On Lie ideals with left derivations in 3-prime near-rings

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Abstract In this paper we improve on our recent results found by using derivations. The new results use the left derivations to relate commutativity of the addition to the commutativity of the whole ring, and only hold for 3-prime-rings. The hypotheses are justified by some examples.

Keywords 3-prime near-rings · Lie ideals · left derivations

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1 Introduction

A right near-ring $\mathcal{N}$ is a triple $(\mathcal{N}, +, \cdot)$ with two binary operations “+” and “.” such that (i) $(\mathcal{N}, +)$ is a group (not necessarily abelian), (ii) $(\mathcal{N}, \cdot)$ is a semigroup, (iii) $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in \mathcal{N}$. We denote by $Z(\mathcal{N})$ the multiplicative center of $\mathcal{N}$, and usually $\mathcal{N}$ will be 3-prime, that is, for $a, b \in \mathcal{N}$, $a \mathcal{N} b = \{0\}$ implies $a = 0$ or $b = 0$. $\mathcal{N}$ is a zero symmetric if $a_0 = 0$ for all $a \in \mathcal{N}$ (recall that right distributive yields $0 \cdot a = 0$). For any pair of elements $a, b \in \mathcal{N}$, $[a, b] = ab - ba$ and $a \circ b = ab + ba$ stand for Lie product and Jordan product respectively. Recall that $\mathcal{N}$ is called 2-torsion free if $2a = 0$ implies $a = 0$ for all $a \in \mathcal{N}$. The Lie ideal $U$ of $\mathcal{N}$ is an additive subgroup that has the property $[u, n] \in U$ for all $u \in U, n \in \mathcal{N}$. An additive mapping $H : \mathcal{N} \to \mathcal{N}$ is a multiplier, if $H(ab) = aH(b) = H(a)b$ for all $a, b \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a left derivation (resp. Jordan left derivation) if $d(ab) = ad(b) + bd(a)$ (resp. $d(a^2) = 2ad(a)$) holds for all $a, b \in \mathcal{N}$. Obviously, every left derivation is a Jordan left derivation, but the converse need not be true in general (see [7], Example 1.1.). Recently, M. Ashraf et al. [1] proved that the converse statement is true in the case when the underlying ring is prime and 2-torsion free. In [4], Bergen showed that if $U$ is a nonzero Lie ideal of a 2-torsion free prime ring $\mathcal{R}$ and $d$ a
nonzero derivation of \( R \) such that \( d(U) \subseteq Z(R) \), then \( U \subseteq Z(R) \). In [5], [6] the authors used Lie ideals and derivations to get a number of important discoveries, including the commutativity of addition in near-rings. However, we could not prove anything about the second law “multiplication,” which made us think about changing derivations by left derivations. Our aim in this article is to prove the commutativity of the two laws of the near-ring. A careful examination of the action of left derivations on Lie ideals led us to what we were aiming. This shows the importance of choosing this notion to facilitate the search for the commutativity of near-rings.

2 Some preliminaries

The proof of our main results is based essentially on the following lemmas.

**Lemma 2.1** Let \( \mathcal{N} \) be a 3-prime near-ring.

(i) [3, Lemma 1.2 (iii)] If \( z \in Z(\mathcal{N}) \setminus \{0\} \) and \( xz \in Z(\mathcal{N}) \) or \( zx \in Z(\mathcal{N}) \), then \( x \in Z(\mathcal{N}) \).

(ii) [3, Lemma 2.2] If \( \mathcal{N} \subseteq Z(\mathcal{N}) \), then \( \mathcal{N} \) is a commutative ring.

(iii) [2, Theorem 2.9 (i)] If \( [x,y] \in Z(\mathcal{N}) \) for all \( x, y \in \mathcal{N} \), then \( \mathcal{N} \) is a commutative ring.

(iv) [5, Lemma 3] If \( U \subseteq Z(\mathcal{N}) \), then \( (\mathcal{N},+) \) is abelian.

**Lemma 2.2** A right near-ring \( \mathcal{N} \) admits a left derivation if and only if it is zero-symmetric.

*Proof.* Suppose that \( \mathcal{N} \) is a zero-symmetric right near-ring. Then the zero map is a left derivation on \( \mathcal{N} \). Conversely, assume that \( \mathcal{N} \) has a left derivation \( d \). Since \( 0.x = 0 \) for all \( x \in \mathcal{N} \), we get \( 0 = d(0.x) = 0.d(x) + x.d(0) = x.0 \) for all \( x \in \mathcal{N} \). So, \( \mathcal{N} \) must be zero-symmetric.

