ON QUASI-EINSTEIN WARPED PRODUCTS

BY

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Abstract. We study quasi-Einstein warped product manifolds for arbitrary dimension $n \geq 3$.

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1. Introduction

A Riemannian manifold $(M, g)$, $(n \geq 2)$, is said to be an Einstein manifold if its Ricci tensor $S$ satisfies the condition $S = \frac{\tau}{n}g$, where $\tau$ denotes the scalar curvature of $M$. A quasi-Einstein manifold was introduced by Chaki and Maity in [1]. A non-flat Riemannian manifold $(M, g)$, $(n \geq 2)$, is defined to be a quasi-Einstein manifold if the condition

\[(1) \quad S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y)\]

is fulfilled on $M$, where $\alpha$ and $\beta$ are scalar functions on $M$ with $\beta \neq 0$ and $A$ is a non-zero 1-form such that

\[(2) \quad g(X, U) = A(X),\]

for every vector field $X ; U \in \chi(M)$ being a unit vector field, $\chi(M)$ is the space of vector fields on $M$. If $\beta = 0$, then the manifold reduces to an Einstein manifold.

By a contraction from the equation (1), it can be easily seen that $\tau = \alpha n + \beta$, where $\tau$ is the scalar curvature of $M$. 
Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker space-times are quasi-Einstein manifolds. For more information about quasi-Einstein manifolds see [4], [5], [6] and [8].

In [2], Chen and Yano introduced the notion of a Riemannian manifold \((M, g)\) of a quasi-constant sectional curvature as a Riemannian manifold with the curvature tensor satisfies the condition

\[
R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + b\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W)
+ g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)),
\]

where \(a\) and \(b\) are scalar functions with \(b \neq 0\), where \(\eta\) is a 1-form denoted by \(g(X, E) = \eta(X)\), \(E\) is a unit vector field. It can be shown that, if the curvature tensor \(R\) is of the form (3), then the manifold is conformally flat. By a contraction from the equation (3), it can be easily seen that every Riemannian manifold of a quasi-constant sectional curvature is a quasi-Einstein manifold.

Let \(M\) be an \(m\)-dimensional, \(m \geq 3\), Riemannian manifold and \(p \in M\). Denote by \(K(\pi)\) or \(K(u, v)\) the sectional curvature of \(M\) associated with a plane section \(\pi \subset T_pM\), where \(\{u, v\}\) is an orthonormal basis of \(\pi\). For any \(n\)-dimensional subspace \(L \subset T_pM\), \(2 \leq n \leq m\), its scalar curvature \(\tau(L)\) is denoted in [3] by \(\tau(L) = 2\sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)\), where \(\{e_1, ..., e_n\}\) is any orthonormal basis of \(L\). When \(L = T_pM\), then the scalar curvature \(\tau(L)\) is just the scalar curvature \(\tau(p)\) of \(M\) at \(p\).

2. Warped product manifolds

Let \((B, g_B)\) and \((F, g_F)\) be two Riemannian manifolds and \(f\) is a positive differentiable function on \(B\). Consider the product manifold \(B \times F\) with its projections \(\pi : B \times F \rightarrow B\) and \(\sigma : B \times F \rightarrow F\). The warped product \(B \times_f F\) is the manifold \(B \times F\) with the Riemannian structure such that \(\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2\), for any vector field \(X\) on \(M\). Thus we have

\[
g = g_B + f^2 g_F
\]

holds on \(M\). The function \(f\) is called the warping function of the warped product [10].
Since \( B \times_f F \) is a warped product, then we have \( \nabla_X Z = \nabla_Z X = (X \ln f)Z \) for unit vector fields \( X, Z \) on \( B \) and \( F \), respectively. Hence, we find \( K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = (1/f)\{(\nabla_X X)f - X^2 f\}. \) If we chose a local orthonormal frame \( e_1, ..., e_n \) such that \( e_1, ..., e_n \) are tangent to \( B \) and \( e_{n+1}, ..., e_n \) are tangent to \( F \), then we have

\[
\frac{\Delta f}{f} = \sum_{i=1}^{n} K(e_j \wedge e_s),
\]

for each \( s = n_1 + 1, ..., n \) [10].

