta)y_2 \leq \theta(x_1 \otimes Q) < y_1 > + \theta(x_2 \otimes Q) < y_2 \leq N^*(P)\) as \(N^*(P)\) is a subsemimodule of \(P\). Thus \(x_1 + x_2 \in N^*(M)\).

Let \(x \in N^*(M)\). Then there exists \(y \in P - M\) such that \(\theta(x \otimes Q) < y \leq N^*(P)\). Therefore \(\theta(Rx \otimes Q) < y \leq N^*(P)\) and \(\theta(xS \otimes Q) < y \leq N^*(P)\). Thus \(Rx, xS \subseteq N^*(M)\) for all \(x \in N^*(M)\). Hence \(N^*(M)\) is a subsemimodule of \(P\).

\[\Box\]

Proposition 3.21 Let \(R, S\) be two Morita equivalent semirings via Morita context \(< R, S, P_S, Q_R, \theta, \phi >\) and \(M\) a semi-reduced prime subsemimodule of \(P\). Then

(i) \(N^*(M)\) is a subsemimodule of \(P\);

(ii) \(N^*(M)\) is a completely semiprime subsemimodule of \(P\) if and only if \(N^*(M) = N^*(M)\).

Proof. (i)

Since \(N^*(M)\) is a subsemimodule of \(P\) and \(N^*(M) \subseteq \overline{N^*(M)}\), \(N^*(M)\) is a nonempty subset of \(P\). Let \(x, y \in N^*(M)\). Then \(\theta(x \otimes Q)^{n-1}x, \theta(y \otimes Q)^{m-1}y \leq N^*(M)\) for some \(n, m \in \mathbb{Z}^+\). Therefore the elements of the form

\[
\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots \theta(x \otimes \alpha_{k-1})\theta(p_{k-1} \otimes \beta_{k-1})x
\]

\((k \geq n)\) and \(\theta(y \otimes \gamma_1)\theta(p_1 \otimes \delta_1)\theta(y \otimes \gamma_2)\theta(p_2 \otimes \delta_2)\ldots \theta(y \otimes \gamma_{r-1})\theta(p_{r-1} \otimes \delta_{k-1})y \in N^*(M)\) i.e. an expression containing at least \(n\) \(x\)'s or \(m\) \(y\)'s must belong to \(N^*(M)\). Since \(N^*(M)\) is a subsemimodule \(\theta((x+y) \otimes Q)^{m+n}(x+y)\) is contained in \(N^*(M)\). So \(x + y \in N^*(M)\). Since \(N^*(M)\) is a subsemimodule, \(\theta(Rx \otimes Q)^{n-1}Rx, \theta(xS \otimes Q)^{m-1}xS \subseteq N^*(M)\) i.e. \(Rx, xS \subseteq N^*(M)\). Therefore \(N^*(M)\) is a subsemimodule of \(P\).

(ii)

Suppose that \(N^*(M)\) is a completely semiprime subsemimodule of \(P\). Clearly, \(N^*(M) \subseteq N^*(M)\). Let \(a \in N^*(M)\). Then \(\theta(a \otimes Q)^{n-1}a \subseteq N^*(M)\) for some positive integer \(n\). As \(N^*(M)\) is completely semiprime subsemimodule of \(P\), \(\theta(a \otimes Q)^{n-1}a \subseteq N^*(M)\) implies \(a \in N^*(M)\). Therefore \(N^*(M) = N^*(M)\). The converse part is obvious.

\[\Box\]

Definition 3.22 Let \(R, S\) be two Morita equivalent semirings via Morita context \(< R, S, P_S, Q_R, \theta, \phi >\). For a semi-reduced prime subsemimodule \(M\) of \(P\), we define the following subsets of \(P\):
\( (i) \ 0^*(M) = \{ x \in P : \theta(x \otimes Q)y = 0 \text{ for some } y \in P - M \}; \)
\( (ii) \ 0^*(M) = \{ x \in P : \theta(x \otimes Q)^{n-1}x \subseteq 0^*(M), \text{ for some positive integer } n \} \)

We obtain the following propositions involving the properties of \( 0^*(M) \) and \( 0^*(\tilde{M}) \) which are also used to establish proposition 3.28. Since the proofs of these propositions are similar to those of propositions 3.18, 3.19, 3.20, 3.21 we omit them.