**Lemma 2.3** Let \( \mathcal{N} \) be a 2-torsion free 3-prime near-ring, and \( U \) a nonzero Lie ideal of \( \mathcal{N} \). If \( \mathcal{N} \) has one of the following properties:

(i) \( U^2 \subseteq Z(\mathcal{N}) \);

(ii) \( u \circ n \in Z(\mathcal{N}) \) for all \( u \in U, n \in \mathcal{N} \);

(iii) \( [u,n] \in Z(\mathcal{N}) \) for all \( u \in U, n \in \mathcal{N} \),

then \( (\mathcal{N},+) \) is abelian.

*Proof.* (i) Suppose that \( U^2 \subseteq Z(\mathcal{N}) \), then \( ([u,n]u)v \in Z(\mathcal{N}) \) for all \( u, v \in U, n \in \mathcal{N} \). Using Lemma (2.1)(i), we deduce that

\[
\exists v \in Z(\mathcal{N}) \text{ or } [u,n]u = 0 \text{ for all } u, n \in \mathcal{N}. \tag{2.1}
\]

If \( [u,n]u = 0 \) for all \( u \in U, n \in \mathcal{N} \), then

\[
unu = nu^2 \text{ for all } u \in U, n \in \mathcal{N}. \tag{2.2}
\]
Replacing \( n \) by \( nm \) in (2.2) and invoking it, we find \([u, n]N = \{0\}\), and by 3-primeness of \( N \), we conclude that \( u \in Z(N) \). Then (2.1) becomes \( U \subseteq Z(N) \), which implies that \((N, +)\) is abelian by Lemma 2.1(iv).

(ii) Suppose that

\[
\forall u \in U, n \in N : u \circ n \in Z(N).
\]

(2.3)

Taking \( nu \) instead of \( n \) in (2.3) and using it, we can see that \((u \circ n)u \in Z(N)\) for all \( u \in U, n \in N \).
From Lemma 2.1(i), we deduce that

\[
\exists u \in Z(N) \quad \text{or} \quad u \circ n = 0 \quad \forall u \in U, n \in N.
\]

(2.4)

If there exists an element \( u_0 \in U \) such that

\[
u_0 \circ n = 0 \quad \forall n \in N.
\]

(2.5)

Replacing \( n \) by \( u_0 \) in (2.5) and using 2-torsion freeness of \( N \), we find that \( u_0^2 = 0 \), substituting \( n \) by \( nu_0 \) in (2.5) with the fact that \( u_0^2 = 0 \), we deduce that \( u_0N u_0 = \{0\} \). By 3-primeness of \( N \), we obtain that \( u_0 = 0 \), then (2.4) becomes \( U \subseteq Z(N) \), which forces \((N, +)\) is abelian by Lemma 2.1(iv).

(iii) Suppose that

\[
[u, n] \in Z(N) \quad \forall u \in U, n \in N.
\]

(2.6)

Replacing \( n \) by \( nu \) in (2.6) and using it, we obtain \([u, n]u \in Z(N)\).
From lemma 2.1(i), we deduce that

\[
u \in Z(N) \quad \text{or} \quad [u, n] = 0 \quad \forall u \in U, n \in N.
\]

(2.7)

Then (2.7) becomes \( U \subseteq Z(N) \), which forces that \((N, +)\) is abelian by Lemma 2.1(iv).

\(\square\)

3 Some results in 3-prime near-rings involving left derivations

In [5] and [6] the authors prove several algebraic identities on Lie ideals of 3-prime near-rings \( N \) involving derivations and generalized derivations, but they only got to \((N, +)\) is abelian. However, we remark that by using the concept of left derivations rather than derivations, we can obtain good and significant results in relation to the second law. The next result is crucial to our work in this article, since it helps us summarize a large number of complex steps, which is a valuable result of this paper.