We need the following two lemmas from [10], for later use:

**Lemma 2.1.** Let \( M = B \times_f F \) be a warped product, with Riemannian curvature tensor \( M^R \). Given fields \( X, Y, Z \) on \( B \) and \( U, V, W \) on \( F \), then:

1. \( M^R(X, Y)Z = B^R(X, Y)Z \),
2. \( M^R(V, X)Y = -(H^f(X, Y)/f)V \), where \( H^f \) is the Hessian of \( f \),
3. \( M^R(X, Y)V = M^R(V, W)X = 0 \),
4. \( M^R(X, V)W = -(g(V, W)/f)\nabla_X (\text{grad } f) \),
5. \( M^R(V, W)U = F^R(V, W)U + (\|\text{grad } f\|^2/f^2)\{g(V, U)W - g(W, U)V\}. \)

**Lemma 2.2.** Let \( M = B \times_f F \) be a warped product, with Ricci tensor \( M^S \). Given fields \( X, Y \) on \( B \) and \( V, W \) on \( F \), then:

1. \( M^S(X, Y) = B^S(X, Y) - \frac{d}{2} H^f(X, Y) \), where \( d = \text{dim } F \),
2. \( M^S(X, V) = 0 \),
3. \( M^S(V, W) = F^S(V, W) - g(V, W) \left[ \frac{\Delta f}{f} + (d - 1)\|\text{grad } f\|^2/f^2 \right] \), where \( \Delta f \) is the Laplacian of \( f \) on \( B \).

Moreover, the scalar curvature \( M^\tau \) of the manifold \( M \) satisfies the condition

\[
M^\tau = B^\tau + \frac{1}{f^2} F^\tau - \frac{2d}{f} \Delta f - \frac{d(d - 1)}{f^2} \|\text{grad } f\|^2,
\]

where \( B^\tau \) and \( F^\tau \) are scalar curvatures of \( B \) and \( F \), respectively.

In [7], GEbaraowski studied Einstein warped product manifolds and proved the following three theorems:
Theorem 2.3. Let \((M, g)\) be a warped product \(I \times_f F\), \(\dim I = 1\), \(\dim F = n - 1\) \((n \geq 3)\). Then \((M, g)\) is an Einstein manifold if and only if \(F\) is Einstein with constant scalar curvature \(F\tau\) in the case \(n = 3\) and \(f\) is given by one of the following formulae, for any real number \(b\),

\[
f^2(t) = \begin{cases} 
\frac{4a}{K} \sinh^2 \sqrt{a}(t+b) & (a > 0), \\
K(t+b)^2 & (a = 0), \\
-\frac{4a}{K} \sin^2 \sqrt{-a}(t+b) & (a < 0), 
\end{cases}
\]

for \(K > 0\), \(f^2(t) = b \exp (at)\) \((a \neq 0)\), for \(K = 0\), \(f^2(t) = -\frac{4a}{K} \cosh^2 \sqrt{-a}(t+b)\), \((a > 0)\), for \(K < 0\), where \(a\) is the constant appearing after first integration of the equation \(q''e^q + 2K = 0\) and \(K = \frac{F}{(n-1)(n-2)}\).

Theorem 2.4. Let \((M, g)\) be a warped product \(B \times_f F\) of a complete connected \(r\)-dimensional \((1 < r < n)\) Riemannian manifold \(B\) and \((n-r)\)-dimensional Riemannian manifold \(F\). If \((M, g)\) is a space of constant sectional curvature \(K > 0\), then \(B\) is a sphere of radius \(\frac{1}{\sqrt{K}}\).

Theorem 2.5. Let \((M, g)\) be a warped product \(B \times_f I\) of a complete connected \((n-1)\)-dimensional Riemannian manifold \(B\) and one-dimensional Riemannian manifold \(I\). If \((M, g)\) is an Einstein manifold with scalar curvature \(M\tau > 0\) and the Hessian of \(f\) is proportional to the metric tensor \(g_B\), then

1. \((B, g_B)\) is an \((n-1)\)-dimensional sphere of radius \(\rho = \left(\frac{b}{(n-1)(n-2)}\right)^{-\frac{1}{2}}\).
2. \((M, g)\) is a space of constant sectional curvature \(K = \frac{M\tau}{n(n-1)}\).

