CONNECTIONS AND REGULARITY
ON THE TANGENT BUNDLE

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Due to a better knowledge of the geometry of the tangent bundle $T^*M$ of a manifold $M$ it began, after 1960, a period of intense research in Lagrangian Mechanics. The introduction of the Frolicher-Nijenhuis calculus on vector forms like the natural tangent structure $J$ on $T^*M$, the Liouville vector field $C$ or semisprays $S$ (second order differential equations) and especially sprays has completely renewed the study of Lagrangian dynamics, in particular the geometrization of the trajectories.

The tangent bundle has a naturally defined integrable almost tangent structure $J$ and one can define the notion of a semispray on it. From a semispray one can derive easily a (nonlinear) connection on the tangent bundle. The cotangent bundle $T^*M$ has a naturally defined symplectic structure $\omega$ but one cannot define naturally something similar to the notion of almost tangent structure or semispray.

In [1] one introduces the notion of regularity and of adapted almost tangent structure on the cotangent bundle $T^*M$, used for the study of some geometric properties of the tangent and cotangent bundles, considering some diffeomorphism between them. A regular vector field on $T^*M$ defines an integrable adapted almost tangent structure and a connection on $T^*M$. On the other hand, this regular vector field defines naturally an $M$-bundle diffeomorphism $\Phi$ from $T^*M$ to $TM$ which transfers some results from the differential geometry of the cotangent bundle to that of the tangent bundle, and inversely.

In [2], based on a similar idea, considering a regular 1-form on $TM$ we constructed a semispray which defines a connection on $TM$. The above regular 1-form defines naturally an $M$-bundle diffeomorphism $\Psi$ from $TM$
to $T^*M$, of the Legendre transformation type, which transfers some results from the differential geometry of the tangent bundle to that of the cotangent bundle.

In connection with these correspondences between the tangent and cotangent bundles defined by geometric regular objects (objects which always exist), in this paper we study conditions for existence of some geometric objects, correspondences of these objects with mechanical systems and we also study a connection defined on $TM$.

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1. Preliminaries. We begin by describing the geometric structure of the tangent bundle $TM$ of a smooth $n$-dimensional manifold $M$. Also, we shall present, in short, some result included in [2]. Denote by $\tau : TM \rightarrow M$ the natural projection and let $(U, x^i), i = 1, \ldots, n$; be a local chart on $M$. Then the local chart $(\tau^{-1}(U); x^i, y^i)$ is induced on $TM$ where $x^i = x^i \circ \tau$ (by abuse of notation) and $y^i$ are the vector space coordinates with respect to the natural local frame $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ in $T^*M$ defined by $(U, x^i)$. Due to the special form of the change rules of the induced local charts on $TM, T^*M$, it is possible to define the notion of an $M$-tensor field on these bundles. An $M$-tensor field of type $(k, l)$ on $TM$ (on $T^*M$) is defined by sets of local coordinate components $T_{i_1 \ldots i_k}^{j_1 \ldots j_l}$, assigned to every induced local chart on $TM$ (on $T^*M$) such that the change rule, when a change of induced local charts on $TM$ (on $T^*M$) is performed, is the same with the change rule of the local coordinate components of a tensor field of type $(k, l)$ on $M$.

Consider the natural integrable almost tangent structure on $TM$, defined by the tensor field $J$ of type $(1, 1)$ on $TM$ such that

$$\text{Ker } J = \text{Im } J = VTM,$$

where $VTM = \text{Ker } \tau_\ast$ is the integrable vertical distribution on $TM$. The local coordinate expression of $J$ in an induced local chart $(\tau^{-1}(U); x^i, y^i)$ on $TM$ is

$$J = \frac{\partial}{\partial y^i} \otimes dx^i.$$

Recall that a semispray (a second order differential vector field) on the tangent bundle $TM$ is a vector field $S$ on $TM$ such that $JS = C$, where $C$ is the Liouville vector field on $TM$ (the local coordinate expression is
The local coordinate expression of the semispray $S$ is

$$S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x, y) \frac{\partial}{\partial y^i}.$$ 

There is a connection $\Gamma$ (thought of as an almost product structure) on $TM$ derived from the semispray $S$ by

$$\Gamma = -\mathcal{L}_S J.$$ 

The connection $\Gamma$ is called the canonical connection associated to the semispray $S$. The coefficients of $\Gamma$ are

$$\Gamma^i_j = -\frac{1}{2} \frac{\partial \sigma^i}{\partial y^j}.$$ 

Consider the splitting $TTM = VTM \oplus HTM$ defined by $\Gamma$. The local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial y^j}; \quad i = 1, \ldots, n$$ 

define a local frame in $HTM$.

