Geometry of golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds

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Abstract We study golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds. We find some equivalent conditions for integrability of distributions and investigate the geometry of leaves of distributions.

Keywords golden semi-Riemannian manifolds · golden structure · lightlike submanifolds · golden GCR-lightlike submanifolds

Mathematics Subject Classification (2010) 53C15 · 53C40 · 53C50

1 Introduction

The geometry of submanifolds is one of the most important topics of differential geometry. It is well known that the geometry of semi-Riemannian submanifolds have many similarities with their Riemannian case but the geometry of lightlike submanifolds is different since their normal vector bundle intersect with the tangent bundle making it more difficult and interesting to study. Lightlike geometry has its applications in mathematical physics, in particular, general relativity and electromagnetism [2]. The lightlike submanifolds of semi-Riemannian manifolds were introduced and studied by Duggal and Bejancu [2]. Later, they were developed by Duggal and Şahin [6].

Duggal and Bejancu [2] introduced CR-lightlike submanifolds of indefinite Kaehler manifolds. Since CR-lightlike submanifold does not include the complex and totally real cases therefore Duggal and Şahin introduced screen Cauchy-Riemann (SCR)-lightlike submanifolds of indefinite Kaehler manifolds [3]. Thus to find a class of submanifolds which would behave as an umbrella for CR-lightlike and SCR-lightlike submanifolds of an indefinite Kaehler manifold, Duggal and Şahin introduced GCR-lightlike submanifolds of indefinite Kaehler manifolds which acts as an umbrella of real hypersurfaces, invariant, screen real and CR lightlike submanifolds [4] and then of
indefinite Sasakian manifolds in [5]. These types of submanifolds have been studied in various manifolds by many authors [10,11,15–17].

Golden ratio partitions a line segment into a major subsegment and a minor subsegment in such a way that both the ratio of whole segment and the major subsegment and the ratio of major subsegment and the minor subsegment must equal the number $\phi$ (the Phidias number), which is the real positive root of the equation $x^2 - x - 1 = 0$ (thus, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618...$). The golden ratio is fascinating topic that continually generated news ideas. The golden ratio has been used in many different areas, in architecture, music, arts and philosophies. Being inspired by the golden ratio, Hrețcanu and Crașmăreanu golden manifold was defined a golden Riemannian manifold [12,13]. In [21], Şahin and Akyol introduced golden maps between golden Riemannian manifolds and showed that such maps are harmonic maps. Erdoğan and Yıldırım in [8] studied semi-invariant submanifolds of golden Riemannian manifolds. Gök, Keleş and Kılıç studied some characterizations for any submanifold of a golden Riemannian manifold to be semi-invariant in terms of canonical structures on the submanifold, induced by the golden structure of the ambient manifold [9]. Poyraz and Yaşar introduced light-like hypersurfaces of golden semi-Riemannian manifolds [19]. Erdoğan studied the geometry of screen transversal lightlike submanifolds, radical screen transversal lightlike submanifolds and screen transversal anti-invariant light-like submanifolds of golden semi-Riemannian manifolds [7]. Acet studied screen pseudo-slant lightlike submanifolds of golden semi-Riemannian manifolds [1]. Poyraz introduced golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds [18].

In this paper, we study golden GCR-lightlike submanifolds of golden semi-Riemannian manifolds. We find some equivalent conditions for integrability of distributions and investigate the geometry of leaves of distributions.

2 Preliminaries

Let $\tilde{M}$ be a $C^\infty$-differentiable manifold. If a tensor field $\tilde{P}$ of type $(1,1)$ satisfies the following equation

$$\tilde{P}^2 = \tilde{P} + I$$

then $\tilde{P}$ is named a golden structure on $\tilde{M}$, where $I$ is the identity transformation [14].

Let $(\tilde{M}, \tilde{g})$ be a semi-Riemannian manifold and $\tilde{P}$ be a golden structure on $\tilde{M}$. If $\tilde{P}$ holds the following equation

$$\tilde{g}(\tilde{P}X, Y) = \tilde{g}(X, \tilde{P}Y)$$

then $(\tilde{M}, \tilde{g}, \tilde{P})$ is named a golden semi-Riemannian manifold [20].
Golden GCR-lightlike submanifolds

If \( \tilde{P} \) is a golden structure, then the equation (2.2) is equivalent to
\[
\tilde{g}(\tilde{P}X, \tilde{P}Y) = \tilde{g}(\tilde{P}X, Y) + \tilde{g}(X, Y)
\]
for any \( X, Y \in \Gamma(\tilde{T}M) \).

Let \( (\tilde{M}, \tilde{g}) \) be a real \((m+n)\)-dimensional semi-Riemannian manifold of constant index \( q \), such that \( m, n \geq 1 \) and \( 1 \leq q \leq m + n - 1 \) and \( (M, g) \) be an \( m \)-dimensional submanifold of \( \tilde{M} \), where \( g \) is the induced metric of \( \tilde{g} \) on \( M \). If \( \tilde{g} \) is degenerate on the tangent bundle \( \tilde{T}M \) of \( \tilde{M} \) then \( M \) is called a lightlike submanifold of \( \tilde{M} \). For a degenerate metric \( g \) on \( M \)
\[
TM^\perp = \bigcup \left\{ u \in T_x \tilde{M} : \tilde{g}(u, v) = 0, \forall v \in T_x M, x \in M \right\}
\]
is a degenerate \( n \)-dimensional subspace of \( T_x \tilde{M} \). Thus, both \( T_x M \) and \( T_x M^\perp \) are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace \( \text{Rad}(T_x M) = T_x M \cap T_x M^\perp \) which is known as radical (null) space. If the mapping \( \text{Rad}(TM) : x \in M \rightarrow \text{Rad}(T_x M) \), defines a smooth distribution, called radical distribution on \( M \) of rank \( r > 0 \) then the submanifold \( M \) of \( \tilde{M} \) is called an \( r \)-lightlike submanifold.