**Theorem 3.1** Let \( N \) be a 3-prime near-ring. If \( N \) admits a nonzero left derivation \( d \), then the following properties hold true:

(i) If there exists a nonzero element \( a \) such that \( d(a) = 0 \), then \( a \in Z(N) \),

(ii) \((N, +)\) is abelian, if and only if \( N \) is a commutative ring.
Proof. (i) Using the definition of $d$, we have
\[ d(xay) = xad(y) + yd(xa) = xad(y) + y(xd(a) + ad(x)) = xad(y) + yad(x) \text{ for all } x, y \in \mathcal{N}, \]
and
\[ d(xay) = xd(ay) + ayd(x) = x(ad(y) + yd(a)) + ayd(x) = xad(y) + ayd(x) \text{ for all } x, y \in \mathcal{N}. \]
Combining the two last relations, we obviously obtain
\[ yad(x) = ayd(x) \text{ for all } x, y \in \mathcal{N}. \quad (3.1) \]
Taking $zt$ in place of $y$ in (3.1) and using it, we may write
\[ ztad(x) = aztd(x) \text{ for all } x, z, t \in \mathcal{N}. \]
Which implies that
\[ zatd(x) = aztd(x) \text{ for all } x, z, t \in \mathcal{N}, \]
and therefore,
\[ [z, a]td(x) = 0 \text{ for all } x, z, t \in \mathcal{N}, \]
so $[z, a]Nd(x) = \{0\}$ for all $x, z \in \mathcal{N}$. Since $d \neq 0$, by 3-primeness of $\mathcal{N}$, we conclude that $a \in Z(\mathcal{N})$.
(ii) Let $x, y \in \mathcal{N}$. Since $(\mathcal{N}, +)$ is abelian, then
\[ d(xy) = xd(y) + yd(x) = yd(x) + xd(y) = d(yx) \text{ for all } x, y \in \mathcal{N}. \quad (3.2) \]
Then (3.2) leads to,
\[ d([x, y]) = 0 \text{ for all } x, y \in \mathcal{N}. \quad (3.3) \]
Using Theorem 3.1(i), then (3.3) forces that $[x, y] \in Z(\mathcal{N})$, which assures that $\mathcal{N}$ is a commutative ring by Lemma (2.1) (iii).

**Theorem 3.2** Let $\mathcal{N}$ be a 2-torsion 3-prime near-ring, and $U$ a nonzero Lie ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left derivation $d$ satisfying any one of the following properties:

(i) $d(U^2) = \{0\}$;
(ii) $d(U) = \{0\}$;
(iii) $d(u \circ n) = 0$ for all $u \in U, n \in \mathcal{N}$,
then $\mathcal{N}$ is a commutative ring.

Proof. (i) Suppose that $d(U^2) = \{0\}$. From Theorem 3.1(i), we get $U^2 \subseteq Z(\mathcal{N})$, and using Lemma 2.3(i), we deduce that $(\mathcal{N}, +)$ is abelian, which implies that $\mathcal{N}$ is a commutative ring by Theorem 3.1(ii).

(ii) Assume that $d(U) = \{0\}$. By Theorem 3.1(i), we obtain $U \subseteq Z(\mathcal{N})$, and using Lemma 2.1 (iv), we deduce that $(\mathcal{N}, +)$ is abelian, which forces that $\mathcal{N}$ is a commutative ring by Theorem 3.1(ii).

(iii) Suppose that $d(u \circ n) = 0$ for all $u \in U, n \in \mathcal{N}$. Using Theorem 3.1(i), we obtain $u \circ n \in Z(\mathcal{N})$. By applying Lemma 2.3 (ii), we conclude that $(\mathcal{N}, +)$ is abelian, which forces that $\mathcal{N}$ is a commutative ring by Theorem 3.1(ii).

$\square$

4 Some polynomial identities in right near-rings involving left derivations

In this section we will study the commutativity of near-rings which admitting left derivations $d_1, d_2$ of $\mathcal{N}$ satisfied conditions $d_1(xu) = d_2(ux)$, $d_1([x, u]) = d_2(x)u$ and $d_1([x, u]) = H([x, u])$ for all $u \in U, x \in \mathcal{N}$, where $H$ is a multiplier of $\mathcal{N}$.

**Theorem 4.1** Let $U$ be a nonzero Lie ideal of a 3-prime near-ring $\mathcal{N}$. If $d_1, d_2$ left derivations on $\mathcal{N}$ such that $d_1(xu) = d_2(ux)$ for all $u \in U, x \in \mathcal{N}$, then $d_1 = d_2 = 0$ or $\mathcal{N}$ is a commutative ring.