Motivated by the above study by Gebarowski, in the present paper our aim is to generalize Theorem 2.3, Theorem 2.4 and Theorem 2.5 for quasi-Einstein manifolds.

3. Quasi-Einstein warped products

In this section, we consider quasi-Einstein warped product manifolds and prove some results concerning these type manifolds.

Now, let begin with the following theorem:

Theorem 3.1. Let \((M, g)\) be a warped product \(I \times_f F\), \(\dim I = 1\), \(\dim F = n - 1\) \((n \geq 3)\), where \(U \in \chi(M)\). If \((M, g)\) is a quasi-Einstein manifold with associated scalars \(\alpha\) and \(\beta\), then \(F\) is a quasi-Einstein manifold.
Proof. Denote by \((dt)^2\) the metric on \(I\). Taking \(f = \exp\left\{\frac{q}{2}\right\}\) and making use of the Lemma 2.2, we can write

\[
M_S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{4}[2q'' + (q')^2]
\]

and

\[
M_S(V, W) = F \, S(V, W) - \frac{1}{4}e^q[2q'' + (n-1)(q')^2]g_F(V, W),
\]

for all vector fields \(V, W\) on \(F\).

Since \(M\) is quasi-Einstein, from (1) we have

\[
M_S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta A\left(\frac{\partial}{\partial t}\right) A\left(\frac{\partial}{\partial t}\right)
\]

and

\[
M_S(V, W) = \alpha g(V, W) + \beta A(V) A(W).
\]

Decomposing the vector field \(U\) uniquely into its components \(U_I\) and \(U_F\) on \(I\) and \(F\), respectively, we can write \(U = U_I + U_F\). Since \(\dim I = 1\), we can take \(U_I = \mu \frac{\partial}{\partial t}\) which gives us \(U = \mu \frac{\partial}{\partial t} + U_F\), where \(\mu\) is a function on \(M\). Then we can write

\[
A\left(\frac{\partial}{\partial t}\right) = g\left(\frac{\partial}{\partial t}, U\right) = \mu.
\]

On the other hand, by the use of (4) and (11), the equations (9) and (10) reduce to

\[
M_S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha + \mu^2 \beta
\]

and

\[
M_S(V, W) = \alpha e^q g_F(V, W) + \beta A(V) A(W).
\]

Comparing the right hand sides of the equations (7) and (12) we get

\[
\alpha + \mu^2 \beta = -\frac{(n-1)}{4}[2q'' + (q')^2].
\]
Similarly, comparing the right hand sides of (8) and (13) we obtain

\[ F S(V, W) = \frac{1}{4} e^{q} [2q'' + (n - 1)(q')^2 + 4\alpha] g_{\alpha}(V, W) + \beta A(V) A(W), \]

which implies that \( F \) is a quasi-Einstein manifold. This completes the proof of the theorem.

Theorem 3.2. Let \((M, g)\) be a warped product \(B \times f F\) of a complete connected \(r\)-dimensional \((1 < r < n)\) Riemannian manifold \(B\) and \((n - r)\)-dimensional Riemannian manifold \(F\).

1. If \((M, g)\) is a space of quasi-constant sectional curvature, the Hessian of \(f\) is proportional to the metric tensor \(g_{B}\) and the associated vector field \(E\) is a general vector field on \(M\) or \(E \in \chi(B)\), then \(B\) is a 2-dimensional Einstein manifold.

2. If \((M, g)\) is a space of quasi-constant sectional curvature and the associated vector field \(E \in \chi(F)\), then \(B\) is an Einstein manifold.

Proof. Assume that \(M\) is a space of quasi-constant sectional curvature. Then from the equation (3) we can write

\[ M R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \]
\[ + b[g(X, W)\eta(Y)\eta(Z) - g(X, Z)\eta(Y)\eta(W)] \]
\[ + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z), \]

for all vector fields \(X, Y, Z, W\) on \(B\).