Let \( S(TM) \) be a screen distribution which is a semi-Riemannian complementary distribution of \( \text{Rad}(TM) \) in \( TM \). This means that
\[
TM = S(TM) \perp \text{Rad}(TM)
\]
and \( S(TM^\perp) \) is a complementary vector subbundle to \( \text{Rad}(TM) \) in \( TM^\perp \). Let \( \text{tr}(TM) \) and \( \text{ltr}(TM) \) be complementary (but not orthogonal) vector bundles to \( TM \) in \( TM|_M \) and \( \text{Rad}(TM) \) in \( S(TM^\perp)^\perp \), respectively. Then, we have
\[
\text{tr}(TM) = \text{ltr}(TM) \perp S(TM^\perp),
\]
\[
TM|_M = TM \oplus \text{tr}(TM)
= \{ \text{Rad}(TM) \oplus \text{ltr}(TM) \} \perp S(TM) \perp S(TM^\perp). \tag{2.7}
\]

**Theorem 2.1** \([2]\) Let \( (M, g, S(TM), S(TM^\perp)) \) be an \( r \)-lightlike submanifold of a semi-Riemannian manifold \( (\tilde{M}, \tilde{g}) \). Suppose \( U \) is a coordinate neighbourhood of \( M \) and \( \{ \xi_i \}, i \in \{1, \ldots, r\} \) is a basis of \( \Gamma(\text{Rad}(TM)\mid_U) \). Then, there exist a complementary vector subbundle \( \text{ltr}(TM) \) of \( \text{Rad}(TM) \) in \( S(TM^\perp)^\perp \mid_U \) and a basis \( \{ N_i \}, i \in \{1, \ldots, r\} \) of \( \Gamma(\text{ltr}(TM)\mid_U) \) such that
\[
\tilde{g}(N_i, \xi_j) = \delta_{ij}, \quad \tilde{g}(N_i, N_j) = 0 \tag{2.8}
\]
for any \( i, j \in \{1, \ldots, r\} \).
We say that a submanifold \((\tilde{M}, g, S(TM), S(TM^⊥))\) of \(\tilde{M}\) is
Case 1: \(r\)-lightlike if \(r < \min\{m, n\}\),
Case 2: Coisotropic if \(r = n < m; S(TM^⊥) = \{0\}\),
Case 3: Isotropic if \(r = m < n; S(TM) = \{0\}\),
Case 4: Totally lightlike if \(r = m = n; S(TM) = \{0\} = S(TM^⊥)\).

Let \(\tilde{\nabla}\) be the Levi-Civita connection on \(\tilde{M}\). Then, using (2.7), the Gauss and Weingarten formulas are given by
\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]
\[
\tilde{\nabla}_X U = -A_U X + \nabla^s_X U,
\]
for any \(X, Y \in \Gamma(TM)\) and \(U \in \Gamma(tr(TM))\), where \(\{\nabla_X Y, A_U X\}\) and \(\{h(X, Y), \nabla^s_X U\}\) belong to \(\Gamma(TM)\) and \(\Gamma(tr(TM))\), respectively. \(\tilde{\nabla}\) and \(\nabla^t\) are linear connections on \(M\) and on the vector bundle \(tr(TM)\), respectively. According to (2.6), considering the projection morphisms \(L\) and \(S\) of \(tr(TM)\) on \(ltr(TM)\) and \(S(TM^⊥)\), respectively, (2.9) and (2.10) become
\[
\tilde{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y),
\]
\[
\tilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N),
\]
\[
\tilde{\nabla}_X W = -A_W X + \nabla^l_X W + D^s(X, W),
\]
for any \(X, Y \in \Gamma(TM)\), \(N \in \Gamma(ltr(TM))\) and \(W \in \Gamma(S(TM^⊥))\), where \(h^l(X, Y) = Lh(X, Y)\), \(h^s(X, Y) = Sh(X, Y)\), \(\nabla_X Y, A_N X, A_W X \in \Gamma(TM)\), \(\nabla^l_X N, D^l(X, W) \in \Gamma(ltr(TM))\) and \(\nabla^l_X W, D^s(X, N) \in \Gamma(S(TM^⊥))\). Then, by using (2.11)-(2.13) and taking into account that \(\tilde{\nabla}\) is a metric connection we obtain
\[
\tilde{g}(h^s(X, Y), W) + \tilde{g}(Y, D^l(X, W)) = g(A_W X, Y),
\]
\[
\tilde{g}(D^s(X, N), W) = \tilde{g}(A_W X, N).
\]

Let \(Q\) be a projection of \(TM\) on \(S(TM)\). Then, using (2.5) we can write
\[
\nabla_X QY = \nabla^s_X QY + h^s(X, QY),
\]
\[
\nabla_X \xi = -A^l_X X + \nabla^l_X \xi,
\]
for any \(X, Y \in \Gamma(TM)\) and \(\xi \in \Gamma(Rad(TM))\), where \(\{\nabla^s_X QY, A^l_X X\}\) and \(\{h^s(X, QY), \nabla^l_X \xi\}\) belong to \(\Gamma(S(TM))\) and \(\Gamma(Rad(TM))\), respectively. Using the equations given above, we derive
\[
\tilde{g}(h^l(X, QY), \xi) = g(A^l_X X, QY),
\]
\[
\tilde{g}(h^s(X, QY), N) = g(A_N X, QY),
\]
\[
\tilde{g}(h^l(X, \xi), \xi) = 0, \quad A^s_X \xi = 0.
\]
Generally, the induced connection \( \nabla \) on \( M \) is not metric connection. Since \( \tilde{\nabla} \) is a metric connection, from (2.11), we obtain
\[
(\nabla_X g)(Y, Z) = \tilde{g}(h^l(X, Y), Z) + \tilde{g}(h^l(X, Z), Y). \tag{2.21}
\]
But, \( \nabla^* \) is a metric connection on \( S(TM) \).

