ON PRIME IDEALS AND PRIME RADICALS
OF A Γ–SEMIRINGS

BY

T.K. DUTTA and S.K. SARDAR

Abstract. In this paper we introduce the notions of Prime ideals and Prime radicals of a Γ–semiring and study them via its operator semirings.

1. Introduction. The notion of Γ-semiring was introduced by M.M.K. RAO [5]. In [2] we introduced the notions of left (right) operator semiring of a Γ-semiring and then established that lattices of all ideals of a Γ-semiring and its left (right) operator semirings are isomorphic and that there exists an inclusion preserving bijection between the sets of all $k$- (h)-ideals of a Γ-semiring and its left (right) operator semiring.

Here we introduce the notions of prime ideals and prime radicals in a Γ-semiring. We obtain a number of results characterizing prime ideal and prime radical of a Γ-semiring. We then obtain an inclusion preserving bijection between the sets of all prime ideals of a Γ-semiring and its left (right) operator semiring. We also obtain relation between the prime radicals of a Γ-semiring and its left (right) operator semiring.

2. Preliminaries. Let $S$ and $Γ$ be two additive commutative semigroups. Then $S$ is called a Γ-semiring if there exists a mapping $S \times Γ \times S \rightarrow S$ (images to be denoted by $aab$ for $a, b \in S$ and $\alpha \in Γ$) satisfying the following conditions: (i) $aa(b + c) = aab + aac$; (ii) $(a + b)\alpha c = aac + b\alpha c$; (iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$; (iv) $aa(b\beta c) = (aab)\beta c$ for all $a, b, c \in S$ and for all $\alpha, \beta \in Γ$.

If $A$ and $B$ are subsets of a Γ-semiring $S$ and $\Delta \subseteq Γ$, we denote by $A\Delta B$ the subset of $S$ consisting of all finite sums of the form $\Sigma a_i\alpha_i b_i$ where...
$a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$. For the singleton subset \{x\} of $S$ we write $x \Delta B$ instead of \{x\} $\Delta B$.

A right (left) ideal of a $\Gamma$-semiring $S$ is an additive subsemigroup of $S$ such that $I \Gamma \subseteq I$ ($S \Gamma \subseteq I$). If $I$ is both a right and a left ideal of $S$, then we say that $I$ is a two–sided ideal or simply an ideal of $S$. For $a \in S$, the principal left ideal (right ideal, ideal) generated by $a$ is denoted by $< a \mid \rangle$ (respectively by $\mid a \rangle$, $\langle a \mid$). Also

\[
\langle a \mid = \left\{ ma + \sum_{i=1}^{n} x_i \alpha_i a : m, n \in \mathbb{Z}^+ \cup \{0\}, \ x_i \in S, \ \alpha_i \in \Gamma \right\},
\]
\[
\mid a \rangle = \left\{ na + \sum_{j=1}^{m} a \beta_j y_j : n, m \in \mathbb{Z}^+ \cup \{0\}, \ y_j \in S, \ \beta_j \in \Gamma \right\}
\]
\[
\langle a \rangle = \left\{ na + \sum_{k=1}^{p} a \gamma_k z_k + \sum_{i=1}^{s} w_i \delta_i a + \sum_{j=1}^{q} u_j \lambda_j a \mu_j v_j : n, p, q \in \mathbb{Z}^+ \cup \{0\}, \ z_k, w_i, u_j, v_j \in S \text{ and } y_k, \delta_i, \lambda_j, \mu_j \in \Gamma \right\},
\]

where $\mathbb{Z}^+$ is the set of all positive integers.

Let $S$ be a $\Gamma$-semiring and $F$ be the free additive commutative semigroup [4] generated by $S \times \Gamma$.