In [2] it is considered a regular 1-form on $TM$ from which one derives, by using the integrable almost tangent structure and the Liouville vector field on $TM$, a semispray on $TM$. This semispray has its associated connection. Then, the regular 1-form defines naturally an $M$-bundle (local) diffeomorphism $\Psi$ from $TM$ to $T^*M$ by which we can establish some correspondences between the differential geometry of these bundles.

Let $\alpha$ be the 1-form, mentioned above, on $TM$ with local coordinate expression

$$(1) \quad \alpha = \alpha^j(x, y)dy^j + \beta^j(x, y)dx^j.$$ 

The components $\alpha^j$ define an $M$-1-form on $TM$, called the associated $M$-1-form of $\alpha$. The form $\alpha$ is called regular if the matrix with the entries $(g_{ij}(x, y))$ defined by

$$(2) \quad g_{ij} = \frac{\partial \alpha_i}{\partial y^j}$$ 

is nondegenerate. Assuming the 1-form $\alpha$ to be regular, we consider the 1-form $\theta$ on $TM$ defined by

$$(3) \quad \theta = \alpha \circ J = i^\alpha,$$
where $i_J$ is the vertical derivation on $TM$. The 1-form $\theta$ corresponds to the Cartan 1-form defined by a regular Lagrangian on $TM$ and has the local coordinate expression

$$\theta = \alpha_i(x, y)dx^i.$$  

The 2-form $d\theta = i_J \alpha$ is a symplectic form. Consider the 1-form $\beta$ on $TM$ defined by

$$(4) \quad \beta = di_C \alpha - \alpha,$$

and define the vector field $S$ on $TM$ by

$$(5) \quad i_S d\theta = -\beta.$$  

We find that $S$ is a semispray on $TM$. Its associated connection is

$$(6) \quad \Gamma = -\mathcal{L}_S J.$$  

The components $\alpha_i(x, y)$ of $\alpha$ may be thought as the last $n$ components of an $M$-bundle map $\Psi : TM \rightarrow T^*M$. This is the (local) diffeomorphism mentioned above. The local coordinate expression of $\Psi$ is

$$(7) \quad q^i = x^i; \quad p_i = \alpha_i(x, y),$$

where $(\pi^{-1}(U); q^i, p_i)$ is the induced local chart on $T^*M$. The inverse $\Psi^{-1}$ of $\Psi$ has the local coordinate expression

$$(8) \quad x^i = q^i; \quad y^i = u^i(q, p).$$

2. Semisprays and connections on $TM$. Considering the semispray $S$ obtained as above from the regular 1-form $\alpha$ on $TM$, we look for sufficient conditions for the semispray $S$ to be a spray (a geodesic spray); in this case its associated connection will be homogeneous (resp. linear). We shall use the terminology of [5] (see also [3]).

Recall that a differential form $\omega$ on $TM$ is said to be homogeneous of degree $r$ if $L_C \omega = r \omega$. A vector field $X$ on $TM$ is said to be homogeneous of degree $r$ if $L_C X = (r - 1)X$. A semispray $S$ is called a spray if its deviation $S^* = [C, S] - S = 0$ and $S$ is $C^1$ on the zero section. If, moreover, $S$ is $C^2$ on the zero section, then $S$ is called a quadratic spray. A connection $\Gamma$ on $TM$ is said to be homogeneous if its tension $H = \frac{1}{2}L_C \Gamma$ vanishes. If, moreover, $\Gamma$ is $C^1$ on the zero section, then $\Gamma$ is said to be a linear connection on $TM$.
(in this case $\Gamma$ defines a linear connection on the base manifold $M$). Recall that the connection associated to a spray is homogeneous and if the spray is quadratic then the connection is linear.

**Proposition 1.** Let $\alpha$ be a regular 1-form on $TM$. If $\alpha$ is homogeneous (of degree $r \in \mathbb{R}$) then the semispray $S$ derived from $\alpha$ is a spray.

