ON QUASI CONFORMALLY FLAT AND QUASI CONFORMALLY CONSERVATIVE RIEMANNIAN MANIFOLDS

BY

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Introduction. The quasi conformal curvature tensor in a Riemannian manifold \((M^n, g)(n > 3)\) is a \((1,3)\) tensor \(\tilde{C}\) defined as follows:

\[
\tilde{C}(X, Y, Z) = aR(X, Y, Z) + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(Z, X)QY] - \frac{\gamma}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y]
\]

where \(a\) and \(b\) are constants such that \(ab \neq 0\), \(R\) is the curvature tensor, \(S\) is the Ricci tensor, \(\gamma\) is the scalar curvature and \(Q\) is the \((1,1)\) Ricci tensor.

Then 2) \(g(QX, Y) = S(X, Y)\) for all \(X, Y\). This paper deals with Riemannian manifolds \((M^n, g)(n > 3)\) for which \(\tilde{C} = 0\) or, \(\tilde{C}\) is conservative [1] i.e. \(\text{div} \tilde{C} = 0\). A manifold \((M^n, g)(n > 3)\) shall be called quasi conformally flat or quasi conformally conservative according as \(\tilde{C} = 0\) or \(\text{div} \tilde{C} = 0\). In this paper it is shown that every quasi conformally flat \((M^n, g)\), in which \(a + (n - 2)b \neq 0\) is a manifold of constant curvature. Further, it is proved that every \((M^n, g)\) of constant curvature is quasi conformally flat. Finally a necessary and sufficient condition for an \((M^n, g)\) to be quasi conformally conservative is obtained.

1. Preliminaries. It can be easily verified that

\[
\tilde{C}(X, Y, Z) = -\tilde{C}(Y, X, Z)
\]
(1.2) \[ \tilde{C}(X, Y, Z) + \tilde{C}(Y, Z, X) + \tilde{C}(Z, X, Y) = 0 \]

(1.3) \[ g[\tilde{C}(X, Y, Z), W] = g[\tilde{C}(Z, W, X), Y]. \]

If \( a = 1 \) and \( b = -\frac{1}{n^2} \), then (1) takes the form

\[ \tilde{C}(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-2} [S(Y, Z)X - S(Y, Z)Y + g(Y, Z)QX - g(Z, X)QY] - \frac{\gamma}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] = C(X, Y, Z). \]

where \( C \) is the conformal curvature tensor \([2]\). Thus the conformal curvature tensor \( C \) is a particular case of the tensor \( \tilde{C} \). For this reason \( \tilde{C} \) is called the quasi conformal curvature tensor. Let

(1.4) \[ L(X, Y) = S(X, Y) - \frac{\gamma}{2(n-1)} g(X, Y). \]

and

(1.5) \[ g(NX, Y) = L(X, Y). \]

From (1.4) and (1.5) we get

(1.6) \[ N(X) = QX - \frac{\gamma}{2(n-1)} X. \]

Using (1.4) and (1.5) we can write (1) as follows.

(1.7) \[ \tilde{C}(X, Y, Z) = aR(X, Y, Z) + b[L(Y, Z)X - L(X, Z)Y + g(Y, Z)NX - g(X, Z)NY] - \lambda \gamma [g(Y, Z)X - g(X, Z)Y] \]

where

(1.8) \[ \lambda = \frac{a + (n-2)b}{n(n-1)}. \]
2. Quasi conformally flat \((M^n, g) (n > 3)\). In this section we shall assume that

\[ 2.1 \quad \bar{C} = 0. \]

Then from (1) we have

\[ 2.2 \quad aR(X, Y, Z) + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(Z, X)QY] - \frac{\gamma}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y] = 0. \]

