3 × 3 idempotent matrices over some domains and a conjecture on nil-clean matrices

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Abstract. A characterization of the 3×3 idempotent matrices over some integral domains is given, in terms of determinant, trace and rank. The conjecture: every nil-clean 3×3 integral matrix is exchange, is revisited. Several new cases are proved.

Keywords. exchange · nil-clean · clean · 3 × 3 integral matrix · similarity · diagonal reduction

Mathematics Subject Classification (2010) 16U10 · 16U60 · 11D04 · 11D09 · 11-04 · 15B36

1 Introduction

Expressing the idempotency of a 3×3 matrix amounts to a quadratic system of 9 equations with 9 unknowns, which is clearly hard to handle. As examples in this note show, Cayley-Hamilton’s theorem, which for a 3×3 matrix $A$ is

$$A^3 - \text{Tr}(A)A^2 + \frac{1}{2}(\text{Tr}^2(A) - \text{Tr}(A^2))A - \text{det}(A)I_3 = 0,$$

does not characterize the idempotents. Therefore a characterization in terms of trace, determinant and rank could be useful.

We did not find any reference for a characterization of the 3×3 idempotent matrices, not over $\mathbb{Z}$, nor over more general conditions on the base ring. In this paper we complete this gap over some special integral (commutative) domains.

We say that a ring $R$ is an ID ring (see [4]) if every idempotent matrix over $R$ is similar to a diagonal one. Examples of ID rings include: division rings, local rings, projective-free rings, PID’s, elementary divisor rings, unit-regular rings and serial rings.
Recall (see [1]) that, since a matrix over an integral domain may be viewed over the corresponding field of fractions, the definition and properties of the rank are the usual ones, well-known from Linear Algebra.

Since diagonal idempotent matrices over domains have only 0 or 1 on the diagonal, and idempotency is invariant to conjugations (similarity, as for square matrices), it follows that a necessary condition for a matrix $E$ (over an ID domain) to be idempotent is $\text{rank}(E) = \text{Tr}(E)$, that is, the rank equals the trace. Actually, this is the motive for considering in the sequel only matrices over ID domains.

An integral domain is a GCD domain if every pair $a$, $b$ of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$. GCD domains include unique factorization domains, Bezout domains and valuation domains.

In Section 2, our main result is the characterization of the idempotent $3 \times 3$ matrices over ID, GCD (commutative) domains (e.g. $\mathbb{Z}$). With this new tool in hand, in Section 3 and 4 we revisit a conjecture made in [2]: Every nil-clean $3 \times 3$ integral matrix is exchange.

2 The characterization

First recall the Sylvester’s rank inequality: if $F$ is a field and $A, B \in \mathbb{M}_n(F)$ then $\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$.

As already mentioned, if $R$ is an integral domain with quotient field $F$ and $A \in \mathbb{M}_n(R)$, $\text{rank}_R(A) = \text{rank}_F(A)$ is the largest integer $t$ such that $A$ contains a $t \times t$ submatrix whose determinant is nonzero. Equivalently, this is the maximum number of linearly independent rows (or columns) of $A$. Therefore Sylvester’s rank inequality holds for matrices over integral domains.

So is the subadditivity of the rank, that is, $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$.

Next we mention a predictable.

Lemma 2.1 Let $R$ be a GCD (commutative) domain and let $C_1$, $C_2$ be two $3 \times 1$ nonzero columns. If $C_1$, $C_2$ are linearly dependent over $R$ there exists a column $C$ and elements $a_1, a_2 \in R$ such that $C_i = a_i C$, $i \in \{1, 2\}$.

Proof. Denote $C_i = \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix}$, $i \in \{1, 2\}$ and assume $b_1 C_1 = b_2 C_2$ for some $0 \neq b_i \in R$, $i \in \{1, 2\}$. Without loss of generality, suppose $c_{11} \neq 0$ and so $c_{21} \neq 0$. Let $d_1 = \gcd(c_{11}; c_{21})$ and $c_{11} = l_1 d_1$, $c_{21} = l_2 d_1$ with $\gcd(l_1; l_2) = 1$.

Since $l_1, l_2$ are coprime, from $b_1 l_1 = b_2 l_2$, $l_1$ divides $b_2$ and $l_2$ divides $b_1$, say $b_1 = l_2 \alpha$, $b_2 = l_1 \beta$. From $b_1 l_1 = b_2 l_2$ it follows that $\alpha = \beta$. Further, since $b_1 c_{12} = b_2 c_{22}$, we obtain $l_2 c_{12} = l_1 c_{22}$. Again, since $l_1, l_2$ are coprime, $l_1$ divides $c_{12}$ and $l_2$ divides $c_{22}$, which we can write (say), $c_{12} = l_1 d_2$ and $c_{22} = l_2 d_2$. Similarly, since $b_1 c_{13} = b_2 c_{23}$ we show that $l_1$ divides $c_{13}$ and $l_2$ divides $c_{23}$.
divides $c_{23}$, which we can write $c_{13} = l_1d_3$ and $c_{23} = l_2d_3$ for some $d_3 \in R$.

Finally, if $C = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ then indeed, $C_i = l_iC$, as desired.

\[ \square \]

An analogous procedure, takes care of the case with three columns.

Recall that for any $n \times n$ matrix $A$, up to sign, the first three coefficients of the characteristic polynomial are $1$, $\text{Tr}(A)$, $\frac{1}{2}(\text{Tr}^2(A) - \text{Tr}(A^2))$ and the last is $\det(A)$. The third coefficient equals the sum of the diagonal $2 \times 2$ minors of $A$, and for $n = 3$ this is $\det(A) = \det(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) + \det(\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}) + \det(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}) = a_{11}a_{22} + a_{13}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}$. To simplify the writing, this coefficient will be denoted by $t$ or even $t_A$, if we need to emphasize the matrix $A$.

Now we can prove our main result.