**Proof.** We shall prove that the deviation $S^*$ of $S$ vanishes, showing that $i_{[C,S]}d\theta = i_SD\theta$ ($d\theta$ being a symplectic form on $TM$). We have $i_{[C,S]}d\theta = L_Ci_SD\theta - i_SL_Cd\theta = -LC\beta - (r - 1)i_SD\theta = -\beta = i_SD\theta$. It follows that $[C,S] = S$. Hence $S^* = [C,S] - S = 0$. We used here the result that, for a homogeneous differential form $\omega$ of degree $r$, the forms $d\omega$ and $i_C\omega$ are also homogeneous of degree $r$, while the form $i_J\omega$ is homogeneous of degree $(r - 1)$.

**Theorem 2.** Consider a semispray $S$ on $TM$ with the local coordinate expression

$$S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x,y) \frac{\partial}{\partial y^i}$$

and let $\beta_i(x,y), i = 1,\ldots,n$ be $n$ smooth real valued functions on the domain $\tau^{-1}(U)$ of an induced local chart on $TM$. If the ordinary differential system on $\tau^{-1}(U)$

$$\frac{dx^1}{y^1} = \ldots = \frac{dx^n}{y^n} = \frac{dy^1}{\sigma^1(x,y)} = \ldots = \frac{dy^n}{\sigma^n(x,y)} = \frac{d\alpha_1}{\beta_1(x,y)} = \ldots = \frac{d\alpha_n}{\beta_n(x,y)},$$

has $n$ prime integrals $C_1(x,y,\alpha),\ldots,C_n(x,y,\alpha)$ such that $\frac{D(C_1,\ldots,C_n)}{D(\alpha_1,\ldots,\alpha_n)} \neq 0$ and $\frac{D(C_1,\ldots,C_n)}{D(y^1,\ldots,y^n)} \neq 0$, then there exists a regular 1-form $\alpha = \alpha_j(x,y)dy^j + \beta_j(x,y)dx^j$ (hence, having the last $n$ components just the given $\beta_i$) such that the semispray $S$ is derived from $\alpha$.

**Proof.** If the semispray $S$ derives from $\alpha$ then

$$\sigma^i(x,y) = g^{ki}(\beta_k - \frac{\partial \alpha_k}{\partial x^l} y^l).$$

From these relations transvecd by $g_{ki}$ we get $\sigma^i g_{ki} = \beta_k - \frac{\partial \alpha_k}{\partial x^h} y^h$. Hence, we obtain the quasilinear partial differential system

$$\frac{\partial \alpha_i}{\partial x^j} y^j + \frac{\partial \alpha_i}{\partial y^j} \sigma^j(x,y) = \beta_i,$$
the unknown functions being $\alpha_i = \alpha_i(x, y)$. The associated characteristic system is
\[
\frac{dx^j}{ds} = y^j, \quad \frac{dy^j}{ds} = \sigma^j(x, y), \quad \frac{d\alpha_i}{ds} = \beta_i(x, y),
\]
which is equivalent with (9). The prime integrals $C_1, ..., C_n$ assure, using the implicit function theorem, the obtaining of the components $\alpha_i(x, y)$.

**Theorem 3.** Let $\alpha_i(x, y), i = 1, ..., n$; be a regular $M$-1-form on $TM$ and $\Gamma$ a nonlinear connection on $TM$ associated to a semispray. There exists a regular 1-form $\alpha$ on $TM$ with the associated $M$-1-form $(\alpha_i)$ for which the connection $\Gamma$ is the connection derived from the 1-form $\alpha$ (as above) if and only if the following conditions are fulfilled

\[
\Gamma^i_k \frac{\partial g_{ij}}{\partial y^k} = \Gamma^h_j \frac{\partial g_{ik}}{\partial y^h}; \quad i, j, k = 1, ..., n.
\]

**Proof.** The semispray $S$ derived from $\alpha$ has the local coordinate expression
\[
S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x, y) \frac{\partial}{\partial y^i}.
\]
Denote by $\sigma^i(x, y) = g^{ki}(\beta_k - \frac{\partial \alpha_k}{\partial x^l} y^l)$; the coefficients $\Gamma^i_j$ of the connection $\Gamma$ are given by
\[
\Gamma^i_j = \frac{1}{2} \frac{\partial \sigma^i}{\partial y^j}.
\]
Then after a straightforward computation, it is obtained the partial differential system:

\[
\frac{\partial \beta_i}{\partial y^j} = \frac{\partial \alpha_i}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^l} y^l - 2\Gamma^h_j g_{ih} + g^{ih} \frac{\partial g_{ih}}{\partial y^j} (\beta_i - \frac{\partial \alpha_l}{\partial x^m} y^m),
\]
in unknowns functions $\beta_i(x, y)$. The complete integrability conditions for this partial differential system are just (10).