**Proposition 3.23** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P_S S Q_R, \theta, \phi > \). Then for any semi-reduced prime subsemimodule \( M \) of \( P \), \( 0^*(M) \subseteq M \) and \( 0^*(\tilde{M}) \subseteq \tilde{M} \).

**Proposition 3.24** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P_S S Q_R, \theta, \phi > \) and \( M \) be a semi-reduced prime subsemimodule of \( P \). Then \( 0^*(M) = \{ x \in P : \theta(x \otimes Q) < y >= 0 \text{ for some } y \in P - M \} \), where \( < y > \) denotes the subsemimodule of \( P \) generated by \( y \).

**Proposition 3.25** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P_S S Q_R, \theta, \phi > \) and \( M \) be any semi-reduced prime subsemimodule of \( P \). Then \( 0^*(M) \) is a subsemimodule of \( P \).

**Proposition 3.26** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P_S S Q_R, \theta, \phi > \) and \( M \) be a semi-reduced prime subsemimodule of \( P \). Then
\( (i) \ 0^*(M) \) is a subsemimodule of \( P; \)
\( (ii) \ 0^*(M) \) is a completely semiprime subsemimodule of \( P \) if and only if \( 0^*(M) = 0^*(\tilde{M}) \).

**Notations** We use \( PSpec(P) \) and \( mPSpec(P) \) to denote the set of all semi-reduced prime subsemimodules and minimal semi-reduced prime subsemimodules of \( P \) in a Morita context \( < R, S, R P_S S Q_R, \theta, \phi > \).

The following theorem shows the existence of a minimal semi-reduced prime subsemimodule of the bisemimodule \( P \).

**Lemma 3.27** Let \( R, S \) be two Morita equivalent semirings via Morita context \( < R, S, R P_S S Q_R, \theta, \phi > \) and \( M \) be a semi-reduced prime subsemimodule of \( P \). Then there exists a minimal semi-reduced prime subsemimodule \( M' \) of \( P \) such that \( M' \subseteq M \).

*Proof.* Let \( \mathfrak{F} = \{ M_\alpha : M_\alpha \text{ is a semi-reduced prime subsemimodule of } P \} \).
Then \( \mathfrak{F} \neq \phi \), since \( M \in \mathfrak{F} \) \( \mathfrak{F} \) is a poset with respect to set inclusion. Let \( \mathcal{C} \) be a chain in \( \mathfrak{F} \) and \( M' = \bigcap_{M_\alpha \in \mathcal{C}} M_\alpha \). Then \( M' \) is a subsemimodule of \( P \).
Let \( A \) and \( B \) be two subsemimodules of \( P \) such that \( g_1(A)B \subseteq M' \). Suppose \( A \not\subseteq M' \) and \( B \not\subseteq M' \). Then \( A \not\subseteq M_i \) and \( B \not\subseteq M_j \) for some \( i \) and \( j \). Since \( \mathcal{C} \) is a chain, either \( M_i \subseteq M_j \) or \( M_j \subseteq M_i \). Hence \( A \not\subseteq M_i \) or \( A \not\subseteq M_j \),
a contradiction as $M' \subseteq M_i$ or $M' \subseteq M_j$ and they are prime. So we must have $A \subseteq M'$ or $B \subseteq M'$. Therefore $M'$ is a prime subsemimodule of $P$.

Since $M'$ is a prime $k$-subsemimodule of $P$, $P/M'$ is a prime $R - S$-bisseminodule. If possible, let $P/M'$ has a nonzero nil bisseminodule say $I/M'$. Then $M' \subseteq I \subseteq P$. Now we show that, $I \nsubseteq M_i$, for some $i$. If not, $I \subseteq M_i$, for all $i$. Then $I \subseteq \bigcap M_i = M'$, a contradiction. Hence we can prove $(I + M_i)/M_i$ is a nonzero nil bisseminodule of $P/M_i$. Let $(x + m)/M_i \in (I + M_i)/M_i$. Then $x \in I$ and $m \in M_i$. Then for $\alpha$ there exists a positive integer $n$ such that $\theta(x \otimes \alpha)^{n-1}x/M' = 0/M'$, i.e., $\theta(x \otimes \alpha)^{n-1}x \in M'$, which implies $\theta(x \otimes \alpha)^{n-1}x \in M_i$ for all $i$. Since $M_i$ is a subsemimodule and $m \in M_i$, $\theta((x + m) \otimes \alpha)^{n-1}(x + m) \in M_i$. Hence $\theta((x + m) \otimes \alpha)^{n-1}(x + m)/M_i = 0/M_i$. Thus $(I + M_i)/M_i$ is a nonzero nil bisseminodule of $P/M_i$, hence a contradiction. Therefore $P/M'$ is a semi-reduced prime $R - S$-bisseminodule. Hence $M' \in \mathfrak{F}$ whence the chain $\mathcal{C}$ has a lower bound in $\mathfrak{F}$. By Zorn’s lemma, $\mathfrak{F}$ has a minimal element. This completes the proof.