Then the relation $\rho$ on $F$, defined by $\sum_{i=1}^{m} (x_i, \alpha_i) \rho \sum_{j=1}^{n} (y_j, \beta_j)$ if and only if $\sum_{i=1}^{m} x_i \alpha_i a = \sum_{j=1}^{n} y_j \beta_j a$ for all $a \in S$ ($m, n \in \mathbb{Z}^+$), is a congruence on $F$. Congruence class containing $\sum_{i=1}^{m} (x_i, \alpha_i)$ is denoted by $\sum_{i=1}^{m} [x_i, \alpha_i]$. Then $F/\rho$ is an additive commutative semigroup. Now $F/\rho$ forms a semiring with the multiplication defined by $\left( \sum_{i=1}^{m} [x_i, \alpha_i] \right) \left( \sum_{j=1}^{n} [y_j, \beta_j] \right) = \sum_{i,j} [x_i \alpha_i y_j, \beta_j]$. We denote the semiring by $L$ and call it the left operator semiring of the $\Gamma$-semiring $S$.

Dually we define the right operator semiring $R$ of the $\Gamma$-semiring $S$ where $R = \left\{ \sum_{i=1}^{m} [\alpha_i, x_i] : \alpha_i \in \Gamma, \ x_i \in S, \ i = 1, 2, \ldots, m; \ m \in \mathbb{Z}^+ \right\}$ and the
For \( N \subseteq S \) and \( \Delta \subseteq \Gamma \) we denote by \([N, \Delta]\) the set of all finite sums \( \sum_{i=1}^{m} [x_i, \alpha_i] \) in \( L \) where \( x_i \in S \) and \( \alpha_i \in \Delta \). Thus in particular \([S, \Gamma] = L\).

Similarly we denote by \([\Delta, N]\) the set of all finite sums \( \sum_{j=1}^{n} [\beta_j, y_j] \) in \( R \) where \( y_j \in S \), \( \beta_j \in \Gamma \) and in particular \([\Gamma, S] = R\).

For simplicity \([\{x\}, \Gamma]\) is written as \([x, \Gamma]\) and \([\Gamma, \{x\}]\) is written as \([\Gamma, x]\). We also have \([x, \Gamma] \subseteq P \) (\([\Gamma, x] \subseteq P\)) if and only if \([x, \alpha] \in P\) (respectively \([\alpha, x] \in P\)) for all \( \alpha \in \Gamma \), where \( P \) is a subset of \( L \) (respectively \( R \)) and \( x \in S \). For \( P \subseteq L \) (\( \subseteq R \)) we define \( P^+ = \{ a \in S : [\Gamma, a] \subseteq P\} \) (respectively \( P' = \{ a \in S : [\Gamma, a] \subseteq P\}\)). For \( Q \subseteq S \) we define

\[
Q^+ = \left\{ \sum_{i=1}^{m} [x_i, \alpha_i] \in L : \left( \sum_{i=1}^{m} [x_i, \alpha_i] \right) \subseteq Q \right\}
\]

where \( \left( \sum_{i=1}^{m} [x_i, \alpha_i] \right) \) \( S \) denotes the set of all finite sums \( \sum_{i,k} x_i \alpha_i s_k \), \( s_k \in S \) and define

\[
Q^+ = \left\{ \sum_{i=1}^{m} [\alpha_i, x_i] \in R : \left( \sum_{i=1}^{m} [\alpha_i, x_i] \right) \subseteq Q \right\}
\]

where \( S \left( \sum_{i=1}^{m} [\alpha_i, x_i] \right) \) is the set of all finite sums \( \sum_{k,i} s_k \alpha_i x_i \), \( s_k \in S \). For \( Q \subseteq S \), \( \sum_{i=1}^{m} [x_i, \alpha_i] \in L \) and \( \sum_{i=1}^{m} [\alpha_i, x_i] \in R \), (i) \( \left( \sum_{i=1}^{m} [x_i, \alpha_i] \right) \subseteq Q \) if and only if \( \sum_{i=1}^{m} x_i \alpha_i x_i \in Q \) for all \( s \in S \); (ii) \( S \left( \sum_{i=1}^{m} [\alpha_i, x_i] \right) \subseteq Q \) if and only if \( \sum_{i=1}^{m} s \alpha_i x_i \in Q \) for all \( s \in S \).
If $P$ is an ideal of $L(R)$ then $P^+$ (respectively $P^*$) is an ideal of $S$. If $Q$ is an ideal of $S$ then $Q^{+\prime}$ ($Q^{\prime}$) is an ideal of $L$ (respectively $R$).

