AREA POTENTIALS FOR THIN PLATES

BY

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Dedicated to Professor Constantin Corduneanu on the occasion of his 70th birthday

Abstract. Classical particular solutions are constructed in terms of area potentials for the nonhomogeneous systems that characterize the equilibrium bending and the high–frequency harmonic flexural oscillations of an elastic plate with transverse shear deformation.

I. Introduction. The bending of thin elastic plates with transverse shear deformation is governed by a nonhomogeneous elliptic system of partial differential equations [1]. A similar nonhomogeneous system is encountered in the theory of high–frequency harmonic oscillations [2]. In this paper, we aim to construct particular solutions of these systems, which can then be used to eliminate the nonhomogeneous terms and thus make these systems amenable to the application of boundary integral equation methods.

We consider a homogeneous and isotropic elastic plate of constant thickness $h_0$, density $\rho$ and Lamé constants $\lambda$ and $\mu$ satisfying

$$\lambda + \mu > 0, \quad \mu > 0.$$ 

The plate occupies a region $S^+ \subset \mathbb{R}^2$ bounded by a simple closed $C^2$–curve $\partial S$. The case when the plate occupies the infinite region $S^- = \mathbb{R}^2 \setminus (S^+ \cup \partial S)$ is analysed in the same manner except for the addition of certain restriction at infinity.

In what follows, Greek subscripts take the values 1,2, summation over repeated indices is understood, a superscript $\top$ denotes matrix transposition, $x = (x_1, x_2)^\top$ and $y = (y_1, y_2)^\top$ are generic points in $\mathbb{R}^2$, and $r = |x - y|$. We emphasize that there is no connection between subscripts...
such as $\alpha, \beta, \rho$ and $\mu$ and the superscripts occurring in the designation of spaces of Hölder continuous functions or physical constants of the plate material. Alternative symbols have not been used in order to keep the notation simple.

We introduce the equilibrium operator $A(\omega_x) = A(\partial/\partial x_1, \partial/\partial x_2)$ defined by [1]

\[
A(\xi_1, \xi_2) = \begin{pmatrix}
h^2 \mu \Delta + h^2(\lambda + \mu)\xi_2^2 - \mu & h^2(\lambda + \mu)\xi_1, \xi_2 - \mu \xi_1 \\
h^2(\lambda + \mu)\xi_1, \xi_2 & h^2 \mu \Delta + h^2(\lambda + \mu)\xi_2^2 - \mu & -\mu \xi_2 \\
\mu \xi_1 & \mu \xi_2 & \mu \Delta
\end{pmatrix},
\]

and the harmonic oscillation operator $A^\omega(\partial_x) = A^\omega(\partial/\partial x_1, \partial/\partial x_2)$ defined by [2]

\[
A^\omega(\xi_1, \xi_2) = A(\xi_1, \xi_2) + \begin{pmatrix}
\rho \omega^2 h^2 & 0 & 0 \\
0 & \rho \omega^2 h^2 & 0 \\
0 & 0 & \rho \omega^2
\end{pmatrix},
\]

where $\omega$ is the oscillation frequency, $h^2 = h_0^2/12$ and $\Delta = \xi_1^2 + \xi_2^2$.

Let $\sigma(a, \rho)$ be the disk of radius $\rho$ centred at $a$, and let $\partial \sigma(a, \rho)$ be its circular boundary. We consider the integrals

\[
I_{\alpha\beta} = \int_{\partial \sigma(x, 1)} (x_\alpha - y_\alpha)(x_\beta - (y_\beta)ds(y)
\]

and

\[
I_{\alpha\beta\rho\eta} = \int_{\partial \sigma(x, 1)} (x_\alpha - y_\alpha)(x_\beta - y_\beta)(x_\rho - y_\rho)(x_\eta - y_\eta)ds(y).
\]