The above lemma is used to prove the following two results.

**Proposition 3.28** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S_R, P_{S,S} Q_R, \theta, \phi >$ and $P$ be a strongly nil semimodule. Then $N(P) \subseteq \bigcap_{M \in P\text{Spec}(P)} \overline{0^*(M)} \subseteq \bigcap_{M' \in mP\text{Spec}(P)} \overline{0^*(M')}$, where $N(P)$ denotes the set of all nilpotent elements of $P$.

**Proof.** We first notice that if $M_1$ and $M_2$ are two semi-reduced prime subsemimodules of $P$ such that $M_1 \subseteq M_2$, then $\overline{0^*(M_2)} \subseteq \overline{0^*(M_1)}$.

Let $M$ be any semi-reduced prime subsemimodule of $P$, then by lemma 3.27, there exists a minimal semi-reduced prime subsemimodule $M'$ of $P$ such that $M' \subseteq M$. Therefore $\bigcap_{M \in P\text{Spec}(P)} \overline{0^*(M)} \subseteq \bigcap_{M' \in mP\text{Spec}(P)} \overline{0^*(M')}$.

Let $a \in N(P)$. Since $P$ is a strongly nil semimodule, $a \in N_Q(P)$. So $\theta(a \otimes Q)^{n-1}a = 0$, for some positive integer $n$. Then $\theta(a \otimes Q)^{n-1}\theta(a \otimes Q)y = 0$ for each $y \in P - M$, where $M$ is a semi-reduced prime subsemimodule of $P$. So $\theta(a \otimes Q)^{n-1}a \subseteq \overline{0^*(M)}$, for each semi-reduced prime subsemimodule $M$ of $P$ i.e. $a \in \overline{0^*(M)}$ for each semi-reduced prime subsemimodule of $P$, which implies that $a \in \bigcap_{M \in P\text{Spec}(P)} \overline{0^*(M)}$. Hence $N(P) \subseteq \bigcap_{M \in P\text{Spec}(P)} \overline{0^*(M)} \subseteq \bigcap_{M' \in mP\text{Spec}(P)} \overline{0^*(M')}$. 

The following result characterizes the unique maximal nil subsemimodule of $P$ in a Morita Context $< R, S_R P_{S,S} Q_R, \theta, \phi >$. 


Theorem 3.29 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S_Q R, \theta, \phi >$. Then $N^*(P) = \bigcap_{M \in P_{\text{Spec}}(P)} N^*(M)$

$$= \bigcap_{M' \in mP_{\text{Spec}}(P)} N^*(M').$$

Proof. Let $a \in N^*(P)$. As $N^*(P)$ is a subsemimodule, $\theta(a \otimes Q)P \subseteq N^*(P)$. Since $M$ is proper, $P - M$ is nonempty. Let $y \in P - M$. Then $\theta(a \otimes Q)y \subseteq N^*(P)$ for every semi-reduced prime subsemimodule $M$ of $P$. This implies $a \in N^*(M)$ for every semi-reduced prime subsemimodule $M$ of $P$, that is, $a \in \bigcap_{M \in P_{\text{Spec}}(P)} N^*(M)$. Hence $N^*(P) \subseteq \bigcap_{M \in P_{\text{Spec}}(P)} M = N^*(P)$.