For a $\Gamma$-semiring $S$, an element $0 \in S$ is said to be zero of $S$ if $0 + x = 0$ and $0ax = x0a = 0$ for all $x \in S$ and for all $\alpha \in \Gamma$. If such an element exists then the $\Gamma$-semiring is said to be with zero. In that case for any $\alpha \in \Gamma$, $[0, \alpha]$ is the zero of the left operator semiring $L$ and $[\alpha, 0]$ is the zero of the right operator semiring $R$. Again if there exists an element $\sum_{i=1}^{m} [e_i, \delta_i] \in L \left( \sum_{j=1}^{n} [\gamma_i, f_i] \in R \right)$ such that $\sum_{i=1}^{m} e_i \delta_i a = \alpha \left( \sum_{j=1}^{n} a \gamma_j f_j = a \right)$ for all $a \in S$ then $S$ is said to have the left unity $\sum_{i=1}^{m} [e_i, \delta_i]$ (respectively the right unity $\sum_{j=1}^{n} [\gamma_j, f_j]$). The left (right) unity of the $\Gamma$-semiring $S$, if it exists, is the left identity (respectively the right identity) of the left operator semiring $L$ (respectively the right operator semiring $R$) of $S$.

For preliminaries we refer to [2] and for preliminaries of semiring we refer to [3]. Throughout the paper the $\Gamma$-semiring $S$ is assumed to have zero.

3. Prime ideal of a $\Gamma$-semiring.

**Definition 3.1.** Let $S$ be a $\Gamma$-semiring. A proper ideal $P$ of $S$ is said to be prime if for any two ideals $H$ and $K$ of $S$, $H \Gamma K \subseteq P$ implies that either $H \subseteq P$ or $K \subseteq P$.

**Lemma 3.2.** Let $S$ be a $\Gamma$-semiring and $L$ be its left operator semiring. If $P$ is a prime ideal of $L$ then $P^+$ is a prime ideal of $S$.

**Proof.** By Proposition 6.1 [2], $P^+$ is an ideal of $S$. Now let $A \Gamma B \subseteq P^+$ where $A$ and $B$ are two ideals of $S$. Then by definition of $P^+$, $[A \Gamma B, \Gamma] \subseteq P$ i.e. $[A, \Gamma] [B, \Gamma] \subseteq P$. Since $[A, \Gamma]$ and $[B, \Gamma]$ are ideals of $L$ and $P$ is a prime ideal of $L$, either $[A, \Gamma] \subseteq P$ or $[B, \Gamma] \subseteq P$ [3] i.e. either $A \subseteq P^+$ or $B \subseteq P^+$. Hence $P^+$ is a prime ideal of $S$.

**Lemma 3.3.** If $Q$ is a prime ideal of $S$ then $Q^{+\prime}$ is a prime ideal of $L$.

**Proof.** By Proposition 6.2 [2], $Q^{+\prime}$ is an ideal of $L$. Now suppose $AB \subseteq Q^{+\prime}$ where $A$ and $B$ are ideals of $L$. Then $ALB \subseteq AB$. This implies that $ALBS \subseteq ABS \subseteq Q$ i.e. $(AS) \Gamma (BS) \subseteq Q$ (since $L = [S, \Gamma]$). Since $AS$ and
BS are ideals of S and Q is a prime ideal of S we have either \( AS \subseteq Q \) or \( BS \subseteq Q \). Hence \( Q^{\prime} \) is a prime ideal of \( L \). Similarly we can prove the following lemmas:

**Lemma 3.4.** Let \( S \) be a \( \Gamma \)-semiring and \( R \) be its right operator semiring. If \( P \) is a prime ideal of \( R \) then \( P^{*} \) is a prime ideal of \( S \).

