CERTAIN NEAR–RINGS ARE RINGS

BY

MOHARRAM A. KHAN

Abstract. In the present note it is shown that $R$ must be commutative if distributively generated $(d-g)$ near-ring $R$ satisfying the property: $xy=p(y,x)$, where $p(y,x)$ is a finite sum of terms of the form $\alpha_i y^{p_i} x^{n_i} y^{q_i}$, where the number of summands and $\alpha_i, p_i, n_i, q_i$ all vary with $x, y \in R$, $\alpha_i, p_i, q_i \geq 1$ and $n_i > 1$.

1. Introduction. Throughout $R$ will denote a left near-ring, $Z(R)$ the center of $R$, $N(R)$ the set of nilpotent elements of $R$. An element $x$ of $R$ is called distributive if $(a + b)x = ax + bx$ for all $a, b \in R$. If all the elements of $R$ are distributive, then $R$ is called distributive near-ring. A left near-ring $R$ is called: (a) periodic near-ring if for each $x \in R$, there exist distinct positive integers $m = m(x), n = n(x)$ such that $x^m = x^n$. (b) a zero-symmetric if $0x = 0$ for all $x \in R$ (left distributivity yields $x0 = 0$). (c) a zero commutative if $xy = 0$ implies that $yx = 0$ for all $x, y \in R$. (d) a distributively generated $(d-g)$ near-ring if it contains a multiplicative subsemigroup of distributive elements which generates additive group $(R, +)$. (e) a strongly distributively generated $(s-d-g)$ near-ring if it contains a set of distributive elements whose squares generate $(R, +)$. (f) a D-near-ring if every non-zero homomorphic image $B$ of $R$ satisfies the following conditions:

(i) $B$ has a non-zero right distributive element.

(ii) $(B, +)$ is abelian implies that $(B, +, \cdot)$ is a ring.

An ideal of a near-ring $R$ is a normal subgroup $I$ of $(R, +)$ such that
(i) $RI \subseteq I$ and
(ii) $(x + \alpha)y - xy \in I$ for all $x, y \in R$ and $\alpha \in I$.

In a $(d - g)$ near-ring (ii) may be replaced by

(ii)$^*$ $IR \subseteq I$.

It is evident by the definition that all distributive and $(d - g)$ near-rings are the examples of $D$-near-rings. However, the Example 2.5 #6 of [7] illustrates that the class of $D$-near-rings is larger than the class of $(d - g)$ near-rings.

A well-known theorem of HERSTEIN [9] asserts that a periodic ring is commutative if its nilpotent elements are central. In order to establish an analogous result in near-rings, BELL [3] proved that ‘if $R$ is a $(d-g)$ near-ring with its nilpotent elements lying in the center, then the set $N(R)$ forms an ideal’, and if $R/N(R)$ is periodic, then $R$ must be commutative. In a recent paper [12], LIKH has remarked that some conditions implying commutativity in near-rings might reduce them to rings. The main purpose of this paper is to examine whether the following property implies that certain near-rings are rings.

(P) For each $x, y \in R$, there exist positive integers $n_i = n_i(x, y) > 1$, $p_i = p_i(x, y) \geq 1, q_i = q_i(x, y) \geq 1, \alpha_i = \alpha_i(x, y) \geq 1$ such that

$$xy = p(y, x)$$

where $p(y, x)$ is a finite sum of terms of the form $\alpha_i y^{p_i} x^{n_i} y^{q_i}$, where the number of summands and $\alpha_i, p_i, n_i, q_i$ all vary with $x, y$.

2. Main result. The main result of the present paper is as follows.

**Theorem 2.1.** Let $R$ be a $(d - g)$ near-ring satisfying (P). Then $R$ is commutative.

We begin with the following lemma which will be used extensively. The proof of (i) is straightforward whereas those of (ii) and (iii) can be found in [8].
Lemma 2.1.
(i) A \((d-g)\) near-ring is always zero-symmetric.
(ii) A \((d-g)\) near-ring \(R\) is distributive if and only if \(R^2\) is additively commutative.
(iii) If \(N(R)\) is a two-sided ideal in a \((d-g)\) near-ring \(R\), then the elements of the quotient group \((R,+)/N(R)\) form a \((d-g)\) near-ring, which will be represented by \(R/N(R)\).

Before proving our theorem, we establish the following results called steps.

Step 2.1. Let \(R\) satisfy \((P)\). Then \(R\) is zero-commutative.

Proof. For a pair of near-ring elements \(x, y \in R\), we have \(xy = 0\). By hypothesis, we get \(yx = p(x, y) = 0\), because \(\alpha, x^{p_1-1}y^{n_1-1}x^{q_1} = 0\). Hence, \(R\) is zero-commutative as well as zero-symmetric.

Step 2.2. Let \(R\) satisfy \((P)\). Then \(N(R) \subseteq Z(R)\).