From (2.2) we get

\[ a'R(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] + g(Y, Z)g(QX, W) - g(Z, X)g(QY, W)] - \frac{\gamma}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] = 0. \]

where

\[ 2.3 \quad 'R(X, Y, Z, W) = g[R(X, Y, Z), W] \]

or,

\[ a'R(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W)] + g(Y, Z)S(X, W) - g(Z, X)S(Y, W)] - \frac{\gamma}{n} \left[ \frac{a}{n-1} + 2b \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] = 0. \]

Contracting (2.4) we get

\[ aS(Y, Z) + b[(n - 2)S(Y, Z) - \gamma g(Y, Z)] + \frac{\gamma}{n} \left[ \frac{a}{n-1} + 2b \right] (n - 1)g(Y, Z) = 0. \]

or,

\[ 2.5 \quad [a + b(n - 2)] \left[ S(Y, Z) - \frac{\gamma}{n} g(Y, Z) \right] = 0. \]
If \( a + b(n - 2) \neq 0 \), then from (2.5) we get

\[
S(Y, Z) = \frac{\gamma}{n} g(Y, Z). 
\]

Hence from (2.4) it follows that

\[
a \left[ 'R(X, Y, Z, W) - \frac{\gamma}{n(n - 1)} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \right] = 0. 
\]

Therefore from (2.7) we get

\[
'R(X, Y, Z, W) = \frac{\gamma}{n(n - 1)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \text{[for } a \neq 0].
\]

From (2.8) it follows that the manifold \((M^n, g)\) is of constant curvature. Thus we can state the following:

**Theorem 1.** A quasi conformally flat \((M^n, g)(n > 3)\) in which \(a + b(n - 2) \neq 0\) is a manifold of constant curvature.

Next we enquire if a manifold of constant curvature is quasi conformally flat. From (1) we have

\[
'\tilde{C}(X, Y, Z, W) = a'aR(X, Y, Z, W) + b[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) - \gamma \left( \frac{a}{n(n - 1)} + 2b \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] - g(Y, W) \{S(X, Z) - \frac{\gamma}{n} g(Y, Z)\} - g(Z, X) \{S(Y, W) - \frac{\gamma}{n} g(Y, W)\}].
\]

We can express (2.9) as follows:

\[
(2.11) \quad '\tilde{C}(X, Y, Z, W) =
\]

\[
= a \left[ 'R(X, Y, Z, W) - \frac{\gamma}{n(n - 1)} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \right] +
\]

\[
+ b \left[ g(X, W) \{S(Y, Z) - \frac{\gamma}{n} g(Y, Z)\} + g(Y, Z) \{S(X, W) - \frac{\gamma}{n} g(X, W)\} - g(Y, W) \{S(X, Z) - \frac{\gamma}{n} g(X, Z)\} - g(Z, X) \{S(Y, W) - \frac{\gamma}{n} g(Y, W)\} \right].
\]
If an \((M^n, g)(n > 3)\) is a manifold of constant curvature, then it is an Einstein manifold. In virtue of this the equation (2.11) takes the form

\[ \tilde{C}(X, Y, Z, W) = 0. \]

From this it follows that

\[ \tilde{C}(X, Y, Z) = 0. \]

Hence the manifold is quasi conformally flat. This leads to the following theorem:

**Theorem 2.** A manifold \((M^n, g)(n > 3)\) of constant curvature is quasi conformally flat.

3. Quasi conformally conservative \((M^n, g)(n > 3)\). In this section we assume that

\[ \text{div} \tilde{C} = 0. \]

From (1.6) we get \(N = Q - \frac{\gamma}{2(n-1)}. \)

Hence

\[ \text{div} N = \text{div} Q - \frac{d\gamma}{2(n-1)}. \]

But \(\text{div} Q = \frac{1}{2} d\gamma.\) Therefore

\[ \text{div} N = \frac{n-2}{2(n-1)} d\gamma. \]