**Lemma 3.5.** If \( Q \) is a prime ideal of \( S \) then \( Q^{*} \) is a prime ideal of \( R \).

**Theorem 3.6.** If \( P \) is an ideal of a \( \Gamma \)-semiring \( S \) then the following conditions are equivalent:

(i) If \( A \) and \( B \) are ideals of \( S \) such that \( AB \subseteq P \) then either \( A \subseteq P \) or \( B \subseteq P \).

(ii) If \( a \Gamma Sb \subseteq P \) then either \( a \in P \) or \( b \in P \) where \( a, b \in S \).

(iii) For \( a, b \in S \) if \( < a > \Gamma < b > \subseteq P \) then either \( a \in P \) or \( b \in P \).

(iv) If \( I_{1} \) and \( I_{2} \) are two right ideals of \( S \) such that \( I_{1} \Gamma I_{2} \subseteq P \) then either \( I_{1} \subseteq P \) or \( I_{2} \subseteq P \).

(v) If \( J_{1} \) and \( J_{2} \) are two left ideals of \( S \) such that \( J_{1} \Gamma J_{2} \subseteq P \) then either \( J_{1} \subseteq P \) or \( J_{2} \subseteq P \).

**Proof.** We omit, because it is a matter of routine verification.

**Lemma 3.7.** If \( a \) and \( b \) are elements of a \( \Gamma \)-semiring \( S \) then the following conditions on a prime ideal \( P \) of \( S \) are equivalent:

(i) If \( a \Gamma b \subseteq P \) then either \( a \in P \) or \( b \in P \).

(ii) If \( a \Gamma b \subseteq P \) then \( b \Gamma a \subseteq P \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( a \Gamma b \subseteq P \). Then by (i) either \( a \in P \) or \( b \in P \). This implies that \( b \Gamma a \subseteq P \).

(iii) \( \Rightarrow \) (i) Let \( a \Gamma b \subseteq P \). Then \( ab \in P \) for all \( \alpha \in \Gamma \). So \( ab \beta s \in P \) for all \( \alpha, \beta \in \Gamma \) and for all \( s \in S \). Now by (ii) \( b \beta sa \in P \) for all \( \alpha, \beta \in \Gamma \), \( s \in S \). So \( b \Gamma S \Gamma a \subseteq P \). Hence by Theorem 3.6, either \( a \in P \) or \( b \in P \).

**Definition 3.8.** A \( \Gamma \)-semiring \( S \) is said to be commutative if \( ab = ba \) for all \( a, b \in S \), \( \alpha \in \Gamma \).

**Example 3.9.** Let \( S \) be the set of all even positive integers and \( \Gamma \) be set of all positive integers divisible by 3. Then with usual addition and multiplication of integers \( S \) is a commutative \( \Gamma \)-semiring.
Theorem 3.10. An ideal \( P \) of a commutative \( \Gamma \)-semiring \( S \) is prime if and only if \( a\Gamma b \subseteq P \) implies that either \( a \in P \) or \( b \in P \).

Proof. Let \( P \) be prime ideal of \( S \) and \( a\Gamma b \subseteq P \). Since \( S \) is commutative this implies that \( b\Gamma a \subseteq P \). So by Lemma 3.7, either \( a \in P \) or \( b \in P \). Conversely, suppose \( a\Gamma b \subseteq P \) implies that either \( a \in P \) or \( b \in P \). Let \( a \leq b \). Then \( a\Gamma b \subseteq P \). Hence either \( a \in P \) or \( b \in P \). So by Theorem 3.6, \( P \) is a prime ideal of \( S \).

Proposition 3.11. Let \( S \) be a \( \Gamma \)-semiring. If \( P \) is a prime ideal of \( S \) and \( a \in S \) is such that \( S\Gamma a\Gamma S \subseteq P \) then \( a \in P \).

Proposition 3.12. Let \( S \) be a \( \Gamma \)-semiring. If \( I \) is an ideal of \( S \) and \( P \) is prime ideal of \( S \) then \( I \cap P \) is a prime ideal of \( I \), considering \( I \) as a \( \Gamma \)-semiring.