It is easily seen that

\[
I_{\alpha\beta} = \pi \delta_{\alpha\beta}, \quad I_{\alpha\beta\rho\eta} = \frac{1}{4} \pi(\delta_{\alpha\beta}\delta_{\rho\eta} + \delta_{\alpha\rho}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\rho}).
\]

2. The Newtonian potentials. Let $k(x, y)$ denote either of the $(3 \times 3)$-matrices of fundamental solutions $D(x, y)$ for $A(\partial_x)$ and $D^\omega(x, y)$ for $A^\omega(\partial_x)$ (see [1] and [3]). The Newtonian potential is defined by

\[
K(x) = \int_{S^+} k(x, y)f(y)da(y),
\]
where \( f \) is some \((3 \times 1)\)-vector function. A similar definition can be given in \( S^- \), subject to certain restriction on \( f \) at infinity to ensure the existence of the improper integral. We also consider the function

\[
M(x) = \int_{S^+} m(x, y) f(y) da(y),
\]

where \( m(x, y) = \partial^2 k(x, y)/\partial x_\alpha x_\beta \). The following assertion is proved in [4].

**Theorem 1.** (i) If \( f \in L^\infty(S^+) \), then \( \partial^{\alpha} K(x) \) exists at each point \( x \in S^+ (x \in \partial S) \)

and

\[
\frac{\partial}{\partial x_\alpha} K(x) = \int_{S^+} \frac{\partial}{\partial x_\alpha} k(x, y) f(y) da(y).
\]

(ii) If \( f \in L^\infty(S^+) \), then \( K \in C^{1,\alpha}(\partial S), \alpha \in (0,1) \).

(iii) If \( f \in C^{0,\alpha}(S^+) \), \( \alpha \in (0,1] \), then \( M(x) \) exists in the sense of principal value for every \( x \in S^+ \) and is given by

\[
M(x) = \int_{S^+} m(x, y)[f(y) - f(x)] da(y) + \left[ \int_{S^+ \setminus \sigma(x,\delta)} m(x, y) da(y) + \lim_{\varepsilon \to 0} \int_{\sigma(x, \delta) \setminus \sigma(x, \varepsilon)} m(x, y) da(y) \right] f(x),
\]

where \( \delta > \varepsilon \) is sufficiently small so that \( \sigma(x, \delta) \subseteq S^+ \).

(iv) If \( f \in C^{0,\beta}(S^+) \), \( \beta \in (0,1) \), then \( M \in C^{0,\alpha}(S_0) \), where \( 0 < \alpha < \beta < 1, S_0 \subseteq S^+ \) and \( M(x) \) is understood in the sense of principal value.

**3. Particular solutions.** We want to construct a particular solution of the system

\[
B(\partial_x) u(x) + f(x) = 0, \ x \in S^+,
\]

where the matrix operator \( B(\partial_x) \) represents either \( A(\partial_x) \) or \( A^\omega(\partial_x) \).

We introduce the symmetric \((3 \times 3)\)-matrix \( \gamma(\alpha, \beta) \) of elements

\[
g_{\rho\eta}(\alpha, \beta) = \pi d_1 \delta_{\alpha\beta} \delta_{\rho\eta} + \frac{1}{2} \pi d_2 \delta_{\alpha\rho} \delta_{\beta\eta} + \delta_{\alpha\eta} \delta_{\beta\rho} - \delta_{\alpha\beta} \delta_{\rho\eta},
\]

\[
\gamma_{\rho\beta}(\alpha, \beta) = \gamma_{3^\rho}(\alpha, \beta) = 0,
\]

\[
\gamma_{33}(\alpha, \beta) = -(1/2\mu) \delta_{\alpha\beta},
\]
where

\[
\begin{align*}
    d_1 &= -\frac{\lambda + 3\mu}{4\pi h^2 \mu (\lambda + 2\mu)}, \\
    d_2 &= \frac{\lambda + \mu}{4\pi h^2 \mu (\lambda + 2\mu)}.
\end{align*}
\]

**Theorem 2.** If \( f \in C^{0,\beta}(S^+), \ \beta \in (0, 1) \), then \( K \in C^{2,\alpha}(\Omega) \), where \( \Omega \) is an arbitrary domain in \( \mathbb{R}^2 \) whose closure lies in \( S^+ \), and \( 0 < \alpha < \beta < 1 \). Also,

\[
\frac{\partial^2}{\partial x_\alpha \partial x_\beta} K(x) = \gamma(\alpha, \beta)(x) + M(x),
\]

where \( M(x) \) and \( \gamma(\alpha, \beta) \) are given by (7) and (8)–(10), respectively.