**Theorem 2.2** A $3 \times 3$ matrix $E$ over an ID, GCD domain $R$ is nontrivial idempotent if and only if $\det(E) = 0$, $\text{rank}(E) = \text{Tr}(E) = 1 + \frac{1}{2}(\text{Tr}^2(E) - \text{Tr}(E^2))$ and $\text{rank}(E) + \text{rank}(I_3 - E) = 3$.

**Proof.** Suppose $E = [e_{ij}]$, $1 \leq i, j \leq 3$. Then $t := t_E = e_{11}e_{22} + e_{13}e_{33} + e_{22}e_{33} - e_{12}e_{21} - e_{13}e_{31} - e_{23}e_{32}$.

By Cayley-Hamilton’s theorem, we can write

$$E^3 - \text{Tr}(E)E^2 + tE - \det E \cdot I_3 = 0.$$ 

To show the conditions are necessary, suppose $E = E^2$. Then $\det(E)^2 = \det(E) \in \{0, 1\}$ and by replacement we get

$$(1 - \text{Tr}(E) + t)E = \det E \cdot I_3.$$ 

We go into two cases.

If $1 - \text{Tr}(E) + t \neq 0$, then $E$ is a scalar matrix and we can show that $E \in \{0, I_3\}$. Indeed, either $\det(E) = 0$ and then $E = 0$, or else, $\det(E) = 1$ and if $E = aI_3$, the equality $E = E^2$ gives $a = a^2$ and since $\det E = 1$, $a = 1$ and $E = I_3$ follow.

In the remaining case, $1 - \text{Tr}(E) + t = 0$ and so $\det(E) = 0$, i.e. all nontrivial idempotents satisfy these two (necessary) conditions.

As for the third condition, we use the Sylvester’s rank inequality $\text{rank}(E) + \text{rank}(I_3 - E) - 3 \leq \text{rank}(E(I_3 - E)) = 0$, for $\text{rank}(E) + \text{rank}(I_3 - E) \leq 3$ and the subadditivity $\text{rank}(E + I_3 - E) = \text{rank}(I_3) = 3 \leq \text{rank}(E) + \text{rank}(I_3 - E)$, for the opposite inequality.

Next, we show the conditions are sufficient. Since $\det(E) = 0$, $\text{rank}(E) \leq 2$. Further, $\text{Tr}(E) = 1 + t$ shows that $E \neq 0$, so $\text{rank}(E) \in \{1, 2\}$.

In the first case, notice that if $\text{rank}(E) = 1$ then $t = 0$ and so $\text{Tr}(E) = 1$ follows from $\text{Tr}(E) = 1 + t$. 

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In this case, by Cayley-Hamilton’s theorem, we have $E^3 = E^2$ which generally does not imply $E^2 = E$ (see example 4 below).

However, if rank($E$) = Tr($E$) = 1, it does.

A 3×3 matrix $A$ has rank 1 if and only if any two (say) columns are linearly dependent. As shown in the previous lemma, the columns are multiples of a common column. Simplifying the writing, we can suppose $E$ has one of the three following forms: $[C, sC, vC]$, $[0, C, sC]$, $[0, 0, C]$ where $s$ and $v$ are elements of $R$ and $C$ is a column with at least one nonzero entry. If $C = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and we fulfill the condition Tr($E$) = 1, it follows that $E$ is in one of the following three forms:

$E_1 = \begin{bmatrix} 1 - sa_2 - va_3 & sa_2 & va_2 \\ a_2 & sa_3 & va_3 \\ a_3 & sa_3 & va_3 \end{bmatrix}$,

$E_2 = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1 - sa_2 s(1 - sa_3) & sa_2 \\ 0 & a_3 & sa_3 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$. It can be checked that all these (rank 1) matrices are indeed, idempotent.

Notice that in this case, we do not use rank($E$) + rank($I_3 - E$) = 3.

In the second case, rank($E$) = Tr($E$) = 2 and Tr($E$) = 1 + $t_E$ yields $t_E = 1$.

Observe that in this case Tr($I_3 - E$) = 3 - 2 = 1 and $t_{I_3 - E} = t_E + 3 - 2\text{Tr}(E) = 0$.

Since rank($E$) = 2 implies rank($I_3 - E$) = 1 by the additional hypothesis, this case reduces to the first one. This is because, if $I_3 - E$ is idempotent, so is $E$ (its complementary idempotent).

In this case, by Cayley-Hamilton’s theorem, we have $E(E - I_3)^2 = 0_3$ which generally does not imply $E^2 = E$ (see example 5 below).

\[ \square \]

By $E_{ij}$ we denote the 3×3 matrix with all entries zero excepting the $(i, j)$ entry which is 1.

**Examples.** 1) $E_{11} + E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has trace 1 but rank 2 so it is not idempotent: the square is $E_{11}$.

2) $2E_{11} + E_{23} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has both trace and rank 2, but $t = 0$ so it is not idempotent: the square is $4E_{11}$.
3) $E = E_{11} + E_{22} + E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has both trace and rank 2 and also $t = 1$. Moreover, $I_3 - E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ so rank$(E) + \text{rank}(I_3 - E) = 2 + 1 = 3$.

It is (indeed) idempotent.

4) Take $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Then $A^3 = A^2 = E_{11} + E_{33} \neq A$ (i.e. $A$ is not idempotent) but Tr$(A) = 2 = \text{rank}(A)$, $t = 1$ but rank$(A) = \text{rank}(I_3 - A) = 2$.

5) The matrix $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has both trace and rank 2 and also $t = 1$. It verifies $C(C - I_3)^2 = 0_3$ but it is not idempotent. Again, rank$(C) = \text{rank}(I_3 - C) = 2$.

Actually, all matrices of type $C = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}$ satisfy rank$(C) = \text{Tr}(C) = 2$ and $t = 1$ but $C^2 = \begin{bmatrix} 1 & 2a + ac & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix} \neq C$ for many choices of $a, b, c$.

6) Observe that if char$(R) = 2$, there are idempotents $E \neq 0_3$ with det$(E) = \text{Tr}(E) = 0$. An example is $E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ with $E^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, det$E = \text{Tr}(E) = 0$ and Tr$(E^2) = 2 = 0$.