Definition 3.13. Let \( S \) be a \( \Gamma \)-semiring. A non–empty subset \( H \) of \( S \) is said to be an \( m \)-system of \( S \) if and only if \( c, d \in H \) implies that there exist \( p \in S \) and \( \alpha, \beta \in \Gamma \) such that \( c\alpha p\beta d \in H \).

Proposition 3.14. Let \( S \) be a \( \Gamma \)-semiring. An ideal \( P \) of \( S \) is prime if and only if its complement \( P^c \) is an \( m \)-system of \( S \).

Proof. Let \( P \) be a prime ideal of \( S \). Let \( a \notin P \), \( b \notin P \). Then by Theorem 3.6, \( a\Gamma S\Gamma b \notin P \). Thus, \( a, b \in P^c \) implies that there exists \( \alpha, \beta \in \Gamma \), \( s \in S \) such that \( a\alpha s\beta b \in P^c \). So \( P^c \) is an \( m \)-system of \( S \). Conversely, let \( P^c \) be an \( m \)-system of \( S \). Then \( a, b \in P^c \) implies that there exists \( \alpha, \beta \in \Gamma \) and \( s \in S \) such that \( a\alpha s\beta b \in P^c \). Thus \( a, b \notin P \) implies that \( a\Gamma S\Gamma b \notin P \). Hence by Theorem 3.6, \( P \) is a prime ideal of \( S \).

Proposition 3.15. Let \( S \) be a \( \Gamma \)-semiring. Let \( A \) be an ideal of \( S \) disjoint from an \( m \)-system \( H \) of \( S \). Then there exists an \( m \)-system \( T \supseteq H \) which is maximal in the class of \( m \)-systems of \( S \) disjoint from \( A \).

Proof. \( T \) is obtained by applying Zorn’s Lemma to the class of \( m \)-systems of \( S \) each containing \( H \) and disjoint from \( A \).

Proposition 3.16. Let \( S \) be a \( \Gamma \)-semiring. Let \( A \) be an ideal of \( S \) disjoint from an \( m \)-system \( H \) of \( S \). Then there exists an ideal \( P \supset A \) which is maximal in the class of ideals of \( S \) each containing \( A \) and disjoint from \( H \). Moreover \( P \) is prime.
Proof. Applying Zorn’s Lemma we get an ideal \( P \) which is maximal in the class of ideals containing \( A \) and disjoint form \( H \). Now we show that \( P \) is prime. Let \( a \notin P \) and \( b \notin P \). Then the ideals \( P_1 = P+ < a > \) and \( P_2 = P+ < b > \) contain \( P \) and hence contain \( A \). Since \( P \) is maximal in the class of ideals each of which contains \( A \) and disjoint from \( H \), therefore \( P_1 \cap H \neq \emptyset \) and \( P_2 \cap H \neq \emptyset \). So there exists \( h_1, h_2 \in H \) such that \( h_1 \in P_1 \) and \( h_2 \in P_2 \). Since \( H \) is an \( m \)-system there exist \( s \in S \) and \( \alpha, \beta \in \Gamma \) such that \( h_1 \alpha s \beta h_2 \in H \). Moreover, since \( P_1 \) and \( P_2 \) are ideals of \( S \), so \( h_1 \alpha s \beta h_1 \in P_1 \Gamma P_2 \). Now let \( < a > \Gamma < b > \subseteq P \). Then \( P_1 \Gamma P_2 = (P+ < a >)\Gamma(P+ < b >) \subseteq P \). Hence \( h_1 \alpha s \beta h_1 \in P \) i.e. \( P \cap H \neq \emptyset \) – a contradiction. Hence \( a \notin P \) and \( b \notin P \) imply that \( < a > \Gamma < b > \not\subseteq P \). So by Theorem 3.6, \( P \) is prime in the \( \Gamma \)-semiring \( S \).

**Theorem 3.17.** Let \( S \) be a \( \Gamma \)-semiring and \( L \) be its left operator semiring. Then there exists an inclusion preserving bijection \( Q \rightarrow Q^+ \) between the set of all prime ideals of \( S \) and the set of all prime ideals of \( L \).