### 3. Regular 1-forms and mechanical systems on $TM$. In this section we study the relations between regular 1-forms and regular mechanical systems on $TM$. Recall that a mechanical system on $TM$ is a triple $m = (M, F, \rho)$, where $F : TM \to \mathbb{R}$ is a smooth real valued function and $\rho$ is a semibasic 1-form on $TM$ (i.e. $\rho(V) = 0$, for any vertical vector field
Consider the form \( \omega_F = dd_J F \). If \( \omega_F \) is a symplectic form (an equivalent condition is for the Hessian \( \left( \frac{\partial^2 F}{\partial y^i \partial y^j} \right) \) of \( F \) to be nondegenerate) the mechanical system \( m \) is said to be regular. In this case, the vector field \( S \) defined by

\[
(12) \quad i_S \omega_F = -dE_F + \rho,
\]

where \( E_F = CF - F \) (the energy function associated to \( F \)) is a semispray. \( m \) is said to be conservative if \( \rho \) is a closed form. In this case \( \rho = -dV \), where \( V \) is a real valued function defined (at least locally) on \( M \) (naturally lifted on \( TM \)). Considering, in this case, the function \( L = F - V \), the relation (12) becomes

\[
(13) \quad i_S \omega_L = -dE_L,
\]

(where \( \omega_L = dd_J L, E_L = CL - L \)), so the above mechanical system is Lagrangian.

**Theorem 4.** Let \( \alpha \) be a regular 1-form on \( TM \) such that its associated \( M \)-1-form defined by \( \alpha_i \) satisfies the conditions

\[
(14) \quad \frac{\partial \alpha_i}{\partial y^j} = \frac{\partial \alpha_j}{\partial y^i}; \quad i, j = 1, ..., n \quad (i.e. \; g_{ij} = g_{ji}).
\]

Then the 1-form \( \alpha \) defines a regular mechanical system on \( TM \).

**Proof.** The conditions (14) may be expressed by

\[
\alpha_i = \frac{\partial F}{\partial y^i},
\]

where \( F \) is a smooth real valued function on a domain \( \tau^{-1}(U) \subset TM \). Using a smooth partition of unity on \( M \) we may get the desired function \( F \), globally defined on \( TM \). Generally, \( \beta_i \neq \frac{\partial F}{\partial x^i} \). We may write \( \beta_i = \frac{\partial F}{\partial x^i} + \rho_i \). We have

\[
\alpha = dF + \rho = \frac{\partial F}{\partial y^i} dy^i + \left( \frac{\partial F}{\partial x^i} + \rho_i \right) dx^i.
\]
The regularity of the form $\alpha$ (of the $M$-1-form $\alpha_i$) is equivalent to the regularity of $F$, i.e. the form $\omega_F = dd_j F$ is symplectic. $\frac{\partial \alpha_i}{\partial y^j} = \frac{\partial^2 F}{\partial y^i \partial y^j}$; so, the matrix $(\frac{\partial^2 F}{\partial y^i \partial y^j})$ is nondegenerate. It follows that the triple $m = (M, F, \rho)$ defines a regular mechanical system on $TM$.

**Theorem 4’ (the inverse).** If $m = (M, F, \rho)$ is a regular mechanical system on $TM$ then the 1-form $\alpha = dF + \rho$ is regular and its associated $M$-1-form $\alpha_i$ satisfy (14).

So, we have a one-to-one correspondence between the set of the regular mechanical system on $TM$ and the set of the regular 1-forms on $TM$ whose associated $M$-1-form satisfy the conditions (14). Moreover, the associated semispray to a mechanical system is just the semispray derived from the associated 1-form to this mechanical system. In order to prove this assertion, let us remark that

$$\theta = i_J \alpha = i_j dF = dJ F; \quad (i_J \rho = 0).$$

Then

$$d\theta = dd_J F = \omega_F,$$

$$\beta = d(i_C \alpha) - \alpha = d(\alpha(C)) - \alpha,$$

$$\alpha(C) = y^j \alpha_j = y^j \frac{\partial F}{\partial y^j} = CF.$$

It follows that

$$\beta = d(CF) - dF - \rho = d(CF - F) - \rho = dE_F - \rho.$$ 

Consequently, the relations (5) and (12) are equivalent.