Decomposing the vector field \(E\) uniquely into its components \(E_{B}\) and \(E_{F}\) on \(B\) and \(F\), respectively, we have

\[ E = E_{B} + E_{F}. \]

By making use of (4) and (16), we can write

\[ \eta(Y) = g(Y, E) = g(Y, E_{B}) = g_{B}(Y, E_{B}). \]

In view of Lemma 2.1 and by the use of (4) and (17), we obtain

\[ B R(X, Y, Z, W) = a[g_{B}(Y, Z)g_{B}(X, W) - g_{B}(X, Z)g_{B}(Y, W)] \]
\[ + b[g_{B}(X, W)g_{B}(Y, E_{B})g_{B}(Z, E_{B}) - g_{B}(X, Z)g_{B}(Y, E_{B})g_{B}(W, E_{B}) \]
\[ + g_{B}(Y, Z)g_{B}(X, E_{B})g_{B}(W, E_{B}) - g_{B}(Y, W)g_{B}(X, E_{B})g_{B}(Z, E_{B})]. \]
By a contraction from the last equation over $X$ and $W$ and making use of the equation (17) again, we get

\[ B S(Y, Z) = [a(r - 1) + bg_B(E_B, E_B)]g_B(Y, Z) + b(r - 2)\eta(Y)\eta(Z), \]

which shows us $B$ is a quasi-Einstein manifold. Contracting from (19) over $Y$ and $Z$, it can be easily seen that

\[ B \tau = (r - 1)[ar + 2bg_B(E_B, E_B)]. \]

Since $M$ is a space of quasi-constant sectional curvature, in view of (5) and (18) we get

\[ \Delta f = \frac{ar + bg_B(E_B, E_B)}{2}. \]

On the other hand, since the Hessian of $f$ is proportional to the metric tensor $g_B$, it can be written as follows

\[ H^f(X, Y) = \frac{\Delta f}{r} g_B(X, Y). \]

Then, by the use of (20) and (21) in (22) we obtain $H^f(X, Y) + Kf g_B(X, Y) = 0$, where $K = \frac{(r-1)g_B(E_B, E_B) - a}{2(r-1)}$ holds on $B$. So by Obata’s theorem [9], $B$ is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r + 1)$-dimensional Euclidean space. This gives us $B$ is an Einstein manifold. Since $b \neq 0$ this implies that $r = 2$. Hence $B$ is a 2-dimensional Einstein manifold.

Assume that the associated vector field $E \in \chi(B)$. Then in view of Lemma 2.1 and by making use of (4) and (15) we can write

\[ B R(X, Y, Z, W) = a[g_B(Y, Z)g_B(X, W) - g_B(X, Z)g_B(Y, W)] + b[g_B(X, W)g_B(Y, E)g_B(Z, E) - g_B(X, Z)g_B(Y, E)g_B(W, E) + g_B(Y, Z)g_B(X, E)g_B(W, E) - g_B(Y, W)g_B(X, E)g_B(Z, E)]. \]

By a contraction from the last equation over $X$ and $W$, we obtain

\[ B S(Y, Z) = [a(r - 1) + bg_B(Y, Z) + b(r - 2)g_B(Y, E)g_B(Z, E)], \]

which gives us $B$ is a quasi-Einstein manifold.

By a contraction from (24) over $Y$ and $Z$, we get

\[ B \tau = (r - 1)[ar + 2b]. \]
Since $M$ is a space of quasi-constant sectional curvature, in view of (5) and (23) we have

\begin{equation}
\frac{\Delta f}{f} = \frac{ar + b}{2}.
\end{equation}

On the other hand, since the Hessian of $f$ is proportional to the metric tensor $g_B$, it can be written as follows

\begin{equation}
H^f(X, Y) = \frac{\Delta f}{r}g_B(X, Y).
\end{equation}

Then, by the use of (25) and (26) in (27) we obtain $H^f(X, Y) + Kf g_B(X, Y) = 0$, where $K = \frac{(r-1)b - b}{2r(r-1)}$ holds on $B$. So by Obata’s theorem [9], $B$ is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$-dimensional Euclidean space. This shows us $B$ is an Einstein manifold. Since $b \neq 0$ this implies that $r = 2$. Hence $B$ is a 2-dimensional Einstein manifold.