**Proof.** The matrix of fundamental solutions \( k(x, y) \) is in fact a function of \( x - y \) [3]. Therefore, by Theorem 1 (i) with \( k(x, y) \equiv k(x - y) \), we can write

\[
\frac{\partial^2}{\partial x_\alpha \partial x_\beta} k(x) = \frac{\partial}{\partial x_\alpha} \int_{S^+} \frac{\partial}{\partial x_\beta} k(x - y)f(y)da(y).
\]

In view of Theorem 2.8 in [5], we find that

\[
\begin{align*}
    \frac{\partial}{\partial x_\alpha} \int_{S^+ \setminus \sigma(x, \delta)} \frac{\partial}{\partial x_\beta} k(x - y)f(y)da(y) &= \\
    = \int_{S^+ \setminus \sigma(x, \delta)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} k(x - y)f(y)da(y) - \\
    &\quad - \int_{\partial \sigma(x, \delta)} \frac{y_\alpha - x_\alpha}{r} \frac{\partial}{\partial x_\beta} k(x - y)f(y)ds(y).
\end{align*}
\]

The second integral on the right–hand side of (13) can be written as

\[
\begin{align*}
    - \int_{\partial \sigma(x, \delta)} \frac{y_\alpha - x_\alpha}{r} \frac{\partial}{\partial x_\beta} k(x - y)f(y)ds(y) &= \\
    = \delta \int_{\partial \sigma(x, 1)} (x_\alpha - y_\alpha) \frac{\partial}{\partial x_\beta} k(\delta(x - y))f(x + \delta(y - x))ds(y).
\end{align*}
\]

Using the result established in [4] on the singularities of \( \partial k(x, y)/\partial x_\beta \) as \( r \to 0 \), we deduce that, on \( \partial \sigma(x, 1) \) as \( \delta \to 0 \),

\[
\begin{align*}
    \delta \frac{\partial}{\partial x_\beta} k_{\rho\eta} &\delta(x - y) = d_1 \delta_{\rho\eta}(x_\beta - y_\beta) + d_2 \{ \delta_{\rho\beta}(x_\eta - y_\eta) + \delta_{\eta\beta}(x_\rho - y_\rho) - \\
    &\quad - 2(x_\rho - y_\rho)(x_\eta - y_\eta)(x_\beta - y_\beta) \} + O(\delta \ln \delta),
\end{align*}
\]
with \(d_1\) and \(d_2\) defined in (11). Also, on \(\partial \sigma(x, 1)\) as \(\delta \to 0\),

\[
\delta \frac{\partial}{\partial x_{\beta}} k_{\rho \beta}(\delta(x - y)) = O(\delta \ln \delta)
\]

and

\[
\delta \frac{\partial}{\partial x_{\beta}} k_{33}(\delta(x - y)) = -\frac{1}{2\pi \mu}(x_{\beta} - y_{\beta}) + O(\delta^2 \ln \delta).
\]