3 Golden generalized Cauchy-Riemann (GCR)-lightlike submanifolds

**Definition 3.1** Let \((M, g, S(TM))\) be a real lightlike submanifold of a golden semi-Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\). Then we say that \( M \) is a golden generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied:

(A) There exist two subbundles \( D_1 \) and \( D_2 \) of \( \text{Rad}(TM) \) such that
\[
\text{Rad}(TM) = D_1 \oplus D_2, \quad \tilde{P}(D_1) = D_1, \quad \tilde{P}(D_2) \subset S(TM). \tag{3.1}
\]
(B) There exist two subbundles \( D_0 \) and \( D' \) of \( S(TM) \) such that
\[
S(TM) = \{ \tilde{P}D_2 \oplus D' \} \perp D_0, \quad \tilde{P}(D_0) = D_0, \quad \tilde{P}(L_1 \perp L_2) = D', \tag{3.2}
\]
where \( D_0 \) is a non-degenerate distribution on \( M \), \( L_1 \) and \( L_2 \) are vector subbundles of \( \text{ltr}(TM) \) and \( S(TM^\perp) \), respectively.

Let \( \tilde{P}(L_1) = M_1 \) and \( \tilde{P}(L_2) = M_2 \). Then we have
\[
D' = \tilde{P}(L_1) \perp \tilde{P}(L_2) = M_1 \perp M_2. \tag{3.3}
\]
Thus we have the following decomposition:
\[
TM = D \oplus D', \quad D = \text{Rad}(TM) \perp D_0 \perp \tilde{P}(D_2). \tag{3.4}
\]
We say that \( M \) is a proper golden GCR-lightlike submanifold of a golden semi-Riemannian manifold if \( D_0 \neq \{0\} \), \( D_1 \neq \{0\} \), \( D_2 \neq \{0\} \) and \( L_2 \neq \{0\} \).

Let \( M \) be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold \((M, \tilde{g}, \tilde{P})\). Thus, for any \( X \in \Gamma(TM) \) we derive
\[
\tilde{P}X = PX + wX, \tag{3.5}
\]
where \( PX \) and \( wX \) are tangential and transversal parts of \( \tilde{P}X \).

For \( V \in \Gamma(tr(TM)) \) we write
\[
\tilde{P}V = BV + CV, \tag{3.6}
\]
where \( BV \) and \( CV \) are tangential and transversal parts of \( \tilde{P}V \).
Lemma 3.2 Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, one has

\begin{align*}
P^2X &= PX + X - BWX, \tag{3.7} \\
wPX &= wX - CWX, \tag{3.8} \\
PBV &= BV - BCV, \tag{3.9} \\
C^2V &= CV + V - wBV, \tag{3.10} \\
g(PX, Y) - g(X, PY) &= g(X, wY) - g(wX, Y), \tag{3.11} \\
\end{align*}

for any $X, Y \in \Gamma(TM)$ \cite{18}.

Theorem 3.3 Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then, $P$ is golden structure on $D_0$.

Proof. Definition of golden GCR-lightlike submanifold we have $wX = 0$, for any $X \in \Gamma(D_0)$. From (3.7) we have $P^2X = PX + X$. Thus $P$ is golden structure on $D_0$. \hfill \Box

Example 3.1 Let $(\tilde{M} = \mathbb{R}^{14}_1, \tilde{g})$ be a 14-dimensional semi-Euclidean space with signature $(+, +, +, - , - , - , - , + , + , + , + , + , +)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14})$ be the standard coordinate system of $\mathbb{R}^{14}_1$. If we define a mapping $\tilde{P}$ by

\[ \tilde{P}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}) = \left( x_1 + x_2, x_1, x_3 + x_4, x_3, x_5 + x_6, x_5 + x_7 + x_8, x_7, x_9 + x_{10}, x_9, x_{11} + x_{12}, x_{11} + x_{13} + x_{14}, x_{13} \right) \]

then $\tilde{P}^2 = \tilde{P} + I$ and $\tilde{P}$ is a golden structure on $\mathbb{R}^{14}_1$. Let $M$ be a submanifold of $\tilde{M}$ given by

\begin{align*}
x_1 &= u_2 + \sqrt{2}u_4, x_2 = u_1 + \sqrt{2}u_3 + \frac{1}{2\sqrt{2}}u_5, x_3 = u_2, x_4 = u_1, \\
x_5 &= \sqrt{2}u_2 + u_4, x_6 = \sqrt{2}u_1 + u_3 - \frac{1}{4}u_5, x_7 = u_4, x_8 = u_3 - \frac{1}{4}u_5, \\
x_9 &= u_7 - u_9, x_{10} = u_6 - u_8, x_{11} = u_7 + u_9, x_{12} = u_6 + u_8, \\
x_{13} &= u_{10}, x_{14} = 0, \\
\end{align*}
where $u_i, 1 \leq i \leq 10$, are real parameters. Thus $TM = \text{Span}\{U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9, U_{10}\}$, where