**Proof.** Case 1: Suppose that $d_2 = 0$, then $d_1(xu) = 0$ for all $u \in U, x \in \mathcal{N}$. Using Theorem 3.1(i), then $xu \in Z(\mathcal{N})$ for all $u \in U, x \in \mathcal{N}$, replacing $x$ by $xt$ in the last equation and applying it, with Lemma (2.1)(i), we may write

$$x \in Z(\mathcal{N}) \text{ or } tu = 0 \text{ for all } u \in U, t \in \mathcal{N}.$$  \hspace{1cm} (4.1)

Since $U \neq \{0\}$, by Lemma 2.1(ii) we conclude that $\mathcal{N}$ is a commutative ring or $d_1 = 0$.

Case 2: Now assume that $d_1 = 0$, then $d_2(ux) = 0$ for all $u \in U, x \in \mathcal{N}$. Using the same techniques as used in case 1, we find that $\mathcal{N}$ is a commutative ring or $d_2 = 0$. Now supposing that $d_1(xu) = d_2(ux)$ for all $u \in U, x \in \mathcal{N}$ with $d_1 \neq 0$ and $d_2 \neq 0$. Taking $xu$ instead of $x$ in the last expression, we obtain

$$d_1(xuu) = d_2(uxu) \text{ for all } u \in U, x \in \mathcal{N}.$$  \hspace{1cm} (4.1)

Using the definition of $d_1$ and $d_2$, we get

$$xud_1(u) + ud_1(xu) = uxd_2(u) + ud_2(ux) \text{ for all } u \in U, x \in \mathcal{N}.$$  \hspace{1cm} (4.1)

By our assumption, we have

$$xud_1(u) = uxd_2(u) \text{ for all } u \in U, x \in \mathcal{N}.$$  \hspace{1cm} (4.1)
Replacing \( x \) by \( xy \) in the last equation and applying it, we may write

\[
xyud_1(u) = uxyd_2(u)
\]

for all \( u \in U, x, y \in \mathcal{N} \).

Which obviously leads to

\[
[x, u]yd_2(u) = 0
\]

for all \( u \in U, x, y \in \mathcal{N} \).

Equivalently, \([x, u]Nd_2(u) = \{0\}\) for all \( u \in U, x \in \mathcal{N} \). By 3-primeness of \( \mathcal{N} \), we arrive at \([x, u] = 0\) or \( d_2(u) = 0\) for all \( u \in U, x \in \mathcal{N} \). If there exists \( u_0 \in U \) such that \( d_2(u_0) = 0 \), then \( u_0 \in Z(\mathcal{N}) \) by Theorem 3.1(i). Then both cases force that \( U \subseteq Z(\mathcal{N}) \). From Lemma 2.1(iv), we deduce that \((\mathcal{N}, +)\) is abelian, which forces that \( \mathcal{N} \) is a commutative ring by Theorem 3.1(ii).

Taking \( d_1 = d_2 = d \), we find the following result:

**Corollary 4.2** Let \( \mathcal{N} \) be a 3-prime near-ring and \( U \) a nonzero Lie ideal of \( \mathcal{N} \). If \( \mathcal{N} \) admits a nonzero left derivation \( d \), then the following assertions are equivalent:

(i) \( d([u, n]) = 0 \) for all \( u \in U, n \in \mathcal{N} \).

(ii) \( \mathcal{N} \) is a commutative ring.

**Theorem 4.3** Let \( \mathcal{N} \) be a 3-prime near-ring and \( U \) be a nonzero Lie ideal of \( \mathcal{N} \). If \( \mathcal{N} \) admits left derivations \( d_1 \) and \( d_2 \) such that \( d_1([x, u]) = d_2(x)u \) for all \( u \in U, x \in \mathcal{N} \), then \( \mathcal{N} \) is a commutative ring.

**Proof.** If \( d_2 = 0 \), then \( d_1([x, u]) = 0 \) for all \( u \in U, x \in \mathcal{N} \). Using Theorem (3.1)(i), then \([u, n] \in Z(\mathcal{N})\), which implies that \((\mathcal{N}, +)\) is abelian by Lemma 2.3(iii), thus \( \mathcal{N} \) is a commutative ring by Theorem (3.1)(ii).