**Proof.** Let \( Q \) be a prime ideal of \( S \). Then by Lemma 3.3, \( Q^+ \) is a prime ideal of \( L \) and so by Lemma 3.2, \( (Q^+)^+ \) is a prime ideal of \( S \). Now \( (Q^+)^+ = \{ x \in S : [x, \Gamma] \subseteq Q^+ \} = \{ x \in S : [x, \Gamma] \subseteq Q \} \). Hence \( (Q^+)^+ \subseteq Q \). Since \( Q \) is a prime ideal of \( S \), this implies that \( (Q^+)^+ \subseteq Q \). Again since \( QT \subseteq Q \) so \( Q \subseteq (Q^+)^+ \). Hence \( (Q^+)^+ = Q \). Next let \( P \) be a prime ideal of \( L \). Then by Lemma 3.2, \( P^+ \) is a prime ideal of \( S \) and so by Lemma 3.3, \( (P^+)^+ \) is a prime ideal of \( L \). Now

\[
(P^+)^+ = \left\{ \sum_{i=1}^m [x_i, \alpha_i] \in L : \left( \sum_{i=1}^m [x_i, \alpha_i] \right) S \subseteq P^+ \right\} = \left\{ \sum_{i=1}^m [x_i, \alpha_i] \in L : \left[ \sum_{i=1}^m [x_i, \alpha_i] \right] S, \Gamma \subseteq P \right\}.
\]

Since \( [S, \Gamma] = L \), this implies that \( (P^+)^+ \subseteq P \). \( P \) being prime ideal in \( L \) this implies that \( (P^+)^+ \subseteq P \). Again since \( PL \subseteq P \), \( P \subseteq (P^+)^+ \). Hence \( (P^+)^+ = P \). Now let \( P \subseteq Q \) where \( P \) and \( Q \) are two prime ideals of \( S \). Then for any \( \sum_{i=1}^n [x_i, \alpha_i] \in P^+ \), \( \sum_{i=1}^n [x_i, \alpha_i] \subseteq P \subseteq Q \). Hence \( \sum_{i=1}^n [x_i, \alpha_i] \in Q^+ \) whence \( P \subseteq Q^+ \). Hence the theorem.

Dually we can prove the following theorem:

**Theorem 3.18.** Let \( S \) be a \( \Gamma \)-semiring and \( R \) be its right operator semiring. Then there exists an inclusion preserving bijection between the set
of all prime ideals of \(S\) and the set of all prime ideals of \(R\) via the mapping \(Q \rightarrow Q^\ast\) where \(Q\) is a prime ideal of \(S\).

Throughout the rest of the paper the \(\Gamma\)-semiring \(S\) is said to have the left unity and the right unity too.

**Definition 3.19.** An equivalence relation \(r\), defined on a \(\Gamma\)-semiring \(S\) satisfying the condition that if \(rpr'\) and \(spst\) in \(S\) then \((r + s)\rho(r' + s')\) and \((ras)\rho(r'as')\) for all \(\alpha \in \Gamma\), is called a \(\Gamma\)-congruence on the \(\Gamma\)-semiring \(S\).

**Remark 3.20.** The equality relation on a \(\Gamma\)-semiring \(S\) defines a \(\Gamma\)-congruence on \(S\). We call this the trivial \(\Gamma\)-congruence on \(S\). All other \(\Gamma\)-congruences are called non–trivial.

**Remark 3.21.** The universal relation on a \(\Gamma\)-semiring \(S\) gives rise to \(\Gamma\)-congruence on \(S\) what we call the improper \(\Gamma\)-congruence on \(S\). All other congruences on \(S\) are called proper.

**Definition 3.22.** For a proper ideal \(A\) of a \(\Gamma\)-semiring \(S\) the \(\Gamma\)-congruence on \(S\), denoted by \(\rho_A\), defined as \(s\rho_A s'\) if and only if \(s + a_1 = s' + a_2\) for some \(a_1, a_2 \in A\), is called the Bourne \(\Gamma\)-congruence on \(S\) defined by the ideal \(A\).