$$U_1 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + \sqrt{2} \frac{\partial}{\partial x_6}, U_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + \sqrt{2} \frac{\partial}{\partial x_5},$$

$$U_3 = \sqrt{2} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, U_4 = \sqrt{2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7},$$

$$U_5 = \frac{1}{4}(\sqrt{2} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}), U_6 = \frac{\partial}{\partial x_{10}}, U_7 = \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}}, U_8 = -\frac{\partial}{\partial x_{12}}, U_9 = -\frac{\partial}{\partial x_{11}}, U_{10} = \frac{\partial}{\partial x_{13}}.$$

Then $M$ is a 3–lightlike submanifold with $\text{Rad}(TM) = \text{Span}\{U_1, U_2, U_3\}$ and $PU_1 = U_2$. Thus, $D_1 = \text{Span}\{U_1, U_2\}$. On the other hand, $PU_3 = U_4 \in \Gamma(S(TM))$ implies that $D_2 = \text{Span}\{U_3\}$. Moreover, $PU_6 = U_7$ and $PU_8 = U_9$ thus $D_0 = \text{Span}\{U_6, U_7, U_8, U_9\}$. We can easily obtain

$$\text{ltr}(TM) = \text{Span}\{N_1 = \frac{1}{4}(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} - \sqrt{2} \frac{\partial}{\partial x_6}),$$

$$N_2 = \frac{1}{4}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} - \sqrt{2} \frac{\partial}{\partial x_5}),$$

$$N_3 = \frac{1}{4}(\sqrt{2} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8})\}$$

and

$$S(TM^\perp) = \text{Span}\{W = \frac{\partial}{\partial x_{14}}\}.$$

Moreover, $\text{Span}\{N_1, N_2\}$ is invariant with respect to $\tilde{P}$. Since $\tilde{P}N_3 = U_5$ and $\tilde{P}W = U_{10}$, then $L_1 = \text{Span}\{N_3\}, L_2 = \text{Span}\{W\}, M_1 = \text{Span}\{U_3\}$ and $M_2 = \text{Span}\{U_{10}\}$. Thus $M$ is a proper golden GCR-lightlike submanifold of $M$.

**Theorem 3.4** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution $D_0$ is integrable iff

(i) $\tilde{g}(h^*(X,Y), N) = \tilde{g}(h^*(Y,X), N),$

(ii) $\tilde{g}(h^*(X, \tilde{P}Y), N') = \tilde{g}(h^*(Y, \tilde{P}X), N'),$

(iii) $\tilde{g}(h^*(X, \tilde{P}Y), W) = \tilde{g}(h^*(Y, \tilde{P}X), W),$

(iv) $g(\nabla_X Y, \tilde{P}\xi) = g(\nabla_Y X, \tilde{P}\xi),$$

for any $X,Y \in \Gamma(D_0), \xi \in \Gamma(D_2), N \in \Gamma(\text{ltr}(TM)), N' \in \Gamma(L_1)$ and $W \in \Gamma(L_2).$
Proof. Using the definition of golden GCR-lightlike submanifolds, $D_0$ is integrable iff

$$\tilde{g}([X,Y], N) = \tilde{g}([X,Y], \tilde{P}N') = \tilde{g}([X,Y], \tilde{P}W) = \tilde{g}([X,Y], \tilde{P}\xi) = 0$$

for any $X,Y \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $N' \in \Gamma(L_1)$ and $W \in \Gamma(L_2)$. Then, from (2.2), (2.11) and (2.16) we derive

$$\tilde{g}([X,Y], N) = \tilde{g}(\tilde{\nabla}_XY - \tilde{\nabla}_YX, N) = \tilde{g}(h^*(X,Y) - h^*(Y,X), N), \quad (3.13)$$

$$g([X,Y], \tilde{P}N') = g(\tilde{\nabla}_XY - \tilde{\nabla}_YX, \tilde{P}N') = \tilde{g}(\tilde{\nabla}_X\tilde{P}Y - \tilde{\nabla}_Y\tilde{P}X, N')$$

$$= \tilde{g}(h^*(X,\tilde{P}Y) - h^*(Y,\tilde{P}X), N'), \quad (3.14)$$

$$g([X,Y], \tilde{P}W) = g(\tilde{\nabla}_XY - \tilde{\nabla}_YX, \tilde{P}W) = \tilde{g}(\tilde{\nabla}_X\tilde{P}Y - \tilde{\nabla}_Y\tilde{P}X, W)$$

$$= \tilde{g}(h^*(X,\tilde{P}Y) - h^*(Y,\tilde{P}X), W), \quad (3.15)$$

$$g([X,Y], \tilde{P}\xi) = g(\tilde{\nabla}_XY - \tilde{\nabla}_YX, \tilde{P}\xi) = g(\nabla^*_XY, \tilde{P}\xi) - g(\nabla^*_YX, \tilde{P}\xi). \quad (3.16)$$

Thus the proof is completed.

□

From Theorem 3.4 we obtain the following corollary.