3 A conjecture revisited

In [2], we can find the following

**Conjecture 3.1** Every nil-clean $3 \times 3$ integral matrix is exchange.

When writing the paper, this characterization of $3 \times 3$ idempotents was not known to the authors.

The characterization allows a different approach in order to prove this conjecture. Indeed, idempotents appear twice in this conjecture: in the definition of nil-clean matrices, i.e. these are sums of idempotents and nilpotents, and in the characterization of exchange elements, i.e. in a ring $R$, $a \in R$ is exchange if and only if there exists $m \in R$ (called exchanger in [2]) such that $a + m(a - a^2)$ is idempotent.

Since
Proposition 3.2 Let $R$ be any ring, $a \in R$, and suppose that $a = e + t$ where $e^2 = e$ and $t^2 = 0$. Then $a$ is exchange in $R$.

in the remaining nonzero case, we will assume the nilpotent, in the nil-clean decomposition of the matrix $A$, has index 3, i.e. $A = E + T$ with $E^2 = E$ and $T^2 \neq 0_3 = T^3$. As for $E$ we can suppose it is nontrivial idempotent: indeed, nilpotents and unipotents are clean and so exchange.

Recall that every nilpotent matrix over a field is similar to a block diagonal matrix

\[
\begin{bmatrix}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k
\end{bmatrix},
\]

where each block $B_i$ is a shift matrix (possibly of different sizes). Actually, this form is a special case of the Jordan canonical form for matrices. A shift matrix has 1’s along the superdiagonal and 0’s everywhere else, i.e. $S = \begin{bmatrix} 1 & 1 & \cdots & \cdots \\
0 & 1 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \end{bmatrix}$, as $n \times n$ matrix.

The following result is proved in [3]:

Theorem 3.3 The following are equivalent for a ring $R$:

(i) Every nilpotent matrix over $R$ is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).

(ii) $R$ is a division ring.

In the sequel, we prove the conjecture for all nil-clean matrices whose nilpotent (of index 3) is similar to the $3 \times 3$ shift.

This is a special case (over $\mathbb{Z}$), because over any commutative domain $D$, there are plenty of nilpotent nonzero matrices which are not similar to the corresponding shift. For example, $\begin{bmatrix} 0 & 2 \\
0 & 0 \end{bmatrix}$ is a nonzero nilpotent of $M_2(\mathbb{Z})$ which is not similar to $E_{12}$, the nonzero $2 \times 2$ shift.

However, it can be proved that

Proposition 3.4 Every nonzero nilpotent $2 \times 2$ matrix over a commutative GCD domain $R$ is similar to $rE_{12}$, for some $r \in R$.

Proof. We are looking for an invertible matrix $U = (u_{ij})$ such that $TU = U(rE_{12})$ with $T = \begin{bmatrix} x & y \\
z & -x \end{bmatrix}$ and $x^2 + yz = 0$.

Let $d = \gcd(x; y)$ and denote $x = dx_1$, $y = dy_1$ with $\gcd(x_1; y_1) = 1$. Then $d^2x_1^2 = -dy_1z$ and since $\gcd(x_1; y_1) = 1$ implies $\gcd(x_1^2; y_1) = 1$, it follows $y_1$ divides $d$. Set $d = y_1y_2$ and so $T = \begin{bmatrix} x_1y_1y_2 & y_1^2y_2 \\
-x_1^2y_2 & -x_1y_1y_2 \end{bmatrix} = y_2 \begin{bmatrix} x_1y_1 & y_1^2 \\
-x_1^2 & -x_1y_1 \end{bmatrix} = y_2T'$. 96
A conjecture on 3 × 3 nil-clean matrices

Since \(\gcd(x_1; y_1) = 1\) there exist \(s, t \in R\) such that \(sx_1 + ty_1 = 1\). Take
\[
U = \begin{bmatrix} y_1 & s \\ -x_1 & t \end{bmatrix}
\]
which is invertible (indeed, \(U^{-1} = \begin{bmatrix} t & -s \\ x_1 & y_1 \end{bmatrix}\)). One can check
\[
T'U = \begin{bmatrix} 0 & y_1 \\ 0 & -x_1 \end{bmatrix} = UE_{12}, \text{ so } r = y_2.
\]

The 3 × 3 analogue is

**Proposition 3.5** Every index 3 nilpotent 3 × 3 matrix over a GCD domain \(R\) is similar to \(rE_{12} + uE_{23}\), for some \(r, u \in R\).

Notice that the possible nonzero 3 × 3 block diagonal matrices with each
block a shift matrix are
\[
S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } S' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } S' \text{ has index two and only } S \text{ has index three } (S^2 = E_{13} \neq 0_3).

Here is what we prove

**Theorem 3.6** The nil-clean 3 × 3 integral matrices whose nilpotent (of index 3) is similar to the shift \(S\), are exchange.

**Proof.** For \(A = E + S\) we have to find an exchanger \(M\) such that \(A + M(A - A^2)\) is an idempotent. As observed in the previous section, it suffices
to consider \(E\) any (nontrivial) trace = rank = 1, 3 × 3 idempotent matrix. Also noticed in the previous section, it suffices to find exchangers for \(E\), any
of the following matrices: \([0, 0, C]\), \([0, C, sC]\), \([C, sC, vC]\) where \(s\) and \(v\) are
some integers and \(C\) is a column with at least one nonzero entry.

There are three cases to discuss.