Hence, $N^*(P) = \bigcap_{M \in P_{\text{Spec}}(P)} N^*(M)$. Now for each $M \in P_{\text{Spec}}(P)$ there exists $M' \in mP_{\text{Spec}}(P)$ such that $M' \subseteq M$ and $M' \subseteq M$ (cf. lemma 3.27) which implies $\Rightarrow N^*(M) \subseteq N^*(M')$. Then $N^*(P) = \bigcap_{M \in P_{\text{Spec}}(P)} N^*(M')$. Again $N^*(P) = \bigcap_{M' \in mP_{\text{Spec}}(P)} M'$ and $\bigcap_{M' \in mP_{\text{Spec}}(P)} N^*(M') \subseteq \bigcap_{M' \in mP_{\text{Spec}}(P)} M'$. Therefore we conclude that $\bigcap_{M' \in mP_{\text{Spec}}(P)} N^*(M') \subseteq N^*(P)$. Hence $N^*(P) = \bigcap_{M \in P_{\text{Spec}}(P)} N^*(M)$

$$= \bigcap_{M' \in mP_{\text{Spec}}(P)} N^*(M').$$

Our aim in this part is to characterize $NI$—semimodules related to a Morita context.

Definition 3.30 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S_Q R, \theta, \phi >$. Then $P$ is said to be a $NI$—semimodule if $N^*(P) = N(P)$, where $N^*(P)$ denotes the unique maximal nil subsemimodule of $P$, i.e., the nil radical of $P$ [5] and $N(P)$ is used to denote the set of all nilpotent elements of $P$.

Definition 3.31 Let $R, S$ be two Morita equivalent semirings via Morita context $< R, S, R_P S, S_Q R, \theta, \phi >$. Then $P$ is said to be reduced if it has no nonzero nilpotent elements.
Lemma 3.32 Let $R$, $S$ be two Morita equivalent semirings via Morita context $<R,S,R,P_S,S,Q_R,\theta,\phi>$ and $P$ is reduced semimodule. Then $P$ is NI- semimodule.

Proof. The proof of the above lemma follows from Definitions 3.30, 3.31, 3.5.

In related to the strongly nilpotent elements of $P$, we consider $P$ to be a strongly nil semimodule.

Recall that $P$ is a strongly nil semimodule if $N(P) = N_Q(P)$, where $N_Q(P)$ is the set of all strongly nilpotent elements of $P$.

Theorem 3.33 Let $R$, $S$ be two Morita equivalent semirings via Morita context $<R,S,R,P_S,S,Q_R,\theta,\phi>$ and $P$ be an strongly nil semimodule. Then

(1) $P$ is a NI-semimodule.
(2) For a semi-reduced prime subsemimodule $M$ of $P$, $0^*(M) = M$ implies that $M$ is a completely prime subsemimodule of $P$.

Proof. (1)
To prove that $P$ is a NI-semimodule, it suffices to show that $N(P) \subseteq N^*(P)$. Let $a \in N(P)$. Since $P$ is a strongly nil semimodule, $\theta(a \otimes Q)^{n-1}a = 0$ for some positive integer $n$. Suppose if possible, $a \notin N^*(P)$, then there exists a semi-reduced prime subsemimodule $M$ of $P$ such that $a \notin M$ i.e. $a \in P - M$. Since $M$ is a semi-reduced prime subsemimodule of $P$. So $P - M$ is an $m$-system of $P$. Then there exist $p_1 \in P$ and $\alpha_1, \beta_1 \in Q$ such that $\theta(a \otimes Q)^{n-1}a \in P - M$. Again since $\theta(a \otimes Q)^{n-1}a$, $a \not\in P - M$, applying $m$-system property $\theta(a \otimes Q)^{n-1}a \in P - M$, for some $\alpha_2, \beta_2 \in Q$ and $p_2 \in P$. Applying $m$-system property after finite step, we have $\theta(a \otimes Q)^{n-1}a \in P - M$ for some $\alpha_3, \beta_3 \in Q$, where $i = 1, 2, \ldots, n - 1$. Since $\theta(a \otimes Q)^{n-1}a = 0 \not\subseteq 0^*(M)$ implies that $\theta(a \otimes Q)^{n-1}a$ is in particular $\theta(a \otimes Q)^{n-1}a \in P - M$ and so $0^*(M) \subseteq N(P)$ and so $N(P) = N^*(P)$ that is $P$ is a NI-semimodule.