Now suppose that \( d_2 \neq 0 \) and

\[
d_1([x, u]) = d_2(x)u \quad \text{for all} \quad x \in \mathcal{N}, u \in U. \tag{4.2}
\]

Replacing \( x \) by \( u \) in (4.2), we get

\[
d_2(u)u = 0 \quad \text{for all} \quad u \in U. \tag{4.3}
\]

Substituting \( xu \) instead of \( x \) in (4.2), we may write

\[
d_1([xu, u]) = d_2(xu)u \quad \text{for all} \quad u \in U, x \in \mathcal{N}.
\]

Notice that \([xu, u] = [x, u]u\), the last relation can be rewritten as

\[
d_1([x, u]u) = (xd_2(u) + ud_2(x))u \quad \text{for all} \quad u \in U, x \in \mathcal{N}.
\]

The definition of \( d_1 \) gives us

\[
[x, u]d_1(u) + ud_1([x, u]) = xd_2(u)u + ud_2(x)u \quad \text{for all} \quad u \in U, x \in \mathcal{N}.
\]
Using our assumption and (4.3), we obviously obtain
\[ xud_1(u) = uxd_2(u) \quad \text{for all } u \in U, x \in \mathcal{N}. \quad (4.4) \]
Replacing \( x \) by \( yt \) in (4.4) and invoking it, we can see that
\[ yutd_1(u) = uytd_2(u) \quad \text{for all } u \in U, y, t \in \mathcal{N}. \]
The last equation gives us \([y, u]N d_1(u) = \{0\}\) for all \( u \in U, x \in \mathcal{N} \).
By the 3-primeness of \( \mathcal{N} \), we get \([y, u] = 0 \) or \( d_1(u) = 0 \) for all \( u \in U, y \in \mathcal{N} \). \quad (4.5)

Suppose that there exists an element \( u_0 \in U \) such that \( d_1(u_0) = 0 \). Using Theorem (3.1)(i), we obtain \( u_0 \in Z(\mathcal{N}) \). In this case, (4.5) becomes \( U \subseteq Z(\mathcal{N}) \), which forces that \( \mathcal{N} \) is abelian by Lemma 2.1(iv), and in the light of Theorem (3.1)(ii), we conclude that \( \mathcal{N} \) is a commutative ring.

Note that if we put \( d_2 = 0 \), we find the Corollary 4.2.

The following example proves that the 3-primeness of \( \mathcal{N} \) in Theorems 3.2 and 4.1 cannot be omitted.

**Example 4.1** Let \( \mathcal{S} \) be a 2-torsion right-near ring which is not abelian. Define \( \mathcal{N}, U, d_1 \) and \( d_2 \) by:
\[
\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & t \\ 0 & y & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z, t, 0 \in \mathcal{S} \right\}, \quad U = \left\{ \begin{pmatrix} 0 & i & 0 \\ 0 & j & 0 \\ 0 & k & 0 \end{pmatrix} \mid i, j, k, 0 \in \mathcal{S} \right\},
\]
\[
d_1 = \begin{pmatrix} 0 & x & t \\ 0 & y & 0 \\ 0 & z & 0 \end{pmatrix} \quad \text{and} \quad d_2 = d_1.
\]
Then \( \mathcal{N} \) is a right near-ring which is not 3-prime, \( U \) is a nonzero Lie ideal of \( \mathcal{N} \) and \( d_1, d_2 \) are nonzero left derivations of \( \mathcal{N} \). We easily can be seen that
(i) \( d(U) = \{0\} \);
(ii) \( d(u \circ u) = 0 \);
(iii) \( d(U^2) = \{0\} \),
(iv) \( d_1(xu) = d_2(ux) \),
but \( \mathcal{N} \) is not a commutative ring.

**Theorem 4.4** Let \( \mathcal{N} \) be a 2-torsion free 3-prime near-ring and \( U \) a nonzero Lie ideal of \( \mathcal{N} \). If \( \mathcal{N} \) admits a nonzero left derivation \( d \) and a multiplier \( H \), then the following assertions are equivalent:
(i) \( d([x, u]) = H([x, u]) \) for all \( u \in U, x \in \mathcal{N} \);
(ii) \( \mathcal{N} \) is a commutative ring.
Proof. It is easy to see that (ii) \(\Rightarrow\) (i).
(i) \(\Rightarrow\) (ii) Suppose that \(H = 0\), then \(d([x,u]) = 0\) for all \(u \in U, x \in \mathcal{N}\), according to Theorem 4.1 with \(d_2 = 0\), we conclude that \(\mathcal{N}\) is a commutative ring.