**Case 1.** The idempotent is of form \([0, 0, C]\), that is, \(E = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}\) and
\[
A = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & b + 1 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Here } ES = 0_3, SE = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^2 = E + SE + E_{13} =
\begin{bmatrix} 0 & 0 & a + b + 1 \\ 0 & 0 & b + 1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A - A^2 = \begin{bmatrix} 0 & 1 & -1 - b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Denoting } M = [m_{ij}], 1 \leq i, j \leq 3
\]
we get \(A + M(A - A^2) = \begin{bmatrix} 0 & 1 + m_{11} & a - (1 + b)m_{11} \\ 0 & m_{21} & (1 + b)(1 - m_{21}) \\ 0 & m_{31} & 1 - (1 + b)m_{31} \end{bmatrix}\). We choose \(m_{21} = m_{31} = 0\) in order to have trace = 1, and \(m_{11} = -1\) in order to vanish the second column. Since the second and third columns of \(M\) play no rôle, we choose
these zero. Hence for \(M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\), \(A + M(A - A^2) = \begin{bmatrix} 0 & a + b + 1 \\ 0 & b + 1 \\ 0 & 1 \end{bmatrix}\)
which is indeed idempotent of the same type as \(E\).
Case 2. Take $E = [0, C, sC] = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1 - sa_3 & s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}$. Now $A - A^2 = (E + S) - (E + S)^2 = S - E_{13} - ES - SE = \begin{bmatrix} 0 & sa_3 & -1 - a_1 - s(1 - sa_3) \\ 0 & -a_3 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}$ and, denoting $b := -1 - a_1 - s(1 - sa_3)$ we obtain

$$M(A - A^2) = \begin{bmatrix} 0 & (m_{11}s - m_{12})a_3 & m_{11}b - m_{13}a_3 \\ 0 & (m_{21}s - m_{22})a_3 & m_{21}b - m_{23}a_3 \\ 0 & (m_{31}s - m_{32})a_3 & m_{31}b - m_{33}a_3 \end{bmatrix}.$$  

Here $\text{Tr}(M(A - A^2)) = (m_{21}s - m_{22} - m_{33})a_3 + m_{31}b$. An exchanger must be found for arbitrary $a_1$, $a_3$ and $s$. For any choice such that $a_3$ and $b$ are not coprime, there are no $m_{ij}$’s such that $\text{Tr}(M(A - A^2)) = 1$ (e.g., $a_1 = -3$, $a_3 = 2, s = 0$ and so $b = 2$). Hence the $m_{ij}$’s must be chosen to give $\text{Tr}(M(A - A^2)) = 0$ for arbitrary $a_1$, $a_3$ and $s$. Hence

$$m_{21}s = m_{22} + m_{33} \text{ and } m_{31} = 0.$$  

Moreover, since then $\text{Tr}(A + M(A - A^2)) = 1$ we also need $\text{rank}(A + M(A - A^2)) = 1$.

Here $A + M(A - A^2) = \begin{bmatrix} 0 & 1 + a_1 & (m_{11}s - m_{12})a_3 & sa_1 + m_{11}b - m_{13}a_3 \\ 0 & 1 + (m_{33} - s)a_3 & 1 + s(1 - sa_3) + m_{21}b - m_{23}a_3 \\ 0 & (1 - m_{32})a_3 & (s - m_{33})a_3 \end{bmatrix}$ has trace 1. For rank 1, we need dependent columns (or rows).

We will choose the other entries in the third row of $M$, in order to have zero 3-rd row in $A + M(A - A^2)$, that is $m_{32} = 1$ and $m_{33} = s$.

Then $m_{22} = (m_{21} - 1)s$ and $A + M(A - A^2) = \begin{bmatrix} 0 & 1 + a_1 & (m_{11}s - m_{12})a_3 & sa_1 + m_{11}b - m_{13}a_3 \\ 0 & 1 & 1 + s(1 - sa_3) + m_{21}b - m_{23}a_3 \\ 0 & 0 & 0 \end{bmatrix}$ and we have to choose $m_{11}, m_{12}, m_{13}, m_{21}$ and $m_{23}$ in order to get the rank 1, that is,

$$\det \begin{bmatrix} 1 + a_1 & (m_{11}s - m_{12})a_3 & sa_1 + m_{11}b - m_{13}a_3 \\ 1 & 1 + s(1 - sa_3) + m_{21}b - m_{23}a_3 \\ 1 & 1 + s(1 - sa_3) + m_{21}b - m_{23}a_3 \end{bmatrix} = 0.$$  

Equivalently, $sa_1 + m_{11}b - m_{13}a_3 = [1 + a_1 + (m_{11}s - m_{12})a_3][1 + s(1 - sa_3) + m_{21}b - m_{23}a_3]$.

Further we choose $m_{21} = 1$s and $m_{11} = a_1$

(and so $m_{22} = 0$). The equality reduces to $sa_1 - a_1[1 + a_1 + s(1 - sa_3)] - m_{13}a_3 = [1 + a_1 + (a_1s - m_{12})a_3][-a_1 - m_{23}a_3]$ and, by taking

$$m_{23} = 0.$$
to (dividing by $a_3$) $m_{13} = s^2 a_1 + s a_1^2 - m_{12} a_1$ with infinitely many possible choices for $m_{12}$. For

$$m_{12} = 0$$

we get $m_{13} = sa_1(s + a_1)$.

Hence finally $M = \begin{bmatrix} a_1 & 0 & sa_1(s + a_1) \\ 1 & 0 & 0 \\ 0 & 1 & s \end{bmatrix}$ and

$$A + M(A - A^2) = \begin{bmatrix} 0 & 1 + a_1 + sa_1 a_3 - a_1(1 + a_1 + sa_1 a_3) \\ 0 & 1 & -a_1 \\ 0 & 0 & 0 \end{bmatrix}.$$ The conditions in Theorem 2.2 can be easily checked: rank = trace = 1, $t = 0$ and

$$\text{rank} \begin{bmatrix} -1 & 1 + a_1 + sa_1 a_3 - a_1(1 + a_1 + sa_1 a_3) \\ 0 & 1 & -a_1 \\ 0 & 0 & -1 \end{bmatrix} = 2.$$ One can verify directly that $A = \begin{bmatrix} 1 & sa_2 - va_3, s(1 - sa_2 - va_3) v(1 - sa_2 - va_3) \\ a_2 & sa_2 & va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix}$ and so matrices of this form are indeed idempotent, of the same type as $E$.