The first two components of the integral over \(\partial \sigma(x, 1)\) are given by

\[
\begin{align*}
\delta \int_{\partial \sigma(x, 1)} & (x_{\alpha} - y_{\alpha}) \frac{\partial}{\partial x_{\beta}} k_{\rho \mu}(\delta(x - y)) f_{\eta}(x + \delta(y - x)) ds(y) + \\
+\delta \int_{\partial \sigma(x, 1)} & (x_{\alpha} - y_{\alpha}) \frac{\partial}{\partial x_{\beta}} k_{\rho \beta}(\delta(x - y)) f_{3}(x + \delta(y - x)) ds(y) = \\
= & \int_{\partial \sigma(x, 1)} (x_{\alpha} - y_{\alpha}) \{d_1 \delta_{\rho \mu} I_{\alpha \beta} + d_2 \delta_{\rho \beta} I_{\alpha \eta} + \delta_{\rho \beta} I_{\alpha \eta} - 2 I_{\alpha \beta \rho \eta}\} \\
& - 2(x_{\rho} - y_{\rho})(x_{\mu} - y_{\mu})(x_{\beta} - y_{\beta}) + O(\delta \ln \delta) f_{\eta}(x + \delta(y - x)) ds(y) + \\
& + \int_{\partial \sigma(x, 1)} (x_{\alpha} - y_{\alpha}) O(\delta \ln \delta) f_{3}(x + \delta(y - x)) ds(y).
\end{align*}
\]

Passing to the limit as \(\delta \to 0\) in this expression and taking (3)–(5), (8) and (9) into account yields the first two components in the form

\[
f_{\eta}(x)[d_1 \delta_{\rho \mu} I_{\alpha \beta} + d_2 (\delta_{\beta \eta} I_{\alpha \rho} + \delta_{\beta \rho} I_{\alpha \eta} - 2 I_{\alpha \beta \rho \eta})] =
\]

\[
= f_{\eta}(x) [\pi d_1 \delta_{\rho \mu} \delta_{\alpha \beta} + \pi d_2 \{\delta_{\beta \eta} \delta_{\alpha \rho} + \delta_{\beta \rho} \delta_{\alpha \eta} - \frac{1}{2} (\delta_{\alpha \beta} \delta_{\rho \eta} + \delta_{\alpha \rho} \delta_{\beta \eta} + \delta_{\alpha \eta} \delta_{\beta \rho})\}] =
\]

\[
= \gamma_{\rho \eta}(\alpha, \beta) f_{\eta}(x) + \gamma_{\rho \beta}(\alpha, \beta) f_{3}(x).
\]

The third component of the integral over \(\partial \sigma(x, 1)\) is found analogously; thus,

\[
\begin{align*}
\delta \int_{\partial \sigma(x, 1)} & (x_{\alpha} - y_{\alpha}) \frac{\partial}{\partial x_{\beta}} k_{33}(\delta(x - y)) f_{\mu}(x + \delta(y - x)) ds(y) + \\
+\delta \int_{\partial \sigma(x, 1)} & (x_{\alpha} - y_{\alpha}) \frac{\partial}{\partial x_{\beta}} k_{33}(\delta(x - y)) f_{3}(x + \delta(y - x)) ds(y) = \\
= & \int_{\partial \sigma(x, 1)} (x_{\alpha} - y_{\alpha}) O(\delta \ln \delta) f_{\mu}(x + \delta(y - x)) ds(y) + \\
& + \int_{\partial \sigma(x, 1)} (x_{\alpha} - y_{\alpha}) \left[ -\frac{1}{2\pi \mu}(x_{\beta} - y_{\beta}) + O(\delta^2 \ln \delta) \right] f_{3}(x + \delta(y - x)) ds(y).
\end{align*}
\]
As \( \delta \to 0 \), equalities (3), (5), (9) and (10) lead to the third component

\[-\frac{1}{2\pi\mu}I_{\alpha\beta}f_3(x) = -\frac{1}{2\mu}\delta_{\alpha\beta}f_3(x) = \gamma_{3\mu}(\alpha, \beta)f_\mu(x) + \gamma_{33}(\alpha, \beta)f_3(x).\]

In [4] it is shown that the limit of the first integral on the right–hand side of (13) as \( \delta \to 0 \) yields the principal value of \( M(x) \) given by (7) (with \( \delta \) replaced by \( \varepsilon \)). Therefore, passing to the limit as \( \delta \to 0 \) in (13), we obtain (12). The fact that \( K \in C^{2,\alpha}(\Omega) \) follows from Theorem 1 (iv) and the assumption that \( f \in C^{0,\beta}(S^+) \).