Now assume that \(H \neq 0\) and
\[
d([x,u]) = H([x,u]) \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
(4.6)
Replacing \(x\) by \(xu\) in (4.6), we get
\[
d([xu,u]) = H([xu,u]) \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Since \([xu,u] = [x,u]u\), it follows that
\[
d([x,u]u) = H([x,u])u \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Using the definition of \(d\), we obtain
\[
[x,u]d(u) + ud([x,u]) = H([x,u])u \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Which reduces to
\[
[x,u]d(u) = [H([x,u]), u] \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
(4.7)
Applying \(H\) to (4.7), we get
\[
H([x,u])d(u) = H([H([x,u]), u]) \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
(4.8)
Replacing \(x\) by \(H([x,u])\) in (4.6) and applying (4.8), we obtain
\[
d(H([x,u]), u) = H([H([x,u]), u]) \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
So that
\[
d(H([x,u])u) - d(uH([x,u])) = H([H([x,u]), u]) \quad \text{for all } u \in U, x \in \mathcal{N},
\]
which implies that
\[
H([x,u])d(u) + ud([H([x,u])]) - (ud(H([x,u])) + H([x,u])d(u)) = H([H([x,u]), u]).
\]
From (4.6) and by a simplification, we find that
\[
ud(H([x,u])) - H([x,u])d(u) - ud(H([x,u])) = 0 \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Which gives
\[
H([x,u])d(u) = 0 \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Since \(H\) is additive, we may write
\[
H(xu)d(u) = H(ux)d(u) \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Applying the definition of \(H\), we thereby obtaining
\[
H(x)ud(u) = H(u)xd(u) \quad \text{for all } u \in U, x \in \mathcal{N}.
\]
Replacing $x$ by $xy$ in the above expression, we can easily arrive at
\[ xH(u)yd(u) = H(u)xyd(u) \] which leads to $[x, H(u)]N\{u\} = \{0\}$, and since $N$ is 3-prime, the above expression yields
\[ H(u) \in Z(N) \text{ or } d(u) = 0 \text{ for all } u \in U. \] (4.9)

If there exists $u_0 \in U$ such that $H(u_0) \in Z(N)$, then
\[ H(u_0)n = nH(u_0) \text{ for all } n \in N. \] (4.10)

Replacing $n$ by $nmt$ in (4.10) and invoking it, we get
\begin{align*}
u_0nH(mt) &= nmtH(u_0) \\
&= nH(u_0)mt \\
&= nu_0H(mt) \text{ for all } n, m, t \in N.
\end{align*}

This equation can be rearranged to yield
\[ [u_0, n]H(mt) = 0 \text{ for all } n, m, t \in N. \]

Which implies $[u_0, n]NH(t) = \{0\}$ for all $n, t \in N$. Since $H$ is a nonzero multiplier, we conclude that $u_0 \in Z(N)$. Hence, (4.9) becomes $U \subseteq Z(N)$, which forces that $(N, +)$ is abelian by lemma 2.1(iv), and from Theorem (3.1)(ii), we conclude that $N$ is a commutative ring.

The next result is a consequence immediate of Theorem 4.4, just to take $H = id_N$ in Theorem 4.4.

**Corollary 4.5** Let $N$ be a 3-prime near-ring and $U$ a nonzero Lie ideal of $N$. If $N$ admits a nonzero left derivation $d$, then the following assertions are equivalent.

(i) $d([x, u]) = [x, u]$ for all $u \in U, x \in N$;

(ii) $N$ is a commutative ring.

The following example shows that the ”3-primeness of $N$” in Theorem 4.4 is crucial.

**Example 4.2** Let $S$ be a right near-ring, which is not abelian. Define $N, U, d$ and $H$ by:
\[ N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, t, 0 \in S \right\}, U = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in S \right\}, \]
\[ d\begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H = \text{id}. \]

Then $N$ is a right near-ring which is not 3-prime, $U$ is a nonzero Lie ideal of $N$, $H$ is a nonzero multiplier of $N$, and $d$ is a left derivation of $N$. We can easily see that $d([x, u]) = H([x, u])$ for all $u \in U, n \in N$.

But $N$ is not a commutative ring.
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References


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