(2)
Let $\theta(x \otimes Q)y \subseteq M = 0^*(M)$. Then there exists $b \in P - M$ such that $\theta(x \otimes Q)_{y \otimes Q}b = 0$. If possible, let $x \notin M$. Therefore $\theta(x \otimes Q)_{y \otimes Q}b = 0 \subseteq N^*(P)$ that $\theta(x \otimes Q)_{y \otimes Q}b \subseteq N^*(P) \subseteq M$. Since $M$ is prime and $x \notin M$, $\theta(y \otimes Q)b \subseteq M$. Again since $b \notin M$, $y \in M$ as $M$ is a prime subsemimodule of $P$. Therefore, either $x \in M$ or $y \in M$. Hence $M$ is a completely prime subsemimodule of $P$. 

\qed
The following theorem characterizes an $NI$–semimodule related with Morita context.

**Theorem 3.34** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S_1, P_1, S_2, Q_2, R, \theta, \phi >$ and $P$ is a strongly nil semimodule. The following statements are equivalent:

1. $P$ is a $NI$–semimodule.
2. $N^*(P)$ is a completely semiprime subsemimodule of $P$.
3. $N^*(P)$ is a symmetric subsemimodule of $P$.

Proof.

1. $\Rightarrow$ (2)

Let $\theta(a \otimes Q)a \subseteq N^*(P)$, where $a \in P$. Since $P$ is a strongly nil and $NI$–semimodule, $\theta(a \otimes Q)a \subseteq N_Q(P)$. Then there exists a positive integer $n$ such that $\theta(\theta(a \otimes Q)(a \otimes Q)^n - 1(a \otimes Q)a) = 0$. This implies that $\theta(a \otimes Q)^{2n-1}a = 0$. So $a \in N_Q(P) = N(P) = N^*(P)$. Therefore $N^*(P)$ is a completely semiprime subsemimodule of $P$.

2. $\Rightarrow$ (3)

Let $\theta(a \otimes Q)(b \otimes Q)c \subseteq N^*(P)$, where $a, b, c \in P$. Then $\theta(\theta(c \otimes Q)(a \otimes Q)b \otimes Q)\theta(c \otimes Q)\theta(a \otimes Q) = \theta(c \otimes Q)(a \otimes Q)\theta(b \otimes Q)c \otimes Q\phi(Q \otimes a) \subseteq N^*(P)$. Since $N^*(P)$ is completely semiprime, so $\theta(c \otimes Q)(a \otimes Q)b \subseteq N^*(P)$.

Now $\theta(b \otimes Q)\theta(a \otimes Q)c \subseteq N^*(P)$ and $\theta(a \otimes Q)b \subseteq N^*(P)$ as $N^*(P)$ is a subsemimodule of $P$. This implies $\theta(b \otimes Q)\theta(a \otimes Q)c \subseteq N^*(P)$.

Again by using similar argument, we have

$$
\theta(\theta(b \otimes Q)(a \otimes Q)b \otimes Q)\theta(c \otimes Q)a \subseteq N^*(P) \Rightarrow \theta(b \otimes Q)\theta(a \otimes Q)c \subseteq N^*(P).
$$

This implies that

$$
\theta(\theta(a \otimes Q)b \otimes Q)^3 \theta(a \otimes Q)b = \theta(a \otimes Q)b \otimes Q)\theta(b \otimes Q)a \otimes Q)c \subseteq N^*(P) \Rightarrow \theta(b \otimes Q) \phi(Q \otimes Q) \subseteq N^*(P)
$$

as $N^*(P)$ is completely semiprime. Hence, $N^*(P)$ is a right symmetric subsemimodule of $P$. Also $\theta(\theta(b \otimes Q)(a \otimes Q)c \otimes Q)(a \otimes Q)b \otimes Q)c \subseteq N^*(P) \Rightarrow \theta(b \otimes Q)(a \otimes Q)c \subseteq N^*(P)$. This shows that $N^*(P)$ is a left symmetric subsemimodule of $P$. Similarly, $N^*(P)$ is a right symmetric subsemimodule of $P$.

(3) $\Rightarrow$ (1)

If $\theta(a \otimes Q)b \subseteq N^*(P)$, where $a, b \in P$. Then, $\theta(a \otimes Q)b \subseteq N^*(P)$.