Corollary 3.5 Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(M, \tilde{g}, \tilde{P})$. Then the distribution $D_0$ is integrable iff

(i) $g(A_NX, Y) = g(A_NY, X)$,
(ii) $g(A_NX, \tilde{P}Y) = g(A_NY, \tilde{P}X)$,
(iii) $g(A_WX, \tilde{P}Y) = g(A_WY, \tilde{P}X)$,
(iv) $\tilde{g}(h^l(X,\tilde{P}Y), \xi) = \tilde{g}(h^l(Y,\tilde{P}X), \xi)$,

for any $X,Y \in \Gamma(D_0)$, $\xi \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $N' \in \Gamma(L_1)$ and $W \in \Gamma(L_2)$.

Theorem 3.6 Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(M, \tilde{g}, \tilde{P})$. Then $\text{Rad}(TM)$ is integrable iff

(i) $\tilde{g}(h^l(\xi, Z), \xi') = \tilde{g}(h^l(\xi', Z), \xi)$,
(ii) $\tilde{g}(h^l(\xi, \tilde{P}\xi''), \xi') = \tilde{g}(h^l(\xi', \tilde{P}\xi''), \xi)$,
(iii) $\tilde{g}(h^l(\xi, \tilde{P}N), \xi) = \tilde{g}(h^l(\xi', \tilde{P}N), \xi)$,
(iv) $\tilde{g}(h^h(\xi, \tilde{P}\xi'), W) = \tilde{g}(h^h(\xi', \tilde{P}\xi), W),$

for any $Z \in \Gamma(D_0)$, $\xi'' \in \Gamma(D_2)$, $\xi, \xi', \xi'' \in \Gamma(\text{Rad}(TM))$, $N \in \Gamma(L_1)$, $W \in \Gamma(L_2)$.
\textbf{Proof.} Using the definition of golden GCR-lightlike submanifolds, \( \text{Rad}(TM) \) is integrable iff

\[ \tilde{g}([\xi, \xi'], Z) = \tilde{g}([\xi, \xi'], \tilde{P} \xi''') = \tilde{g}([\xi, \xi'], \tilde{P} t) = \tilde{g}([\xi, \xi'], \tilde{P} W) = 0, \]

for any \( Z \in \Gamma(D_0), \xi'' \in \Gamma(D_2), \xi, \xi', \xi'' \in \Gamma(\text{Rad}(TM)), N \in \Gamma(L_1), W \in \Gamma(L_2) \). Then, taking into account that \( \nabla \) is a metric connection and using (2.2) and (2.11) we get

\begin{align*}
\tilde{g}(\xi, \xi') Z &= \tilde{g}(\nabla_\xi \xi', Z) - \tilde{g}(\nabla_\xi \xi, Z) \\
&= -\tilde{g}(\xi', \nabla_\xi Z) + \tilde{g}(\xi, \nabla_\xi Z) \\
&= -\tilde{g}(h^c(\xi, Z), \xi') + \tilde{g}(h^c(\xi', Z), \xi) \\
&= \tilde{g}(\nabla_\xi \xi', \xi') - \tilde{g}(\nabla_\xi \xi, \xi') + \tilde{g}(h^c(\xi', \xi'), \xi), \tag{3.17}
\end{align*}

\begin{align*}
\tilde{g}(\xi, \xi') \tilde{P} &\xi''' = \tilde{g}(\nabla_\xi \xi', \tilde{P} \xi''') - \tilde{g}(\nabla_\xi \xi, \tilde{P} \xi''') \\
&= -\tilde{g}(\xi', \nabla_\xi \tilde{P} t) + \tilde{g}(\xi, \nabla_\xi \tilde{P} \xi) \\
&= -\tilde{g}(h^c(\xi, \tilde{P} \xi'''), \xi') + \tilde{g}(h^c(\xi', \tilde{P} \xi'''), \xi), \tag{3.18}
\end{align*}

\begin{align*}
\tilde{g}(\xi, \xi') \tilde{P} \xi &= \tilde{g}(\nabla_\xi \xi', \tilde{P} \xi) - \tilde{g}(\nabla_\xi \xi, \tilde{P} \xi) \\
&= -\tilde{g}(\xi', \nabla_\xi \tilde{P} N) + \tilde{g}(\xi, \nabla_\xi \tilde{P} \xi) \\
&= -\tilde{g}(h^c(\xi, \tilde{P} \xi), \xi') + \tilde{g}(h^c(\xi', \tilde{P} \xi), \xi), \tag{3.19}
\end{align*}

\begin{align*}
\tilde{g}(\xi, \xi') \tilde{P} W &= \tilde{g}(\nabla_\xi \xi', \tilde{P} W) - \tilde{g}(\nabla_\xi \xi, \tilde{P} W) \\
&= \tilde{g}(\nabla_\xi \tilde{P} \xi', W) - \tilde{g}(\nabla_\xi \tilde{P} \xi, W) \\
&= \tilde{g}(h^c(\xi, \tilde{P} \xi'), W) - \tilde{g}(h^c(\xi', \tilde{P} \xi), W). \tag{3.20}
\end{align*}

This completes proof. \( \square \)

From Theorem 3.6 we get the following corollary.