**Case 3.** Take $E = [C, sC, vC] = \begin{bmatrix} 1 - sa_2 - va_3, s(1 - sa_2 - va_3) v(1 - sa_2 - va_3) \\ a_2 & sa_2 & va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix}$ and so

$$A = E + S = \begin{bmatrix} 1 - sa_2 - va_3, 1 + s(1 - sa_2 - va_3) v(1 - sa_2 - va_3) \\ a_2 & sa_2 & 1 + va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix}.$$ As above $A - A^2 = S - E_{13} - ES - SE = \begin{bmatrix} -a_2 & va_3 & -1 - s(1 - sa_2 - va_3) - va_3 \\ -a_3 & -a_2 - sa_2 & 1 - sa_2 - va_3 \\ 0 & -a_3 & -sa_3 \end{bmatrix}$ and denoting $b = 1 - sa_2 - va_3$,

\[
\begin{bmatrix}
-a_2 & va_3 & -1 - sb - va_2 \\
-a_3 & -a_2 - sa_3 & b \\
0 & -a_3 & -sa_3
\end{bmatrix}.
\]

Finally the columns of $A + M(A - A^2)$ are

\[
\begin{bmatrix}
1 - sa_2 - va_3 - m_{11} a_2 - m_{12} a_3 \\
a_2 - m_{21} a_2 - m_{22} a_3 \\
a_3 - m_{31} a_2 - m_{32} a_3 \\
1 + s(1 - sa_2 - va_3) + m_{11} va_3 - m_{12}(a_2 + sa_3) - m_{13} a_3 \\
a_2 + m_{21} va_3 - m_{22}(a_2 + sa_3) - m_{23} a_3 \\
a_3 + m_{31} va_3 - m_{32}(a_2 + sa_3) - m_{33} a_3 \\
va_3 - m_{31}(1 + sb + va_2) + m_{32} b - m_{33} sa_3
\end{bmatrix}
\]
and

\[
\begin{bmatrix}
1 + va_2 - m_{21}(1 + sb + va_2) + m_{22} b - m_{23} sa_3 \\
va_3 - m_{31}(1 + sb + va_2) + m_{32} b - m_{33} sa_3
\end{bmatrix}.
\]
Using computer aid, we chose $M = \begin{bmatrix} \cdots & 1 & 0 & v \\ 0 & 1 & 0 \end{bmatrix}$.

Replacing we get $A + M(A - A^2) = \begin{bmatrix} 0 & sa_2 & -s(1 - sa_2) \\ 0 & -a_2 & 1 - sa_2 \end{bmatrix}$ with (so far) the same first row.

Moreover with $m_{11} = -s$, $m_{12} = -v$ we obtain $A + M(A - A^2) = \begin{bmatrix} 1 + s + (v - s^2)a_2 - (sv + m_{13})a_3 s[1 + s + (v - s^2)a_2 - (sv + m_{13})a_3] \\ 0 \\ sa_2 \\ -a_2 \end{bmatrix}$ .

Finally $m_{13}$ is arbitrary since matrices of type $\begin{bmatrix} 1 & \alpha & sa \\ 0 & sa_2 & -s(1 - sa_2) \\ 0 & -a_2 & 1 - sa_2 \end{bmatrix}$ are idempotent for any $\alpha$. Indeed $\text{Tr}(A + M(A - A^2)) = \text{rank}(A + M(A - A^2)) = 2$, $t = 1$ and $\text{Tr}(I_3 - A - M(A - A^2)) = \text{rank}(I_3 - A - M(A - A^2)) = 1$.

Therefore (choosing $m_{13} = 0$) the exchanger in this case is $M = \begin{bmatrix} -s & -v & 0 \\ 1 & 0 & v \\ 0 & 1 & 0 \end{bmatrix}$.

\[\square\]

**Example.** For $A = \begin{bmatrix} -13 & -25 & -39 \\ 1 & 2 & 4 \\ 4 & 8 & 12 \end{bmatrix}$ and $M = \begin{bmatrix} -2 & -3 & -3 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ we have $M(A - A^2) = \begin{bmatrix} 14 & 15 & 19 \\ -1 & 0 & -2 \\ -4 & -9 & -13 \end{bmatrix}$, $A + M(A - A^2) = \begin{bmatrix} 1 -10 & 20 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}$ (here $a_2 = 1$, $a_3 = 4$, $s = 2$, $v = 3$; $b = -13$).

As already noticed in the previous section, any $3 \times 3$ index 3 nilpotent is similar to a generalized shift $S_g = rE_{12} + uE_{23}$.

In trying to prove the (whole) conjecture, one has to replace the shift $S$ by $S_g$.

We were able to do this in the first case of the previous proof, and made some progress with the second and third case.

**Proposition 3.7** The nil-clean $3 \times 3$ integral matrices with idempotent of form $[0, 0, C]$ are exchange.

**Proof.** The proof goes along the lines of the (previous) special case $r = v = 1$. Take $A = E + S_g$ with $E = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 1 \end{bmatrix}$ and $S_g = \begin{bmatrix} 0 & r & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}$ ($S_g^2 = ruE_{13}$).
Then \( ES_g = 0_3, \ S_gE = \begin{bmatrix} 0 & r b \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ A - A^2 = \begin{bmatrix} 0 & r - r(u + b) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \). Denoting \( M = [m_{ij}], 1 \leq i, j \leq 3 \) we get

\[
A + M(A - A^2) = \begin{bmatrix} 0 & r + rm_{11} & a - r(u + b)m_{11} \\ 0 & rm_{21} & b + s - r(u + b)m_{21} \\ 0 & rm_{31} & 1 - r(u + b)m_{31} \end{bmatrix}.
\]

We choose \( m_{21} = m_{31} = 0 \) in order to have trace = 1, and \( m_{11} = -1 \) in order to vanish the second column. Since the second and third columns of \( M \) play no rôle, we choose these zero. Hence for \( M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) (the same exchanger), we get

\[
A + M(A - A^2) = \begin{bmatrix} 0 & 0 & a + r(b + u) \\ 0 & b + u & 1 \end{bmatrix}
\]

which is an idempotent of the same type as \( E \).

\[
\square
\]

4 The other cases

The second (general) case. \( E = [0, C, sC] = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1 - sa_3 & s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix} \) and \( A = E + S_g \), again going along the lines of the previous proof, the following can be done.