Proof. From Step 2.1, it follows easily that \(R\) must have the insertion-of-factors property, that is, any product equal to 0 remains so on the insertion of additional factors between any existing factors; in particular, if \(u^t = 0\), any product of ring elements having at least \(t\) factors equal to \(u\) is 0. Let \(u \in N(R)\) and \(x \in R\), and suppose \(u^t = 0\). Replacing \(y\) by \(x\) and \(x\) by \(u\) in the hypothesis, then there exist positive integers \(\alpha_{i1} = \alpha_i(u, x), p_{i1} = p_i(u, x), q_{i1} = q_i(u, x) \geq 1\) and \(n_{i1} = n_i(u, x) > 1\) such that

\[ux = p(x, u) = \alpha_{i1}x^{p_{i1}}u^{n_{i1}}x^{q_{i1}}.\]

Further, choose integers \(\alpha_{i2} = \alpha(u^{p_{i1}}, x^{n_{i1}}), p_{i2} = p(u^{p_{i1}}, x^{n_{i1}}), q_{i2} = q(u^{p_{i1}}, x^{n_{i1}}) \geq 1\) and \(n_{i2} = n(u^{p_{i1}}, x^{n_{i1}}) > 1\) such that

\[\alpha_{i1}x^{p_{i1}}u^{n_{i1}}x^{q_{i1}} = \alpha_{i1}\alpha_{i2}x^{p_{i1}p_{i2}}u^{n_{i1}n_{i2}}x^{p_{i1}q_{i2}q_{i1}}.\]

By the above equality, one gets

\[ux = \alpha_{i1}\alpha_{i2}x^{p_{i1}p_{i2}}u^{n_{i1}n_{i2}}x^{p_{i1}q_{i2}q_{i1}}.\]

Thus, it is obvious that for arbitrary \(t\), we have \(\alpha_{i1}, \alpha_{i2}, ..., \alpha_{it} \geq 1, p_{i1}, p_{i2}, ..., p_{it} \geq 1, n_{i1}, n_{i2}, ..., n_{it} > 1\) and \(q_{i1}, q_{i2}, ..., q_{it} \geq 1\) such that

\[ux = \alpha_{i1}\alpha_{i2}...\alpha_{it}x^{p_{i1}p_{i2}...p_{it}}u^{n_{i1}n_{i2}...n_{it}}x^{q_{i1}q_{i2}...q_{it}}.\]
But, since $u \in N(R), u^{n_1 n_2 \ldots n_t} = 0$ for sufficiently large $t$. Hence $ux = 0$ for all $x \in R$ and, by Step 2.1, this implies that $xu = 0$, that is,

$$RN(R) = N(R)R = \{0\}.$$  

Equation (1) shows that nilpotent elements of $R$ annihilate $R$ on both sides and hence, in particular, $N^2 = \{0\}$ and $u$ is central.

**Step 2.3.** Let $R$ satisfy $(P)$. Then $N(R)$ forms an ideal.

**Proof.** Let $a, b \in N(R)$. Then, by Step 2.2, $(a - b)^2 = 0$. This yields that $a - b \in N(R)$ and hence $N(R)$ is a subgroup of the additive group $(R, +)$. Further, an application of [1, Lemma 1] gives the required result.

**Step 2.4.** Let $A$ be an arbitrary ring satisfying $(P)$. Then $A$ is a periodic ring.

**Proof.** Taking $y = x$ in $(P)$, we get

$$x^2 = p(x, x) = \sum_{i \in I, \text{finite}} \alpha_i x^{p_i + n_i + q_i},$$

for some positive integer $p_i + n_i + q_i$; hence by Chacron’s theorem [6], $A$ is periodic.

**Remark 2.1.** From the proof of Step 2.4, it is clear that a near–ring $R$ satisfies $(P)$ together with the Chacron’s result is periodic.

**Proof of Theorem 2.1.** By Step 2.2, we have $N(R) \subseteq Z(R)$; by Step 2.3 $N(R)$ is an ideal. We consider the near-ring $\overline{R} = R/N$. Now it is enough to prove that $(\overline{R}, +)$ is abelian. By Fröhlich’s theorem [8] $\overline{R}$ is a ring, and is periodic by Remark 2.1; commutativity of $R$ follows from [3].

3. **Applications.** The following results are corollaries of our main theorem as well as the application of Lemma 2.1 (ii), due to FRÖHLICH [8].

**Result 3.1.** Let $R$ be a $(d - g)$ near-ring satisfying $(P)$. If $R^2 = R$, then $R$ is a commutative ring.
Proof. From the above theorem, one can observe that a \((d - g)\) near-ring satisfying \((P)\) is commutative. Hence for any \(a, b, c \in R\), we have
\[
(b + c)a = a(b + c) = ab + ac = ba + ca.
\]
It follows that \(R\) is distributive and hence, by Lemma 2.1 (ii), \(R^2\) is additively commutative. Next \(R^2 = R\) gives that \((R, +)\) is abelian. Hence \(R\) is a commutative ring.

Result 3.2. Let \(R\) be a \((s - d - g)\) near-ring satisfying \((P)\). Then \(R\) is a commutative ring.

Proof. In view of the above theorem, \(R\) is a commutative \((s - d - g)\) near-ring in which every element is distributive and by, Lemma 2.1 (ii), \(R^2\) is additively commutative. Thus the additive group \((R, +)\) of the \((s - d - g)\) near-ring is also commutative and \(R\) is a commutative ring.

Result 3.3. Let \(R\) be a \(D\)-near-ring with unity 1 satisfying \((P)\). Then \(R\) is a commutative ring.

Proof. From the Step 2.2, one gets \(N(R) \subseteq Z(R)\). Further, by Remark 2.1, \(R\) is periodic and if \(R\) has unity 1, then by a result of BELL [4] \((R, +)\) is abelian. Hence by the definition of \(D\)-near-ring, \(R\) turns out to be a ring which is periodic with central nilpotents. By an application of well-known result of BELL [2, Theorem 2] \(R\) is a commutative ring.

REFERENCES

10. KHAN, M.A. – Commutativity and structure of certain classes of rings and near-rings, Presented the paper conference on nearrings and near fields, July 30–August 6, 1995 at Hamburg (Germany).

Received: 13.VII.1999

Department of Mathematics
Faculty of Science
King Abdulaziz University
P.O. Box 30356, Jeddah 21477
KINGDOM SAUDI ARABIA
E-mail: nassb@hotmail.com