ON HILBERT TYPE INEQUALITY IN SEVERAL VARIABLES

BY

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Abstract. The main aim of the present note is to establish a new inequality of Hilbert type in several variables by using a fairly elementary analysis.

1. Introduction. The well known Hilbert’s double series theorem can be stated as follows (see [1, p.226]).

Theorem H. If $p > 1$, $p' = \frac{p}{p-1}$ and $\sum a_m^p \leq A$, $\sum b_n^{p'} \leq B$, the summation running from 1 to $\infty$, then

$$\sum \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} A^{1/p} B^{1/p'},$$

unless the sequences $\{a_m\}$ or $\{b_n\}$ is null.

Over the years many papers which deals with alternative proofs, various generalizations, numerous variants and applications of the Hilbert’s double series theorem have appeared in the litterature. A survey of some of the earlier developments related to Hilbert’s double series theorem and many important applications in analysis can be found in [1, Chapter IX]. The main purpose of the present note is to establish a new inequality in several variables similar to Hilbert’s inequality involving functions and their finite differences. The analysis used in the proof is elementary and our result provides a new estimate on this type of inequality.
2. Statement of results. In what follows we denote by $R$ the set of real numbers. Let $N = \{1, 2, \ldots\}$, $N_0 = \{0, 1, 2, \ldots\}$. For $x = (x_1, \ldots, x_n)$ in $N_0^n$ and $z : N_0^n \to R$, we define the difference operators

$$
\nabla_1 z(x_1, x_2, \ldots, x_n) = z(x_1, x_2, \ldots, x_n) - z(x_1 - 1, x_2, \ldots, x_n),
$$

$$
\vdots
$$

$$
\nabla_n z(x_1, \ldots, x_{n-1}, x_n) = z(x_1, \ldots, x_{n-1}, x_n) - z(x_1, \ldots, x_{n-1}, x_n - 1).
$$

Similarly, we define

$$
\nabla_1 \nabla_2 z(x_1, x_2, \ldots, x_n) = \nabla_1[z(x_1, x_2, \ldots, x_n) - z(x_1, x_2 - 1, x_3, \ldots, x_n)],
$$

and so on. Let $A$ and $B$ be bounded domains in $N_0^n$ defined by

$$
A = \prod_{i=1}^n [0, a_i],
$$

$$
B = \prod_{i=1}^n [0, b_i],
$$

where $a_i, b_i$ are the elements of $N$. For any real-valued functions $u$ and $v$ defined on $A$ and $B$ respectively, we denote by $\sum_A u(x)$ and $\sum_B v(y)$ the $n$-fold sums

$$
\sum_{x_1=1}^{a_1} \ldots \sum_{x_n=1}^{a_n} u(x_1, \ldots, x_n)
$$

and

$$
\sum_{y_1=1}^{b_1} \ldots \sum_{y_n=1}^{b_n} v(y_1, \ldots, y_n)
$$

and for any $x \in A$ and $y \in B$, we denote by $\sum_{A_x} u(s)$ and $\sum_{B_y} v(t)$ the $n$-fold sums

$$
\sum_{s_1=1}^{x_1} \ldots \sum_{s_n=1}^{x_n} u(s_1, \ldots, s_n)
$$

and

$$
\sum_{t_1=1}^{y_1} \ldots \sum_{t_n=1}^{y_n} v(t_1, \ldots, t_n)
$$

respectively. We denote by $G(A)$ and $G(B)$ the classes of functions $u : A \to R$ and $v : B \to R$ respectively such that

$$
u(0, x_2, \ldots, x_n) = u(x_1, 0, x_3, \ldots, x_n) = \cdots = u(x_1, \ldots, x_{n-1}, 0) = 0,$$

$$
v(0, y_2, \ldots, y_n) = v(y_1, 0, y_3, \ldots, y_n) = \cdots = v(y_1, \ldots, y_{n-1}, 0) = 0.
$$

Or main result is given in the following theorem.