If $\theta(a \otimes Q)b \subseteq N^*(M)$. We have $\theta(a \otimes Q)b \subseteq N^*(M)$. It is clear that $N^*(P) \subseteq N(P)$ always. Let $x \in N(P)$. Since $P$ is a $SN$-semimodule, $x \in N^*(P)$.
$N_Q(P)$. Hence, exists a positive integer \( n \) such that $\theta(x \otimes Q)^{n-1}x = 0$. Suppose if possible, let $x \notin N^*(P)$. Then $x \notin M$ for some semi-reduced prime subsemimodule $M$ of $P$. Hence $x \in P - M$. Since $M$ is prime, $P - M$ is an $m - \text{system}$. Clearly, there exists $p_1 \in P$, $\alpha_1, \beta_1 \in Q$ such that $\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)x \in P - M$. Again since $\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)x \in P - M$, by the property of $m - \text{system}$, we have $\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)x \in \ P - M$, for some $\alpha_2, \beta_2 \in Q$ and $p_2 \in P$. Applying the property of $m - \text{system}$ again after a number of finite steps, we have $\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in P - M$ for some $p_i \in P$, $\alpha_i, \beta_i \in Q$, where $i = 1, 2, \ldots, n - 1$. Also since $\theta(x \otimes Q)^{n-1}x = 0 \subseteq N^*(M)$ implies $\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in N^*(M)$ i.e. $\theta(x \otimes \alpha_1)\theta(p_1 \otimes \beta_1)\theta(x \otimes \alpha_2)\theta(p_2 \otimes \beta_2)\ldots\theta(x \otimes \alpha_{n-1})\theta(p_{n-1} \otimes \beta_{n-1})x \in M[N^*(M) \subseteq M]$, a contradiction. Thus $x \in N^*(P)$ and so $N^*(P) = N(P)$. Hence, we have proved that $P$ is a $NI$-semimodule.

(1) $\Rightarrow$ (4)

Let $M$ be a semi-reduced prime subsemimodule of $P$. Let $a \in \overline{N^*(M)}$. Then $\theta(a \otimes Q)^{n-1}a \subseteq N^*(M)$, for some positive integer $n$. Hence there exists $b \in P - M$ such that $\theta(a \otimes Q)^{n-1}\theta(a \otimes Q)b \subseteq N^*(P)$. By (3), we see that $N^*(P)$ is a left and right symmetric subsemimodule of $P$. Thus $\theta(\theta(a \otimes Q)b \otimes Q)\theta(\theta(a \otimes Q)b \otimes Q)\ldots\theta(\theta(a \otimes Q)b \otimes Q) (n - \text{times}) \subseteq N^*(P)$. Again by (2) $N^*(P)$ is a completely prime subsemimodule of $P$. Then we have $\theta(a \otimes Q)b \subseteq N^*(P)$, which implies that $a \in N^*(M)$. Consequently, we deduce that $\overline{N^*(M)} \subseteq N^*(M)$. Therefore $N^*(M) = \overline{N^*(M)}$.

(4) $\Rightarrow$ (1)

Let $N^*(M) = \overline{N^*(M)}$ for each semi-reduced prime subsemimodule $M$ of $P$. Then by definition, $N^*(P) \subseteq N(P)$. Let $a \in N(P)$. Since $P$ is an $SN$-semimodule, $a \in N_Q(P)$. This implies that $\theta(a \otimes Q)^{n-1}a = 0$ for some positive integer $n$ and so $\theta(a \otimes Q)^{n-1}a \subseteq N^*(M) \Rightarrow a \in N^*(M) \Rightarrow a \in N^*(M) \Rightarrow a \in M (N^*(M) \subseteq M)$ for each prime subsemimodule $M$. Hence, by Proposition 3.16, $a \in N^*(P)$. Thus, $N(P) \subseteq N^*(P)$. This shows that $P$ is an $NI$-semimodule.

\[\square\]

As a consequence of the above theorem, we obtain the following corollary.

**Corollary 3.35** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S; R_P, S_Q, \theta, \phi >$ and $P$ be a $NI$-semimodule. Also assume that $M$ be a semi-reduced prime subsemimodule of $P$ such that $N^*(M) = M$. Then $M$ is a completely prime subsemimodule of $P$.

\[\text{Proof.} \ \theta(x \otimes Q)y \subseteq M \text{ and } y \notin M. \text{ Since } N^*(M) = M, \theta(x \otimes Q)y \subseteq N^*(M), \text{ there exists } b \in P - M \text{ such that } \theta(x \otimes Q)\theta(y \otimes Q)b \subseteq N^*(P) \subseteq M. \text{ Since } M \text{ is prime and } y, b \notin M, x \in M, \text{ the result follows}. \ \square\]

For strongly nil semimodule, we also have the following theorem.
The following is another result characterizing an $NI$–semimodule related with a Morita context.