Now differentiating (1.7) covariantly we get

\[ (\nabla_W \tilde{C})(X, Y, Z) = a(\nabla_W R)(X, Y, Z) + b[(\nabla_W L)(Y, Z)X - \]

\[ - (\nabla_W L)(X, Z)Y + g(Y, Z)(\nabla_W N)(X) - \]

\[ - g(X, Z)(\nabla_W N)(Y)] + \lambda W, \gamma[g(Y, Z)X - g(X, Z)Y]. \]

Contracting (3.4) and using (3.3) we have

\[ (\text{div} \tilde{C})(X, Y, Z) = a(\text{div} R)(X, Y, Z) + b[(\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z)] + \]

\[ + \left[ \frac{n-2}{2(n-1)} b + \lambda \right] [g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)]. \]
But
\[= (\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) + \frac{1}{2(n-1)}[d\gamma(X)g(Y, Z) - d\gamma(Y)g(Z, X)].\]

Hence (3.5) takes the form
\[(3.6) \quad (\text{div } \tilde{C})(X, Y, Z) = (a + b)[(\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z)] + \frac{(n+2)[a + (n-2)b]}{2n(n-1)}[g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)].\]

If \(a + b = 0\), then from (3.6) we get
\[(3.7) \quad (\text{div } \tilde{C})(X, Y, Z) = -\frac{(n+2)(n-3)a}{2n(n-1)}[g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)].\]

Hence if \(\text{div } \tilde{C} = 0\) then \(g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y) = 0\). Consequently \(\gamma\) is constant. Again if \(\gamma\) is constant then from (3.7) it follows that \(\text{div } \tilde{C}(X, Y, Z) = 0\).

Hence we can state as follows:

\textbf{Theorem 3.} If in a quasi conformally conservative manifold \(a + b = 0\), then the manifold is of constant scalar curvature.

\textbf{Theorem 4.} If in a Riemannian manifold \((M^n, g)(n > 3)\), the quasi conformal curvature tensor is such that \(a + b = 0\), then the manifold is quasi conformally conservative if the scalar curvature is constant.

If \(a + b \neq 0\), then from (3.6) we get
\[(3.8) \quad \frac{(\text{div } \tilde{C})(X, Y, Z)}{a + b} = [(\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z)] + \frac{(n+2)[a + (n-2)b]}{2n(n-1)(a + b)}[g(Y, Z)d\gamma(X) - g(X, Z)d\gamma(Y)].\]

From (3.8) we can state as follows:

\textbf{Theorem 5.} If in a Riemannian manifold \((M^n, g)(n > 3)\), the quasi conformal curvature tensor is such that \(a + b \neq 0\), then the manifold is quasi conformally conservative if and only if
\[(\nabla_X L)(Y, Z) - (\nabla_Y L)(X, Z) = \]

\[= \frac{(\text{div } \tilde{C})(X, Y, Z)}{a + b} = \]

\[= \frac{(\text{div } \tilde{C})(X, Y, Z)}{a + b} = \]

\[= \frac{(\text{div } \tilde{C})(X, Y, Z)}{a + b} = \]
\[
= \frac{(n + 2)[a + (n - 2)b]}{2n(n - 1)(a + b)} [g(X, Z)d\gamma(Y) - g(Y, Z)d\gamma(X)].
\]

If \(a = 1\) and \(b = -\frac{1}{n-2}\), then \(a + b = \frac{n - 3}{n - 2} \neq 0\) because \(n > 3\). In this case the tensor \(\tilde{\mathcal{C}}\) reduces to the conformal curvature tensor \(\mathcal{C}\). Further in this case \(a + (n - 2)b = 0\). This leads to the following corollary of the above theorem:

**Corollary.** A Riemannian manifold \((M^n, g)(n > 3)\) is conformally conservative if and only if

\[(\nabla_X L)(Y, Z) = (\nabla_Y L)(X, Z).\]

This result follows easily from a known result in L.P. Eisenhat’s Riemannian geometry [3].

**REFERENCES**


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