For \( S_g = \begin{bmatrix} 0 & r & 0 \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix} \), \( S_g^2 = ruE_{13}, \ A = \begin{bmatrix} 0 & r + a_1 & sa_1 \\ 0 & 1 - sa_3 & u + s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}, \ ES_g = \begin{bmatrix} 0 & 0 & ua_1 \\ 0 & 0 & u(1 - sa_3) \\ 0 & 0 & ua_3 \end{bmatrix} \) and \( S_gE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ua_3 & usa_3 \\ 0 & 0 & 0 \end{bmatrix} \).

So \( A - A^2 = S_g - ruE_{13} - ES_g - S_gE = \begin{bmatrix} 0 & rsa_3 - ua_1 - rs(1 - sa_3) - ru \\ 0 & -ua_3 & 0 \\ 0 & 0 & -ua_3 \end{bmatrix} \). Denoting \( M = [m_{ij}], 1 \leq i, j \leq 3 \) and \( b = -ua_1 - rs(1 - sa_3) - ru \) we get

\[
M(A - A^2) = \begin{bmatrix} 0 & (m_{11}rs - m_{12}u)a_3 & m_{11}b - m_{13}ua_3 \\ 0 & (m_{21}rs - m_{22}u)a_3 & m_{21}b - m_{23}ua_3 \\ 0 & (m_{31}rs - m_{32}u)a_3 & m_{31}b - m_{33}ua_3 \end{bmatrix}
\]

\[
A + M(A - A^2) = \begin{bmatrix} 0 & r + a_1 + (m_{11}rs - m_{12}u)a_3 & sa_1 + m_{11}b - m_{13}ua_3 \\ 0 & 1 - sa_3 + (m_{21}rs - m_{22}u)a_3 & u + s(1 - sa_3) + m_{21}b - m_{23}ua_3 \\ 0 & a_3 + (m_{31}rs - m_{32}u)a_3 & sa_3 + m_{31}b - m_{33}ua_3 \end{bmatrix}.
\]

Here \( \text{Tr}(M(A - A^2)) = (m_{21}rs - m_{22}u - m_{33}u)a_3 + m_{31}b \). An exchanger must be found for arbitrary \( a_1, a_3 \) and \( s \). For any choice such that \( a_3 \) and
b are not coprime, there are no \( m_{ij} \)'s such that \( \text{Tr}(M(A - A^2)) = 1 \) (e.g., \( a_1 = -3, a_3 = 2 \) and \( s = 0; b = 2 \)).

Hence the \( m_{ij} \)'s must be chosen to give \( \text{Tr}(M(A - A^2)) = 0 \) for arbitrary \( a_1, a_3 \) and \( s \). Hence

\[
m_{21}rs = (m_{22} + m_{33})v \quad \text{and} \quad m_{31} = 0.
\]

Moreover, since then \( \text{Tr}(A + M(A - A^2)) = 1 \) we also need \( \text{rank}(A + M(A - A^2)) = 1 \).

Here \( A + M(A - A^2) =
\[
\begin{bmatrix}
0 & r + a_1 + (m_{11}rs - m_{12}u)a_3 & s a_1 + m_{11}b - m_{13}ua_3 \\
0 & 1 + (m_{33}u - s)a_3 & u + s(1 - sa_3) + m_{21}b - m_{23}ua_3 \\
0 & (1 - m_{32}u)a_3 & (s - m_{33}u)a_3
\end{bmatrix}
\]

has trace 1. For rank 1, we need dependent columns (or rows).

This reduces to

\[
\det \begin{bmatrix}
0 + a_1 + (m_{11}rs - m_{12}u)a_3 & sa_1 + m_{11}b - m_{13}ua_3 \\
(1 - m_{32}u)a_3 & (s - m_{33}u)a_3
\end{bmatrix} = 0,
\]

that is

\[
[r + a_1 + (m_{11}rs - m_{12}u)a_3](s - m_{33}u) = [sa_1 + m_{11}b - m_{13}ua_3](1 - m_{32}u)
\]

and

\[
[1 + (m_{33}u - s)a_3](s - m_{33}u) = [u + s(1 - sa_3) + m_{21}b - m_{23}ua_3](1 - m_{32}u)
\]

[both equalities divided by \( a_3 \)].

Notice that, as in the special \( r = u = 1 \) case, the vanishing of the third row of \( A + M(A - A^2) \) cannot be done, unless \( u = 1 \).

We were not able to determine the entries of a suitable exchanger.

By computer aid, the third row of \( M \), \( \{0, m_{32}, m_{33}\} \) could be \( \{0, 1, 1\} \) or \( \{0, 1, s\} \) or \( \{0, 1, 0\} \) or some others. In each case, computation yields a complementary condition on \( a_1, a_3, r, u \) and \( s \).

Trying to find a counterexample for the conjecture, with \( E = [0, C, sC] \) and \( S_g = rE_{12} + uE_{23} \), we have successively gathered the following non-conditions:

\( u \neq 1 \), \( u \) not dividing \( s, a_3 \) not dividing \( u, a_3 \) not dividing \( rs, a_3 \) not dividing \( u + s - 2, s + u \neq a_3 \) and \( a_3 \) not dividing \( ua_1 - 1 \).

The selection \( a_1 = 2, a_3 = 7, s = 3, r = 2 \) and \( u = 5 \) satisfies all these. The resulting \( 3 \times 3 \) matrix is

\[
A = \begin{bmatrix}
0 & 4 & 6 \\
0 & -20 & -55 \\
0 & 7 & 21
\end{bmatrix}
\]

which still is exchange: among the exchangers we find

\[
\begin{bmatrix}
-1 & x & y \\
0 & 0 & -2 \\
0 & 1 & 0
\end{bmatrix}
\]

with \( [x, y] \in \{[-2, -2], [2, -5], [-6, 1]\} \).