**Theorem 3.36** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, R P_S, S Q_R, \theta, \phi >$ and $P$ be a strongly nil semimodule. Then the following statements are equivalent:

1. $P$ is a $NI$–semimodule.
2. $0^*(M) \subseteq M$ for each $M \in PSpec(P)$.
3. $N(P) = \bigcap_{M \in PSpec(P)} 0^*(M) = N^*(P)$.

**Proof.** (1) $\Rightarrow$ (2)

Let $a \in 0^*(M)$. Then there exists a positive integer $n$ such that $\theta(a \otimes Q)^{n-1}a \subseteq 0^*(M)$. Hence $\theta(a \otimes Q)^{n-1}\theta(a \otimes Q)b = 0$ for some $b \in P - M$ i.e. $\theta(a \otimes Q)^{n-1}\theta(a \otimes Q)b \subseteq N^*(P)$, for some $b \in P - M$, which implies $\theta(a \otimes Q)^{n-1}a \subseteq N^*(M)$ i.e., $a \in N^*(M)$. Hence, $0^*(M) \subseteq N^*(M)$ for each semi-reduced prime subsemimodule. Also by Theorem 3.34, $N^*(M) = N^*(P)$ for each semi-reduced prime subsemimodule $M$ of $P$. Again since $N^*(M) \subseteq M$ for each semi-reduced prime subsemimodule $M$ of $P$. Thus $0^*(M) \subseteq N^*(M) = N^*(M) \subseteq M$ for each semi-reduced prime subsemimodule $M$ of $P$.

(2) $\Rightarrow$ (3)

Since $0^*(M) \subseteq M$ for each semi-reduced prime subsemimodule $M$ of $P$,

$$\bigcap_{M \in PSpec(P)} 0^*(M) \subseteq \bigcap_{M \in PSpec(P)} M = N^*(P)$$

Now by lemma 3.28 $N(P) \subseteq \bigcap_{M \in PSpec(P)} 0^*(M) \subseteq N^*(P)$. Also $N^*(P) \subseteq N(P)$. Therefore $N(P) = \bigcap_{M \in PSpec(P)} 0^*(M) = N(P)$.

(3) $\Rightarrow$ (1)

This part is obvious.

We now formulate another Theorem for the strongly nil semimodule.

**Theorem 3.37** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, R P_S, S Q_R, \theta, \phi >$ and $P$ is strongly nil semimodule. If $P$ $0^*(M) = M$ for each $M \in PSpec(P)$: then

1. $0^*(M) = N^*(M)$ for each $M \in PSpec(P)$.
2. Every semi-reduced prime subsemimodule of $P$ is minimal and completely prime.
Proof. (1)
Since $N^*(M) \subseteq M$ and $0^*(M) = M$ for each semi-reduced prime subsemimodule $M$ of $P$, we have $N^*(M) \subseteq 0^*(M)$ for each semi-reduced prime subsemimodule $M$ of $P$. Since $P$ is $NI$, by Theorem 3.34, $N^*(M) = N^*(M)$ for each semi-reduced prime subsemimodule $M$ of $P$. Also $0^*(M) \subseteq N^*(M)$ [see the above theorem] for each semi-reduced prime subsemimodule $M$ of $P$. Thus $0^*(M) \subseteq N^*(M)$ for each semi-reduced prime subsemimodule $M$ of $P$. Therefore $0^*(M) = N^*(M)$ for each semi-reduced prime subsemimodule $M$ of $P$.

(2) Let $M$ be a semi-reduced prime subsemimodule of $P$. From (1) and the given condition $0^*(M) = M$, we get $N^*(M) = M$ for each semi-reduced prime subsemimodule of $M$ of $P$. If $M'$ is a minimal semi-reduced prime subsemimodule of $P$ contained in $M$, then $N^*(M) \subseteq N^*(M') \subseteq M' \subseteq M = N^*(M)$. Thus $M = M'$ that is $M$ is a minimal semi-reduced prime subsemimodule of $P$.