\textbf{Corollary 3.7} Let \( M \) be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold \( (\tilde{M}, \tilde{g}, \tilde{P}) \). Then \( \text{Rad}(TM) \) is integrable iff

(i) \( g(A_{\xi}^c \xi, Z) = g(A_{\xi}^c \xi', Z) \),

(ii) \( g(A_{\xi}^c \xi, \tilde{P} \xi''') = g(A_{\xi}^c \xi', \tilde{P} \xi''') \),

(iii) \( g(A_{\xi}^c \xi, \tilde{P} \xi) = g(A_{\xi}^c \xi', \tilde{P} \xi) \),

(iv) \( h^c(\xi, \tilde{P} W), \xi') = \tilde{g}(h^c(\xi', \tilde{P} W), \xi) \),

for any \( Z \in \Gamma(D_0), \xi'' \in \Gamma(D_2), \xi, \xi', \xi'' \in \Gamma(\text{Rad}(TM)), N \in \Gamma(L_1), W \in \Gamma(L_2) \).
\textbf{Theorem 3.8} Let \( M \) be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\). Then, each leaf of radical distribution is totally geodesic on \( M \) iff 

(i) \( A^*_\xi \xi \in \Gamma(P(D_2) \perp M_2) \), 
(ii) \( \tilde{g}(h^l(\xi, \tilde{P}N), \xi^\prime) = 0 \), 
(iii) \( \tilde{g}(h^s(\xi, \tilde{P}\xi^\prime), W) = 0 \), 

for any \( Z \in \Gamma(D_0), \xi, \xi^\prime, \in \Gamma(\text{Rad}(TM)), N \in \Gamma(L_1), W \in \Gamma(L_2) \).

\textbf{Proof.} Using the definition of golden GCR-lightlike submanifolds, each leaf of radical distribution is totally geodesic iff 

\[
g(\nabla \xi \xi^\prime, Z) = g(\nabla \xi \xi^\prime, \tilde{P} \tilde{P}^\prime) = g(\nabla \xi \xi^\prime, \tilde{P} N) = g(\nabla \xi \xi^\prime, \tilde{P} W) = 0,
\]

for any \( Z \in \Gamma(D_0), \xi^\prime \in \Gamma(D_2), \xi, \xi^\prime, \in \Gamma(\text{Rad}(TM)), N \in \Gamma(L_1), W \in \Gamma(L_2) \). Since \( \tilde{\nabla} \) is a metric connection and using (2.9), (2.11) and (2.17), we have

\[
g(\nabla \xi \xi^\prime, Z) = -g(A^*_\xi \xi, Z), \quad (3.21)
g(\nabla \xi \xi^\prime, \tilde{P} \tilde{P}^\prime) = -g(A^*_\xi \xi, \tilde{P} \tilde{P}^\prime), \quad (3.22)
g(\nabla \xi \xi^\prime, \tilde{P} N) = g(\tilde{\nabla} \xi \xi^\prime, \tilde{P} N) = -g(\xi^\prime, \tilde{\nabla} \xi \xi^\prime) = -\tilde{g}(h^l(\xi, \tilde{P} \tilde{P} N), \xi^\prime), \quad (3.23)
g(\nabla \xi \xi^\prime, \tilde{P} W) = g(\tilde{\nabla} \xi \xi^\prime, \tilde{P} W) = g(h^s(\xi, \tilde{P} \xi^\prime), W). \quad (3.24)
\]

Hence, from (3.21)-(3.24) we complete the proof. \qed

\textbf{Theorem 3.9} Let \( M \) be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold \((\tilde{M}, \tilde{g}, \tilde{P})\). Then the distribution \( D_1 \) is integrable if and only if 

(i) \( \nabla^d_{P^1} \tilde{P} Y - \nabla^d_{P^1} \tilde{P} X \in \Gamma(D_1) \), 
(ii) \( A^*_X Y = A^*_Y X \), 
(iii) \( Bh(X, \tilde{P} Y) = Bh(Y, \tilde{P} X) \),

for any \( X, Y \in \Gamma(D_1) \).

\textbf{Proof.} Since \( \tilde{P} \) is the golden structure of \( \tilde{M} \), we have

\[
\tilde{\nabla}_XY = \tilde{P}\tilde{\nabla}_X \tilde{P}Y - \tilde{\nabla}_X \tilde{P}Y \quad (3.25)
\]

for any \( X, Y \in \Gamma(\text{Rad}(TM)) \). Using (2.9), (2.10) and (2.17) we obtain

\[
\nabla_X Y + h(X, Y) = \tilde{P}(-A^*_P Y X + \nabla^d_X \tilde{P} Y + h(X, \tilde{P} Y))
\]

\[
-(-A^*_P Y X + \nabla^d_X \tilde{P} Y + h(X, \tilde{P} Y)), \quad (3.26)
\]

for any \( X, Y \in \Gamma(D_1) \). Taking the tangential components of above equation both sides, we get

\[
\nabla_X Y = -PA^*_P Y X + P\nabla^d_X \tilde{P} Y - Bh(X, \tilde{P} Y) + A^*_Y X - \nabla^d_X \tilde{P} Y \quad (3.27)
\]

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for any $X,Y \in \Gamma(D_1)$. Replacing $X$ by $Y$ and subtracting resulting equation from this equation, we derive

$$[X,Y] = P(A^*_P X - A^*_P Y) + P(\nabla^\nabla_X^\nabla Y - \nabla^\nabla_Y^\nabla X)
-Bh(X, \tilde{P}Y) + Bh(Y, \tilde{P}X) + A^*_P Y - A^*_P X \quad (3.28)$$

$$-\nabla^\nabla_X^\nabla Y + \nabla^\nabla_Y^\nabla X$$

thus $[X,Y] \in \Gamma(D_1)$ if and only if $\nabla^\nabla_X^\nabla Y - \nabla^\nabla_Y^\nabla X \in \Gamma(D_1)$, $Bh(X, \tilde{P}Y) = Bh(Y, \tilde{P}X)$, $A^*_P Y = A^*_P X$, this completes the proof.

\[\square\]

From Theorem 3.9 we obtain the following corollary.