Next attempt: \( m_{32} = 1, m_{33} = 0 = m_{21} \).

The second equation: \( (1 - sa_3)s = [v + s(1 - sa_3) + m_{21}b - m_{23}ua_3](1 - v) \)
or \( v + m_{21}b - m_{23}ua_3 = u[u + s(1 - sa_3) + m_{21}b - m_{23}ua_3]
\]
A conjecture on $3 \times 3$ nil-clean matrices

If $m_{21} = 0$ (as in example), $1 - m_{23}a_3 = u + s(1 - sa_3) - m_{23}ua_3$ (divided by $u$). Or $(1 - u)(1 - m_{23}a_3) = s(1 - sa_3)$ so now $1 - u$ divides $s(1 - sa_3)$.

Here $a_1 = 2$, $a_3 = 7$, $s = 3$, $r = 2$ and $u = 5$: indeed $4$ divides $20$.

So we add another non-condition: $1 - u$ not dividing $s(1 - sa_3)$: $u = 10$.

So $A = \begin{bmatrix} 0 & 4 & 6 \\ 0 & -20 & -50 \\ 0 & 7 & 21 \end{bmatrix}$.

Nothing until $z = 6$ (inclusive), but for $z = 7$ we found $M = \begin{bmatrix} 6 & 3 & 6 \\ 0 & 3 & 4 \\ 7 & 2 & 5 \end{bmatrix}$,

but also $M = \begin{bmatrix} 7 & x & y \\ -5 & -5 & -7 \\ -7 & 1 & -6 \end{bmatrix} [x, y] \in \{[-5, 5], [-2, -6], [1, 7]\}$.

We did not continue our attempts in this case.

The third case. The computation goes along the lines of the $r = u = 1$

The computation goes along the lines of the $r = u = 1$

case. $A = E + S_g = \begin{bmatrix} 1 - sa_2 - va_3 & r + s(1 - sa_2 - va_3) & v(1 - sa_2 - va_3) \\ a_2 & sa_2 & u + va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix}$,

$ES_g = \begin{bmatrix} 0 & r(1 - sa_2 - va_3) & us(1 - sa_2 - va_3) \\ 0 & ra_2 & usa_2 \\ 0 & ra_3 & usa_3 \end{bmatrix}$,

$S_gE = \begin{bmatrix} ra_2 & rsa_2 & rva_2 \\ ua_3 & usa_3 & usa_3 \\ 0 & 0 & 0 \end{bmatrix}$ and $A - A^2 = S_g - ruE_{13} - ES_g - S_gE = \begin{bmatrix} -ra_2 & rva_3 & -us(1 - sa_2 - va_3) - rva_2 - ru \\ -ua_3 & -ra_2 - usa_3 & u(1 - sa_2 - va_3) \\ 0 & -ra_3 & -usa_3 \end{bmatrix}$.

Denoting $b = u(1 - sa_2 - va_3)$ we have

$A - A^2 = \begin{bmatrix} -ra_2 & rva_3 & -sb - rva_2 - ru \\ -ua_3 & -ra_2 - usa_3 & b \\ 0 & -ra_3 & -usa_3 \end{bmatrix}$. Denoting $M = [m_{ij}]$,

$1 \leq i, j \leq 3$ the columns

of $A + M(A - A^2)$ are

$\begin{bmatrix} 1 - sa_2 - va_3 - m_{11}ra_2 - m_{12}ua_3 \\ a_2 - m_{21}ra_2 - m_{22}ua_3 \\ a_3 - m_{31}ra_2 - m_{32}ua_3 \end{bmatrix}$,

$\begin{bmatrix} r + s(1 - sa_2 - va_3) + m_{11}rva_3 - m_{12}(ra_2 + usa_3) - m_{13}ra_3 \\ sa_2 + m_{21}rva_3 - m_{22}(ra_2 + usa_3) - m_{23}ra_3 \\ sa_3 + m_{31}rva_3 - m_{32}(ra_2 + usa_3) - m_{33}ra_3 \end{bmatrix}$,

and $\begin{bmatrix} v(1 - sa_2 - va_3) - m_{11}(sb + rva_2 + ru) + m_{12}b - m_{13}usa_3 \\ u + va_2 - m_{21}(sb + rva_2 + ru) + m_{22}b - m_{23}usa_3 \\ va_3 - m_{31}(sb + rva_2 + ru) + m_{32}b - m_{33}usa_3 \end{bmatrix}$.
Continuation with $M = \begin{bmatrix} 1 & 0 & v \\ 0 & 1 & 0 \end{bmatrix}$ is not very bad but seems not likely [unlikely to get rank=trace =2]: $A + M(A - A^2) = \begin{bmatrix} (1 - r)a_2 & sa_2 & (-s + r - 1)u + [s^2u + (1 - r)v]a_2 \\ (1 - u)a_3 - ra_2 + (1 - u)sa_3 & u(1 - sa_2) + (1 - u)va_3 \end{bmatrix}$.

For $r = u = 1$ this was already $A + M(A - A^2) = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & sa_2 & -s(1 - sa_2) \\ 0 & -a_2 & 1 - sa_2 \end{bmatrix}$.

We did not continue our attempts in this case.

In trying to find a counterexample for the conjecture, we made the following selection:

**Example.** $A = E + S_g = \begin{bmatrix} -13 & -26 & -39 \\ 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -13 & -21 & -39 \\ 1 & 2 & 9 \\ 4 & 8 & 12 \end{bmatrix}$ no exchanger until (incl.) $z = 11$. Here $a_2 = 1, a_3 = 4, s = 2, v = 3, r = 5, u = 6$ and $b = u(1 - sa_2 - va_3) = -78$. Now $A - A^2 = \begin{bmatrix} -ra_2 & rv_a 3 & -sb - rv_a 2 - ru \\ -ua_3 - ra_2 - usa_3 & b \\ 0 & -ra_3 & -usa_3 \end{bmatrix} = \begin{bmatrix} -5 & 60 & 111 \\ -24 & -53 & -78 \\ 0 & -20 & -48 \end{bmatrix}$.