Before proving the main result, we remark that since both

\[
\int_{S^+} k(x, y)f(y)da(y) \text{ and } \int_{S^+} \frac{\partial}{\partial x_\alpha}k(x, y)f(y)da(y)
\]

exist as improper integrals (provided that \( f \) is bounded), their principal values obviously exist and coincide with the improper integrals themselves. Hence, we can write (cf. (7))

\[
\int_{S^+} k(x, y)f(y)da(y) = \int_{S^+} k(x, y)[f(y) - f(x)]da(y) + \lim_{\varepsilon \to 0} \int_{\sigma(\varepsilon) \setminus \sigma(x, \varepsilon)} k(x, y)da(y)f(x)
\]

and

\[
\int_{S^+} \frac{\partial}{\partial x_\alpha}k(x, y)f(y)da(y) = \int_{S^+} \frac{\partial}{\partial x_\alpha}k(x, y)[f(y) - f(x)]da(y) + \lim_{\varepsilon \to 0} \int_{\sigma(\varepsilon) \setminus \sigma(x, \varepsilon)} \frac{\partial}{\partial x_\alpha}k(x, y)da(y)f(x).
\]

**Theorem 3.** If \( f \in C^{0,\beta}(S^+) \), \( \beta \in (0, 1) \), then \( K(x) \) defined by (6), is a solution of the system of equations

\[
B(\partial_x)u(x) + f(x) = 0, \ x \in S^+,
\]

where \( B(\partial_x) = A(\partial_x) \) and \( k(x, y) = D(x, y) \), or \( B(\partial_x) = A^\omega(\partial_x) \) and \( k(x, y) = D^\omega(x, y) \).

**Proof.** We consider \( B(\partial_x) = A^\omega(\partial_x) \) and \( k(x, y) = D^\omega(x, y) \). The latter case follows from the former with \( \omega = 0 \).
Using (1) and (2), we obtain

\[ [A^\omega(\partial_x)K(x)]_\alpha = \]

\[ = h^2 \mu \sum_{\beta=1}^{2} \frac{\partial^2}{\partial x_\beta^2} K_\alpha(x) + h^2 (\lambda + \mu) \sum_{\rho=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\rho} K_\rho(x) + \]

\[ + (\rho \omega^2 - \mu) K_\alpha(x) - \mu \frac{\partial}{\partial x_\alpha} K_3 x. \]

As was remarked earlier, we can write the integrals with kernels \( k(x, y) \) and \( \partial k(x, y)/\partial x_\alpha \) in terms of their principal values; therefore, (14) together with Theorems 1 (i) and 2 lead to

\[ [A^\omega(\partial_x)K(x)]_\alpha = h^2 \mu \sum_{\beta=1}^{2} [\gamma(\beta, \beta) f(x)]_\alpha + h^2 (\lambda + \mu) \sum_{\rho=1}^{2} [\gamma(\alpha, \rho) f(x)]_\rho + \]

\[ + \int_{S^+} [A^\omega(\partial_x)D^\omega(x,y)]_\alpha da(y) = \]

\[ + \left( \int_{S^+ \setminus \sigma(x,\delta)} [A^\omega(\partial_x)D^\omega(x,y)]_{\alpha k} da(y) \right) f_k(x) = \]

\[ + \lim_{\varepsilon \to 0} \left( \int_{\sigma(x,\delta) \setminus \sigma(x,\varepsilon)} [A^\omega(\partial_x)D^\omega(x,y)]_{\alpha k} da(y) \right) f_k(x). \]