Assume that the associated vector field $E \in \chi(F)$, then the equation (15) reduces to

\begin{equation}
^{M}R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].
\end{equation}

In view of Lemma 2.1 and by the use of (4), the above equation can be written as follows

\begin{equation}
^{B}R(X, Y, Z, W) = a[g_B(Y, Z)g_B(X, W) - g_B(X, Z)g_B(Y, W)].
\end{equation}

By a contraction from the above equation over $X$ and $W$, we get $^{B}\tau = a(r - 1)g_B(Y, Z)$, which implies that $B$ is an Einstein manifold with the scalar curvature $^{B}\tau = ar(r - 1)$. Hence, the proof of the theorem is completed. \hfill \Box

**Theorem 3.3.** Let $(M, g)$ be a warped product $B \times f I$ of a complete connected $(n-1)$-dimensional Riemannian manifold $B$ and one-dimensional Riemannian manifold $I$. If $(M, g)$ is a quasi-Einstein manifold with constant associated scalars $\alpha$ and $\beta$, $U \in \chi(M)$ and the Hessian of $f$ is proportional to the metric tensor $g_B$, then $(B, g_B)$ is an $(n-1)$-dimensional sphere of radius $\rho = \frac{n-1}{\sqrt{b\tau + \alpha}}$. 


Proof. Assume that $M$ is a warped product manifold. Then by the use of the Lemma 2.2 we can write

$$\begin{align*}
\mathcal{B}S(X, Y) &= \mathcal{M}S(X, Y) + \frac{1}{f}H^f(X, Y)
\end{align*}$$

for any vector fields $X, Y$ on $B$. On the other hand, since $M$ is quasi-Einstein we have

$$\begin{align*}
\mathcal{M}S(X, Y) &= \alpha g(X, Y) + \beta A(X)A(Y).
\end{align*}$$

Decomposing the vector field $U$ uniquely into its components $U_B$ and $U_I$ on $B$ and $I$, respectively, we get

$$\begin{align*}
U &= U_B + U_I.
\end{align*}$$

In view of (2), (4), (29) and (30) the equation (28) can be written as

$$\begin{align*}
\mathcal{B}S(X, Y) &= \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) + \frac{1}{f}H^f(X, Y).
\end{align*}$$

By a contraction from the above equation over $X$ and $Y$, we find

$$\begin{align*}
\mathcal{B}\tau &= \alpha(n - 1) + \beta g_B(U_B, U_B) + \frac{\Delta f}{f}.
\end{align*}$$

On the other hand, we know from the equation (29) that

$$\begin{align*}
\mathcal{M}\tau &= \alpha n + \beta g_B(U_B, U_B).
\end{align*}$$

By the use of (32) in (31) we get $\mathcal{B}\tau = \mathcal{M}\tau - \alpha + \frac{\Delta f}{f}$. In view of Lemma 2.2 we also know that

$$\begin{align*}
-\frac{\mathcal{M}\tau}{n} &= \frac{\Delta f}{f}.
\end{align*}$$

The last two equations give us $\mathcal{B}\tau = \frac{(n - 1)}{n} \mathcal{M}\tau - \alpha$. On the other hand, since the Hessian of $f$ is proportional to the metric tensor $g_B$, we can write $H^f(X, Y) = \frac{\Delta f}{(n - 1)^2} g_B(X, Y)$. As a consequence of the equation (33) we have

$$\begin{align*}
\frac{\Delta f}{n-1} &= -\frac{1}{n(n-1)} \mathcal{M}\tau f,
\end{align*}$$

which implies that

$$\begin{align*}
H^f(X, Y) + \frac{\mathcal{B}\tau + \alpha f g_B(X, Y)}{(n - 1)^2} = 0.
\end{align*}$$
So $B$ is isometric to the $(n - 1)$-dimensional sphere of radius $\sqrt{\frac{n-1}{\alpha}}$ (see Obata [9]). Thus our theorem is proved. □

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