**Corollary 3.10** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then $D_1$ defines a totally geodesic foliation in $M$ if and only if

(i) $\nabla^\nabla_X^\nabla Y \in \Gamma(D_1)$,
(ii) $A^*_P Y = 0$,
(iii) $Bh(X, \tilde{P}Y) = 0$,
for any $X,Y \in \Gamma(D_1)$.

**Theorem 3.11** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then the distribution $D_2$ is integrable iff

(i) $P(\nabla^\nabla_X^\nabla Y - \nabla^\nabla_Y^\nabla X) = \nabla^\nabla_X^\nabla Y + \nabla^\nabla_Y^\nabla X \in \Gamma(D_2)$,
(ii) $Bh(X, \tilde{P}Y) = Bh(Y, \tilde{P}X)$,
(iii) $h^*(X, \tilde{P}Y) = h^*(Y, \tilde{P}X)$,
for any $X,Y \in \Gamma(D_2)$.

**Proof.** Since $\tilde{P}$ is the golden structure of $\tilde{M}$, we have

$$\tilde{\nabla}_X Y = \tilde{P}\tilde{\nabla}_X \tilde{P}Y - \tilde{\nabla}_X \tilde{P}Y \quad (3.29)$$

for any $X,Y \in \Gamma\left(\text{Rad}(TM)\right)$. Using (2.9) and (2.16) we get

$$\nabla_X Y + h(X, Y) = \tilde{P}(\nabla^\nabla_X^\nabla Y + h^*(X, \tilde{P}Y) + h(X, \tilde{P}Y))
-(\nabla^\nabla_X^\nabla Y + h^*(X, \tilde{P}Y) + h(X, \tilde{P}Y)),$$

for any $X,Y \in \Gamma(D_2)$. Taking the tangential components of above equation both sides, we obtain

$$\nabla_X Y = P\nabla^\nabla_X^\nabla Y + Ph^*(X, \tilde{P}Y) + Bh(X, \tilde{P}Y) - \nabla^\nabla_X^\nabla Y - h^*(X, \tilde{P}Y) \quad (3.31)$$
for any $X,Y \in \Gamma(D_2)$. Replacing $X$ by $Y$ and subtracting resulting equation from this equation, we derive

$$[X,Y] = P(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X) + P(h^*(X, \tilde{P}Y) - h^*(Y, \tilde{P}X)) - Bh(X, \tilde{P}Y) + Bh(Y, \tilde{P}X) - \nabla_X \tilde{P}Y + \nabla_Y \tilde{P}X$$

(3.32)

thus $[X,Y] \in \Gamma(D_2)$ if and only if $P(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X) - \nabla_X \tilde{P}Y + \nabla_Y \tilde{P}X \in \Gamma(D_2)$, $Bh(X, \tilde{P}Y) = Bh(Y, \tilde{P}X)$, $h^*(X, \tilde{P}Y) = h^*(Y, \tilde{P}X)$, this completes the proof.

From Theorem 3.11 we get the following corollary.

**Corollary 3.12** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then $D_2$ defines a totally geodesic foliation in $M$ if and only if

(i) $P(\nabla_X \tilde{P}Y - \nabla_Y \tilde{P}X) \in \Gamma(D_2)$,

(ii) $Bh(X, \tilde{P}Y) = 0$,

(iii) $h^*(X, \tilde{P}Y) = 0$,

for any $X,Y \in \Gamma(D_2)$.

**Theorem 3.13** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then $P(D_2)$ is integrable iff

(i) $g(A^*_\xi \tilde{P}e', \tilde{P}Z) = g(A^*_\xi \tilde{P}e, \tilde{P}Z)$,

(ii) $\tilde{g}(h^*(\xi, \tilde{P}e'), \xi'') = \tilde{g}(h^*(\xi', \tilde{P}e), \xi'')$,

(iii) $\tilde{g}(h^*(\xi, \tilde{P}e'), W) = \tilde{g}(h^*(\xi', \tilde{P}e), W)$,

(iv) $g(A_N \tilde{P}e, \tilde{P}e') = g(A_N \tilde{P}e', \tilde{P}e)$,

for any $Z \in \Gamma(D_0)$, $\xi, \xi', \xi'' \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_2)$.

**Proof.** Using the definition of golden GCR-lightlike submanifolds, $\tilde{P}(D_2)$ is integrable iff

$$\tilde{g}([\tilde{P}e, \tilde{P}e'], Z) = \tilde{g}([\tilde{P}e, \tilde{P}e', \tilde{P}e''], \tilde{P}W) = \tilde{g}([\tilde{P}e, \tilde{P}e'], N) = 0,$$

(3.33)

for any $Z \in \Gamma(D_0)$, $\xi, \xi', \xi'' \in \Gamma(D_2)$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(L_2)$. Then, from (2.2), (2.3), (2.11), (2.12) and (2.17) we obtain

$$g([\tilde{P}e, \tilde{P}e'], Z) = g(\tilde{\nabla}_{\tilde{P}e} \tilde{P}e' - \tilde{\nabla}_{\tilde{P}e'} \tilde{P}e, \tilde{P}Z) = g(\tilde{\nabla}_{\tilde{P}e} \tilde{P}e' - \tilde{\nabla}_{\tilde{P}e'} \tilde{P}e, \tilde{P}Z) = g(A^*_\xi \tilde{P}e - A^*_\xi \tilde{P}e, \tilde{P}Z),$$