Denoting $M = [m_{ij}], 1 \leq i, j \leq 3$ we get $M(A - A^2) = \begin{bmatrix} -5m_{11} - 24m_{12} 60m_{11} - 53m_{12} - 20m_{13} 111m_{11} - 78m_{12}b - 48m_{13} \\ -5m_{21} - 24m_{22} 60m_{21} - 53m_{22} - 20m_{23} 111m_{21} - 78m_{22}b - 48m_{23} \\ -5m_{31} - 24m_{32} 60m_{31} - 53m_{32} - 20m_{33} 111m_{31} - 78m_{32}b - 48m_{33} \end{bmatrix}$ and the columns of $D := A + M(A - A^2)$ are

$\begin{bmatrix} -13 - 5m_{11} - 24m_{12} \\ 1 - 5m_{21} - 24m_{22} \\ 4 - 5m_{31} - 24m_{32} \end{bmatrix}, \begin{bmatrix} -21 + 60m_{11} - 53m_{12} - 20m_{13} \\ 2 + 60m_{21} - 53m_{22} - 20m_{23} \\ 8 + 60m_{31} - 53m_{32} - 20m_{33} \end{bmatrix}$ and

$\begin{bmatrix} -39 + 111m_{11} - 78m_{12}b - 48m_{13} \\ 9 + 111m_{21} - 78m_{22}b - 48m_{23} \\ 0 + 111m_{31} - 78m_{32}b - 48m_{33} \end{bmatrix}$.

The trace is

$\text{Tr}(D) = 1 + \text{Tr}(M(A - A^2)) = 1 - 5m_{11} - 24m_{12} + 60m_{21} - 53m_{22} - 20m_{23} + 111m_{31} - 78m_{32}b - 48m_{33}$. 
$\text{Tr}(I_3 - D) = 2 - \text{Tr}(M(A - A^2)) = 2 - (-5m_{11} - 24m_{12} + 60m_{21} - 53m_{22} - 20m_{23} + 111m_{31} - 78m_{32}b - 48m_{33})$.

How to prove this cannot be idempotent?

In [2], the nil-clean matrices discussed had (by similarity) the idempotent $E_{11}$ or $E_{11} + E_{22}$. Since $\text{Tr}(E) = \text{rank}(E) = 1$, $E$ is similar to $E_{11}$. We look for a conjugation. $EU = UE_{11}$ amounts to
A conjecture on 3 × 3 nil-clean matrices

\[
\begin{bmatrix}
-13(u_{11} + 2u_{21} + 3u_{31}) & -13(u_{12} + 2u_{22} + 3u_{32}) & -13(u_{13} + 2u_{23} + 3u_{33}) \\
u_{11} + 2u_{21} + 3u_{31} & u_{12} + 2u_{22} + 3u_{32} & u_{13} + 2u_{23} + 3u_{33} \\
4(u_{11} + 2u_{21} + 3u_{31}) & 4(u_{12} + 2u_{22} + 3u_{32}) & 4(u_{13} + 2u_{23} + 3u_{33})
\end{bmatrix}
= \begin{bmatrix}
u_{11} & 0 & 0 \\
u_{21} & 0 & 0 \\
u_{31} & 0 & 0
\end{bmatrix}
\text{ with } \det(U) = \pm 1. \text{ Hence }
\]
\[
-13(u_{11} + 2u_{21} + 3u_{31}) = u_{11} \text{ or } 14u_{11} + 26u_{21} + 39u_{31} = 0
\]
\[
u_{11} + 2u_{21} + 3u_{31} = u_{21} \text{ or } u_{11} + u_{21} + 3u_{31} = 0
\]
\[
4(u_{11} + 2u_{21} + 3u_{31}) = u_{31} \text{ or } 4u_{11} + 8u_{21} + 11u_{31} = 0
\]

and

\[
u_{12} + 2u_{22} + 3u_{32} = u_{13} + 2u_{23} + 3u_{33} = 0.
\]

The first 3 equation form a homogeneous linear system with zero determinant, so we can choose only

\[
u_{11} + u_{21} + 3u_{31} = 0 \text{ (multiplied by } -4 \text{ and added to the next)}
\]
\[
4u_{11} + 8u_{21} + 11u_{31} = 0 \text{ or } 4u_{21} = u_{31} \text{ and } u_{11} = -13u_{21}.
\]

An example is

\[
U = \begin{bmatrix} -13 & 2 & -1 \\ 1 & -1 & -1 \\ 4 & 0 & 1 \end{bmatrix}
\]
for which \(U^{-1}E = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}\) and

\[
U^{-1}EU = E_{11}.
\]

Then the similar nil-clean matrix with \(E_{11}\) idempotent is \(A' = E_{11} + U^{-1}SgU\).

\[
U^{-1}SgU = E_{11} + \begin{bmatrix} 53 & -5 & 7 \\ 241 & -25 & 29 \\ -212 & 20 & -28 \end{bmatrix}
\]

Here \(S_g^2 = \begin{bmatrix} 120 & 0 & 30 \\ 600 & 0 & 150 \\ -480 & 0 & -120 \end{bmatrix}\) and (indeed) \(S_g^3 = 0_3\).

However, for \(A' = \begin{bmatrix} 53 & -5 & 7 \\ 241 & -25 & 29 \\ -212 & 20 & -28 \end{bmatrix}\), an exchanger was fast found for

\[
z = 6: M = \begin{bmatrix} 1 & 0 & 0 \\ 5 & -1 & 6 \\ -4 & 0 & -1 \end{bmatrix}.
\]

The idempotent is \(A' + M(A' - A'^2) = \begin{bmatrix} -119 & -5 & -23 \\ 2856 & 120 & 552 \\ 0 & 0 & 0 \end{bmatrix}\).

References

Received: 08.IV.2021 / Accepted: 15.X.2021

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