We denote the Bourne \(\Gamma\)-congruence \(\rho_A\) class of an element \(r\) of \(S\) by \(r/\rho_A\) or simply by \(r/A\) and denote the set of all such \(\Gamma\)-congruences classes of the elements of the \(\Gamma\)-semiring \(S\) by \(S/\rho_A\) or simply by \(S/A\). It should noted here that for any \(s \in S\) and for any proper ideal \(A\) of \(S\), \(s/A\) is not necessarily equal to \(s + A = \{s + a : a \in A\}\) but surely contains it.

**Definition 3.23.** For any proper \(k\)-ideal \(A\) of a \(\Gamma\)-semiring \(S\) if the Bourne \(\Gamma\)-congruence \(\rho_A\), defined by \(A\), is proper i.e. \(0/A \neq S\) then we define on \(S/A\) the following operations: \(s/A + s'/A = (s + s')/A\) and \((s/A)\alpha(s'/A) = = (s\alpha s')A\) for all \(\alpha \in \Gamma\). Routine verification shows that these operations are well–defined and \(S/A\) is a \(\Gamma\)-semiring with these operations. We call this \(\Gamma\)-semiring the Bourne factor \(\Gamma\)-semiring or simply the factor \(\Gamma\)-semiring of \(S\) by \(A\).

**Definition 3.24.** A \(\Gamma\)-semiring \(S\) is said to be weak zero–divisor free (WZDF) if \(aGb = 0, a, b \in S\), implies that either \(a = 0\) or \(b = 0\).
Definition 3.25. A commutative \( \Gamma \)-semiring \( S \) is said to be a \( \Gamma \)-semi–
integral domain \((\Gamma – S I D)\) if it is WZDF.

Theorem 3.26. Let \( S \) be a commutative \( \Gamma \)-semiring. Then a proper 
k-ideal \( P \) of \( S \) is prime if and only if \( S/P \) is a \( \Gamma – S I D \).

Proof. Let the \( k \)-ideal \( P \) be prime in \( S \). Let \( a/P, b/P \in S/P \) such that 
\((a/P)\Gamma(b/P) = 0/P\). Then \((a/P)\alpha(b/P) = (a\alpha b)/P = 0/P \) for all 
\( \alpha \in \Gamma \). This implies that \( a\alpha b \in P \) for all \( \alpha \in \Gamma \) i.e. \( a\Gamma b \subseteq P \). Since \( S \) is 
commutative and \( P \) is prime in \( S \), by theorem 3.10, \( a \in P \) or \( b \in P \). So 
\( a/P = 0/P \) or \( b/P = 0/P \). Hence \( S/P \) is WZDF. Also commutativity of 
\( S \) implies commutativity of \( S/P \). So \( S/P \) is a \( \Gamma – S I D \).

Conversely, suppose that the proper \( k \)-ideal \( P \) of \( S \) is such that 
\( S/P \) is a \( \Gamma – S I D \). Let \( a \Gamma b \subseteq P \), \( a, b \in S \). Then \( a\alpha b \in P \) for all 
\( \alpha \in \Gamma \). So \( a\alpha b/P = 0/P \) for all \( \alpha \in \Gamma \) i.e. \( a/P \alpha (b/P) = 0/P \) for all \( \alpha \in \Gamma \). So 
\((a/P)\Gamma(b/P) = 0/P \). Hence \( a/P = 0/P \) or \( b/P = 0/P \). So \( a \in P \) or \( b \in P \). So by theorem 3.10, \( P \) is prime in \( S \).

4. Prime Radical.

Definition 4.1. The prime radical of a \( \Gamma \)-semiring \( S \) is defined as the 
intersection of all prime ideals of \( S \) and is denoted by \( P(S) \).

Theorem 4.2. If \( P(L) \) is the prime radical of the left operator semi–
ring \( L \) of a \( \Gamma \)-semiring \( S \) then \( P(S) = P(L)^+ \) and \( P(S)^+ = P(L) \).