(3.34)
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\[ g([\tilde{P}\xi, \tilde{P}\xi'], \tilde{P}\xi'') = g(\nabla_{\tilde{P}\xi'} \tilde{P}\xi, \tilde{P}\xi'') - g(\nabla_{\tilde{P}\xi'} \tilde{P}\xi, \tilde{P}\xi'') = g(\nabla_{\tilde{P}\xi'} \tilde{P}\xi, \tilde{P}\xi'') + g(\tilde{P}\xi', \tilde{P}\xi'') - g(\tilde{P}\xi', \tilde{P}\xi'') = \tilde{g}(h^1(\tilde{P}\xi', \tilde{P}\xi), \xi'') - \tilde{g}(h^1(\tilde{P}\xi, \tilde{P}\xi), \xi'') \]  

Corollary 3.14 Let \( M \) be a golden GCR-lightlike submanifold of a golden \( M \)-\( \tilde{g} \)-\( \tilde{P} \). The distribution \( \tilde{P}(D_2) \) is integrable if and only if for any \( X, Y \in \Gamma(D), \xi \in \Gamma(Rad(TM)) \), \( W \in \Gamma(S(TM^\perp)) \), \( A_\xi X \) has no component in \( \Gamma(D_0 \perp M_1) \) and \( g(A_\xi X, \tilde{P}Y) = \tilde{g}(\tilde{P}Y, D^\xi(X, W)) \).

Definition 3.15 A golden GCR-lightlike submanifold \( M \) of a golden semi-Riemannian manifold \( (M, \tilde{g}, \tilde{P}) \) is said to be \( D \)-geodesic (resp. \( D' \)-geodesic) golden GCR-lightlike submanifold if its the second fundamental form \( h \) satisfies \( h(X, Y) = 0 \) (resp. \( h(Z, W) = 0 \)) for any \( X, Y \in \Gamma(D), (Z, W \in \Gamma(D')) \).

Theorem 3.16 Let \( M \) be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold \( (M, \tilde{g}, \tilde{P}) \). Then the distribution \( D \)-geodesic foliation if and only if for any \( X, Y \in \Gamma(D), \xi \in \Gamma(Rad(TM)), W \in \Gamma(S(TM^\perp)) \), \( A_\xi X \) has no component in \( \Gamma(D_0 \perp M_1) \) and \( g(A_\xi X, \tilde{P}Y) = \tilde{g}(\tilde{P}Y, D^\xi(X, W)) \).
Proof. For any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$, taking into account that $\tilde{\nabla}$ is a metric connection and using (2.9), (2.13) and (2.17) we have
\[
\tilde{g}(h(X, Y), \xi) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, \xi) = -\tilde{g}(\tilde{\nabla}_X \xi, \tilde{P}Y) = g(A_\xi^* X, \tilde{P}Y), \quad (3.42)
\]
(3.38)
\[
\tilde{g}(h(X, \tilde{P}Y), W) = \tilde{g}(\tilde{\nabla}_X \tilde{P}Y, W) - \tilde{g}(\tilde{\nabla}_X W, \tilde{P}Y) = g(A_W X, \tilde{P}Y) - \tilde{g}(\tilde{P}Y, D^i(X, W)). \quad (3.39)
\]
Thus the result follows from (3.38) and (3.39).

\[\square\]

**Theorem 3.17** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then $M$ is $D'$-totally geodesic submanifold iff for any $X \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$ $A_\xi^* X$ and $A_W X$ has no component in $\Gamma(\tilde{P}(D_2) \perp M_2)$.

Proof. Taking into account that $\tilde{\nabla}$ is a metric connection and using (2.9), (2.13) and (2.17) we derive
\[
\tilde{g}(h(X, Y), \xi) = \tilde{g}(A_\xi^* X, Y), \quad (3.40)
\]
(3.41)
\[
\tilde{g}(h(X, Y), W) = \tilde{g}(A_W X, Y),
\]
for any $X, Y \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$. Thus from the equations (3.40) and (3.41) we conclude $h(X, Y) = 0$ iff $A_\xi^* X$ and $A_W X$ has no component in $\Gamma(\tilde{P}(D_2) \perp M_2)$.

\[\square\]

**Definition 3.18** A golden GCR-lightlike submanifold of a golden semi-Riemannian manifold is called mixed-geodesic golden GCR-lightlike submanifold if its second fundamental form $h$ satisfies $h(X, Y) = 0$, for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

**Theorem 3.19** Let $M$ be a golden GCR-lightlike submanifold of a golden semi-Riemannian manifold $(\tilde{M}, \tilde{g}, \tilde{P})$. Then $M$ is mixed geodesic iff for any $X \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$ $A_\xi^* X$ and $A_W X$ has no component in $\Gamma(\tilde{P}(D_2) \perp M_2)$.

Proof. $M$ is mixed geodesic iff
\[
g(h(X, Y), \xi) = 0 \text{ and } g(h(X, Y), W) = 0,
\]
for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM))$, $W \in \Gamma(S(TM^\perp))$. Taking into account that $\tilde{\nabla}$ is a metric connection and using (2.9), (2.13) and (2.17) we have
\[
\tilde{g}(h(X, Y), \xi) = \tilde{g}(A_\xi^* X, Y), \quad (3.42)
\]
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\[ \tilde{g}(h(X,Y), W) = \tilde{g}A_W X, Y). \quad (3.43) \]

Thus \( M \) is mixed geodesic submanifold iff \( A_\xi X \) and \( A_W X \) has no component in \( \Gamma(\tilde{P}(D_2) \perp M_2) \).

\[ \Box \]

References

Received: 21.IV.2021 / Accepted: 05.X.2021

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