**Proposition 5.** The mechanical system defined by the regular 1-form $\alpha$ is conservative if and only if $\alpha$ is a closed form.

**Proof.** The mechanical system $m = (M, F, \rho)$ is conservative if and only if $d\rho = 0$. But $\alpha = dF + \rho$. It follows that $d\alpha = 0$ if and only if $d\rho = 0$.

**Remark.** In this case the mechanical system $m$ is a Lagrangian system and the form $\alpha$ is just the exterior differential of the Lagrangian. Indeed, considering $L = F - V$ we have

$$dL = dF - dV = dF + \rho = \alpha.$$
Let us consider a regular mechanical system \( m = (M, F, \rho) \) on \( TM \). The metric \( \bar{g} \) defined by

\[
(15) \quad \bar{g}(JX, JY) = \omega_F(JX, Y); \quad X, Y \in \chi(TM).
\]

is a pseudo–Riemannian metric on the vertical bundle \( VTM \). In relation with the connection \( \Gamma = -\mathcal{L}_S J \), associated to the semispray \( S \) derived from the mechanical system \( m \), one can prolongate this metric to the metric \( g \) defined on \( TM \) by

\[
(16) \quad g(X, Y) = \bar{g}(JX, JY) + \bar{g}(vX, vY); \quad X, Y \in \chi(TM).
\]

(\( v = \frac{1}{2}(I - \Gamma) \) is the vertical projection operator), called the Riemannian prolongation of \( \bar{g} \) along \( \Gamma \).

We shall define a new connection \( \Gamma' \) on \( TM \) which, generally, differs from \( \Gamma \), by

\[
(17) \quad \omega_F(Z_1, \Gamma'Z_2) = g(JZ_1, Z_2) + g(Z_1, JZ_2); \quad Z_1, Z_2 \in \chi(TM).
\]

In order to prove that \( \Gamma' \) is a nonlinear connection on \( TM \) we shall verify the conditions

\[
(18) \quad J\Gamma = J, \Gamma'J = -J,
\]

which are equivalent to the conditions from the definition of a connection on \( TM \) (see [3]).

We have:

\[
\omega_F(Z_1, \Gamma'JZ_2) = g(JZ_1, JZ_2) = \bar{g}(JZ_1, JZ_2) = \omega_F(JZ_1, Z_2) =
\]

\[
= -\omega_F(Z_1, JZ_2) = \omega_F(Z_1, -JZ_2); \quad Z_1, Z_2 \in \chi(TM).
\]

(We used the relation \( i_J \omega_F = 0 \)). It follows that the first relation (18) is fulfilled.

Then:

\[
\omega_F(Z_1, J\Gamma'Z_2) = -\omega_F(JZ_1, \Gamma'Z_2) = -g(JZ_1, JZ_2) = -\bar{g}(JZ_1, JZ_2) =
\]

\[
= -\omega_F(JZ_1, Z_2) = \omega_F(Z_1, JZ_2); \quad Z_1, Z_2 \in \chi(TM).
\]

Consequently, the condition (18) are fulfilled.
If $\Gamma^i_j$ are the coefficients of the connection $\Gamma = -L_S J$ associated to the mechanical system $m = (M, F, \rho)$, we obtain, by a straightforward computation, the coefficients $\Gamma'_i^j$ of the connection $\Gamma'$:

$$\Gamma'_i^j = \Gamma_i^j + \frac{1}{4} g^{ki} \left( \frac{\partial \rho_k}{\partial y^j} - \frac{\partial \rho_j}{\partial y^k} \right).$$

Remark. In the case of the conservative (Lagrangian) mechanical system, $\Gamma'$ and $\Gamma$ coincide, because $\frac{\partial \rho_i}{\partial y^j} = 0$.

Related to the connection $\Gamma'$ we ask for the existence of a semibasic 1-form $\rho'$ on $TM$ such that the associated connection to the mechanical system $(M, F, \rho')$ to be $\Gamma'$.