Now, since [3]

\[ A^\omega(\partial_x)D^\omega(x,y) = -\delta(|x - y|)E_3, \]

where \( \delta \) is the Dirac delta distribution and \( E_3 \) is the \( (3 \times 3) \)-identity matrix, we find that

\[ \int_{S^+} [A^\omega(\partial_x)D^\omega(x,y)]_\alpha da(y) = \]

\[ = -\int_{S^+} \delta(|x - y|) f_\alpha(y) da(y) + \left( \int_{S^+} \delta(|x - y|) da(y) \right) f_\alpha(x) = \]

\[ = -f_\alpha(x) + f_\alpha(x) = 0. \]

The integrals over \( S^+ \backslash \sigma(x,\delta) \) and \( \sigma(x,\delta) \backslash \sigma(x,\varepsilon) \) are also zero because of (15) and the fact that the point \( x \) is outside the region of integration in each
case. Consequently, from (8) and (9) we obtain

\[
A_\omega (\partial_x K(x))_\alpha = \sum_{\beta=1}^{2} \{ h^2 \mu \gamma_{\alpha \eta}(\beta, \beta) f_\eta(x) + 2 h^2 \mu \gamma_{\alpha \beta}(\beta, \beta) f_\beta(x) + h^2 (\lambda + \mu) \gamma_{\beta \mu}(\alpha, \beta) f_\mu(x) + h^2 (\lambda + \mu) \gamma_{\beta \eta}(\alpha, \beta) f_\beta(x) \} = \\
= \sum_{\beta=1}^{2} \{ h^2 \mu [\pi d_1 \delta_{\alpha \eta} + \frac{1}{2} \pi d_2 (2 \delta_{\alpha \beta} \delta_{\mu \eta} - \delta_{\alpha \mu})] f_\eta(x) + \\
+ h^2 (\lambda + \mu) [\pi d_1 \delta_{\alpha \beta} \delta_{\beta \mu} + \frac{1}{2} \pi d_2 \delta_{\alpha \mu}] f_\mu(x) \},
\]

so, using (11), we find that

\[
A_\omega (\partial_x K(x))_\alpha = \\
= h^2 \mu (2 \pi d_1 - \pi d_2) + h^2 (\lambda + \mu) \pi d_2 + h^2 (\lambda + \mu) \pi d_1] f_\alpha(x) = \\
= h^2 \pi [(\lambda + 3 \mu) d_1 + (\lambda + \mu) d_2] f_\alpha(x) = \\
= \frac{1}{4 \mu (\lambda + 2 \mu)} [(\lambda + \mu)^2 - (\lambda + 3 \mu)^2] f_\alpha(x) = - f_\alpha(x).
\]

Similarly,

\[
A_\omega (\partial_x K(x))_3 = \mu \sum_{\beta=1}^{2} \left\{ \frac{\partial}{\partial x_\beta} K_\beta(x) + \frac{\partial^2}{\partial x_\beta^2} K_3(x) \right\} + \rho \omega^2 K_3(x),
\]

which implies that

\[
A_\omega (\partial_x K(x))_3 = \mu \sum_{\beta=1}^{2} [\gamma(\beta, \eta) f(x)]_3 + \\
+ \int_{S^+} [A_\omega (\partial_x) D_\omega(x, y)(f(y) - f(x))]_3 da(y) + \\
+ \left( \int_{S^+ \setminus \sigma(x, \delta)} [A_\omega (\partial_x) D_\omega(x, y)]_3 k da(y) \right) f_k(x) + \\
+ \lim_{\varepsilon \to 0} \left( \int_{S^\varepsilon \setminus \sigma(x, \delta)} [A_\omega (\partial_x) D_\omega(x, y)]_3 k da(y) \right) f_k(x).
\]
As before, the integrals vanish, and (9) and (10) yield

\[
[A^\omega(\partial_x)K(x)]_3 = \mu \sum_{\beta=1}^{2} [\gamma_{3\alpha}(\beta,\beta)f_\alpha(x) + \gamma_{33}(\beta,\beta)f_3(x)] = \\
= -\mu \sum_{\beta=1}^{2} \frac{1}{2\mu} f_3(x) = -f_3(x),
\]
as required.

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