Let $\theta(x \otimes Q)y \subseteq M = N^*(M)$ and $x \notin M$. Since $\theta(x \otimes Q)y \subseteq N^*(M)$, there exists $b \in P - M$ such that $\theta(\theta(x \otimes Q)y \otimes Q)b \subseteq N^*(P)$ i.e., $\theta(x \otimes Q)(\theta(y \otimes Q)b) \subseteq N^*(P) \subseteq M$. As $x \notin M$, $\theta(y \otimes Q)b \subseteq M$. Again since $b \notin M$, $y \in M$, either $x \in M$ or $y \in M$. This proves that $M$ is a completely prime subsemimodule of $P$.

\[ \square \]

In order to conclude the paper we deduce the following result which shows that the property of being $NI$ is a Morita invariant property.

**Theorem 3.38** Let $R$, $S$ be two Morita equivalent semirings via Morita context $< R, S, R_\phi P, S_\psi S, Q, R, S_\psi S, \theta, \phi >$ and $R$, $S$ are commutative semiring. Then the following are equivalent

(i) $P$ is a $NI$-semimodule.
(ii) $R$ is a $NI$-semiring.
(iii) $S$ is a $NI$-semiring.

Proof. (i) $\Rightarrow$ (ii)
Let $P$ is a $NI$-semimodule. We have to show that $R$ is a $NI$-semiring i.e., $N^*(R) = N(R)$, where $N^*(R)$ denotes the unique maximal nil ideal of $R$ and $N(R)$ is the set of all nilpotent elements of $R$. Here we show that $g_1(N^*(P)) = N^*(R)$. Let $\sum_{i=1}^{m} \theta(p_i \otimes q_i) \in g_1(N^*(P))$ where $p_i \in (N^*(P))$ and for each $p_i$ and $q_i$ there exist $n_i$ such that $\theta(p_i \otimes q_i)^{n_i}p_i = 0$, $i = 1, 2, \ldots, m$. Now we can easily check that $(\sum_{i=1}^{m} \theta(p_i \otimes q_i))^{n_1+n_2+\cdots+n_m+1} = 0$. So $g_1(N^*(P))$ is a nil subsemimodule. The uniqueness and maximalness can be obtained easily from the lattice isomorphism $g_1$. So $g_1(N^*(P))$ is
an unique maximal nil ideal of \( R \) i.e., \( g_1(N^*(P)) = N^*(R) \) and it must be equal to \( N(R) \). If not let \( r \in R \) be a nilpotent element \( \notin N(R) \). Then the nil ideal containing \( N(R) \) and \( r \) contains \( N^*(R) \) (since \( N^*(R) \subseteq N(R) \)) contradicting the maximality of \( N^*(R) \). So \( N^*(R) = N(R) \) i.e., \( R \) is a \( NI \)-semiring.

\((ii) \Rightarrow (i)\)

Let \( R \) is a \( NI \)-semiring. We first show that \( f_1(N^*(R)) = N^*(P) \). First we prove that \( f_1(N^*(R)) \) is a nil ideal. Let \( \sum_{k=1}^{n} i_k p_k | p_k \in P, i_k \in N^*(R) \) and \( n \in \mathbb{Z}^+ \in f_1(N^*(R)) \). Then there exists \( m_1, m_2, \ldots, m_n \in N \) such that \( i_k^{m_k} = 0, \ i_2^{m_2} = 0, \ldots, i_n^{m_n} = 0 \). Let \( m' = \max\{m_1, m_2, \ldots, m_n\} \). We can easily check that \((\theta(\sum_{k=1}^{n} i_k p_k) \otimes q)^{m'} \sum_{k=1}^{n} i_k p_k = 0 \). So \( f_1(N^*(R)) \) is a nil subsemimodule and \( f_1(N^*(R)) = N^*(P) \) follows from the above proof. So \( P \) is a \( NI \)-semimodule.

\((i) \Rightarrow (iii) \) and \((iii) \Rightarrow (i) \) follow in a similar manner.

\( \square \)

4 Concluding remarks

All the results of this paper investigate the relationship between \( S, P \) and \( R \) of a Morita context \( < R, S, R P_S, S P_R, \theta, \phi > \). But similar relationships between \( Q \) and \( S, R, P \) can be established analogously which in turn provide us with alternative proofs for the relevant results discussed in this paper.

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