Proof. Let \( Q \) be a prime ideal of \( S \). Then by lemma 3.3, \( Q^{+'} = P \) is a prime ideal of \( L \). Then \( P^+ = (Q^{+'}) = Q \). Let \( \Omega \) and \( \Lambda \) be the collections 
of all prime ideals of \( S \) and \( L \) respectively. Now by lemma 3.2, \( P \in \Lambda \) 
implies that \( P^+ \in \Omega \). Hence \( P(S) = \bigcap_{Q \in \Omega} Q \subseteq \bigcap_{P \in \Lambda} P^+ = \left( \bigcap_{P \in \Lambda} P \right)^+ = P(L)^+ \).

Again \( P(S) = \bigcap_{Q \in \Omega} Q = \bigcap_{P \in \Lambda} P^+ \) \((\Lambda' \) is a subcollection of \( \Lambda \) \) \( \subseteq \bigcap_{P \in \Lambda} P^+ = \left( \bigcap_{P \in \Lambda} P \right)^+ = P(L)^+ \). Thus \( P(S) = P(L)^+ \).

Similar is the proof of \( P(S)^{+'} = P(L) \).

Corollary 4.3. If \( P(L) \) is the prime radical of the left operator semi–
ring \( L \) of a \( \Gamma \)-semiring \( S \) then \( P(L) = (P(L)^{+'})^+ \) and \( P(S) = (P(S)^{+'})^+ \).
Theorem 4.4. For a Γ-semiring $S$, $P(S) = \{ s \in S : \text{every m-system of } S \text{ contains } s \text{ contains 0 of } S \}$.

Proof. Let $P' = \{ x \in S : \text{every m-system of } S \text{ which contains } x \text{ contains 0} \}$. Let $y \notin P(S)$. Then $y \notin P$ for some prime ideal $P$ of $S$. By Proposition 3.14, $P^c$ is an m-system of $S$. Since $0 \in P$, $0 \notin P^c$. Thus $P^c$ is an m-system of $S$ containing $y$ but not containing 0. So $y \notin P'$ whence $P \subseteq P(S)$. Now let $y \notin P'$. Then there exists an m-system $H$ of $S$ such that $P \cap H = \emptyset$. So $y \notin P$ whence $y \notin P(S)$. So $P(S) \subseteq P'$. This completes the proof.

Theorem 4.5. Let $S$ be a Γ-semiring. If $I$ is an ideal of $S$ then $P(I) = I \cap P(S)$ where $P(I)$ denotes the prime radical of $I$ considering $I$ as a Γ-semiring.

Proof. Let $\Omega$ be the collection of all prime ideals of $S$ and $\Lambda$ be the collection of all prime ideals of $I$. By Proposition 3.12, $P \in \Omega$ implies that $I \cap P(S)$. Let $a \notin P(I)$. Then $0 \notin H$ for some m-system $H$ of $I$ containing $a$. Since $H$ is also an m-system of $S$, by Theorem 4.4, $a \notin P(S)$. Hence $P(S) \subseteq P(I)$. So $I \cap P(S) \subseteq P(I)$. Thus $P(I) = I \cap P(S)$.

Theorem 4.6. Let $S$ be a Γ-semiring. If $I$ is an ideal of $S$ then $P(I)^+ = P(L)$ where $L$ is the left operator semiring of $S$.

Proof. By Theorem 4.5, $P(I)^+ = (I \cap P(S))^+ = I^+ \cap P(S)^+ = I^+ \cap P(L)$ (since by Theorem 4.2, $P(S)^+ = P(L)$).

Analogous results involving relation between the prime radical $P(S)$ of $S$ and the prime radical $P(R)$ of the right operator semiring $R$ of $S$ can be obtained. As a consequence we have the following result analogous to that of [1].

Theorem 4.7. For a Γ-semiring $S$ with left and right operator semirings $L$ and $R$, respectively, $P(L)^+ = P(R)^*$. 
REFERENCES


Received: 30.XI.1999

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta – 700019
INDIA