**Proposition 6.** Let $m = (M, F, \rho)$ be a regular mechanical system, $\Gamma = -L_S J$ its associated connection and $\Gamma'$ the connection defined by (16). There exists a semibasic 1-form $\rho'$ such that $\Gamma'$ coincides with the associated connection to the mechanical system $(M, F, \rho')$ if and only if

$$\frac{\partial g^{hi}}{\partial y^k} \left( \frac{\partial \rho_h}{\partial y^j} - \frac{\partial \rho_j}{\partial y^h} \right) - \frac{\partial g^{hi}}{\partial y^j} \left( \frac{\partial \rho_h}{\partial y^k} - \frac{\partial \rho_k}{\partial y^h} \right) + g^{hi} \left( \frac{\partial^2 \rho_k}{\partial y^h \partial y^j} - \frac{\partial^2 \rho_j}{\partial y^h \partial y^k} \right) = 0.$$

**Proof.** A necessary condition for $\Gamma'$ to be derived from a semispray $S'$ is

$$\frac{\partial \Gamma'_i^j}{\partial y^k} = \frac{\partial \Gamma'_i^j}{\partial y^j},$$

which is equivalent to (20). This condition is also sufficient. In order to prove this, let us consider the 1-form

$$\rho' = i_{S'} \omega_F + dE_F.$$

Then, one can prove easily that $\rho'$ is a semibasic form.

4. **Sections in the tangent bundle tangent to a semispray.** Let $X$ be a vector field on the manifold $M$, thought of as a section $X : M \to TM$ in the tangent bundle. Then $X(M)$ is an $n$-dimensional smooth submanifold in $TM$ and we should be interested in finding the conditions under which a given semispray $S$ on $TM$ is tangent to the submanifold $X(M) \subset TM$. 
in the points of $X(M)$. The tangent space to $X(M)$ in a point $X(x)$ is spanned by

$$X_*(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} + \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial y^j}; \quad i = 1, \ldots, n$$

where $X = X^i(x) \frac{\partial}{\partial x^i}$ is the local coordinate expression of $X$. The condition for $S(X(x))$ to be tangent to $X(M)$ in $X(x)$ is obtained by expressing it as a combination of $X_*(\frac{\partial}{\partial x^i})$.

Let

$$S = y^i \frac{\partial}{\partial x^i} + \sigma^i(x, y) \frac{\partial}{\partial y^i}$$

be the local coordinate expression of $S$. We have

$$S = X^i X_*(\frac{\partial}{\partial x^i}) - X^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial y^j} + \sigma^i \frac{\partial}{\partial y^i} = X^i (\frac{\partial}{\partial x^i}) - (\frac{\partial X^i}{\partial x^j} X^j + \sigma^i) \frac{\partial}{\partial y^i}$$

in the points of $X(M)$. The condition for $S$ to be tangent to $X(M)$ becomes

$$\frac{\partial X^i}{\partial x^j}(x) X^j = \sigma^i(x, X).$$

**Proposition 7.** Let $S$ be a semispray on $TM$ with the local coordinate expression (21). Assume that the ordinary differential system

$$\frac{dx^1}{X^1} = \ldots = \frac{dx^n}{X^n} = \frac{dX^1}{\sigma^1(x, X)} = \ldots = \frac{dX^n}{\sigma^n(x, X)}$$

has $n$ prime integrals $C_1(x, X), \ldots, C_n(x, X)$ such that $\frac{D(C_1, \ldots, C_n)}{D(X^1, \ldots, X^n)} \neq 0$. Then there exists a local vector field $X$ on $M$ such that $S$ is tangent to $X(M)$ in the points of $X(M)$.

**Proof.** Thinking of (22) as a quasilinear partial differential system (the unknowns being $X^i(x)$) we obtain the characteristic system

$$\frac{dx^i}{ds} = X^j; \quad \frac{dX^i}{ds} = \sigma^i(x, X),$$

which is equivalent to (23). Then the local solution $X$ of (22) is obtained from the prime integrals $C_1, \ldots, C_n$ by using the implicit function theorem.

**Remark.** If $S$ is derived from a regular differential form $\alpha$ on $TM$, the result is, obviously, maintained. In this case $\sigma^i(x, y) = g^{ki}(\beta_k - \frac{\partial \alpha_k}{\partial x^i} y^l)$. 

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