**Theorem 1.** Let $u(x) \in G(A)$ and $v(y) \in G(B)$. Then the following inequality holds
\[
\sum_A \left( \sum_B \frac{|u(x)||v(y)|}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i} \right) \leq \frac{1}{2} \left( \prod_{i=1}^{n} a_i \right)^{1/2} \left( \prod_{i=1}^{n} b_i \right)^{1/2}.
\]

(1)

A slightly different version of Theorem 1 is embodied in the following theorem.

**Theorem 2.** Let \( u(x) \in G(A) \) and \( v(y) \in G(B) \). Then the following inequality holds

\[
\sum_A \left( \sum_B \frac{|u(x)||v(y)|}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i} \right) \leq \frac{1}{4} \left( \prod_{i=1}^{n} a_i \right)^{1/2} \left( \prod_{i=1}^{n} b_i \right)^{1/2}.
\]

(2)

3. Proofs of Theorems 1 and 2. From the hypotheses, for any \( u(x) \in G(A), v(y) \in G(B) \), we have the following identities

\[
u(x) = \sum_{A_x} \nabla_1 \ldots \nabla_n u(s),
\]
(4) \[ v(y) = \sum_{B_y} \nabla_1 \cdots \nabla_n v(t). \]

From (3), (4) and using the Schwarz inequality in summation form we observe that

(5) \[ |u(x)| \leq \left( \prod_{i=1}^{n} x_i \right)^{1/2} \left( \sum_{A_x} |\nabla_1 \cdots \nabla_n u(s)|^2 \right)^{1/2}, \]

(6) \[ |v(x)| \leq \left( \prod_{i=1}^{n} y_i \right)^{1/2} \left( \sum_{B_y} |\nabla_1 \cdots \nabla_n v(s)|^2 \right)^{1/2}, \]

From (5), (6) and using the elementary inequality \( c^{1/2}d^{1/2} \leq \frac{1}{2} (c + d) \) (for \( c, d \) nonnegative reals) and rewriting we observe that

(7) \[ \frac{|u(x)||v(y)|}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i} \leq \frac{1}{2} \left( \sum_{A_x} |\nabla_1 \cdots \nabla_n u(s)|^2 \right)^{1/2} \cdot \left( \sum_{B_y} |\nabla_1 \cdots \nabla_n v(t)|^2 \right)^{1/2}, \]

for \( x \in G, y \in B \). Summing both sides of (7) first over \( B \) and then summing both sides of the resulting inequality over \( A \) and using the Schwarz inequality and then interchanging the order of summations (see [2], [5]) we observe that

\[ \sum_{A} \left( \sum_{B} \frac{|u(x)||v(y)|}{\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i} \right) \leq \frac{1}{2} \left( \sum_{A} \left( \sum_{A_x} |\nabla_1 \cdots \nabla_n u(s)|^2 \right)^{1/2} \right) \cdot \left( \sum_{B} \left( \sum_{B_y} |\nabla_1 \cdots \nabla_n v(t)|^2 \right)^{1/2} \right) \leq \]
\[
\frac{1}{2} \left( \prod_{i=1}^{n} a_i \right)^{1/2} \left( \sum_A \left( \sum_{A_x} |\nabla_1 \ldots \nabla_n u(s)|^2 \right) \right)^{1/2} \cdot \\
\cdot \left( \prod_{i=1}^{n} b_i \right)^{1/2} \left( \sum_B \left( \sum_{B_y} |\nabla_1 \ldots \nabla_n v(t)|^2 \right) \right)^{1/2} = \\
\frac{1}{2} \left( \prod_{i=1}^{n} a_i \right)^{1/2} \left( \prod_{i=1}^{n} b_i \right)^{1/2} \cdot \\
\cdot \left( \sum_A \prod_{i=1}^{n} (a_i - x_i + 1) |\nabla_1 \ldots \nabla_n u(x)|^2 \right)^{1/2} \cdot \\
\cdot \left( \sum_B \prod_{i=1}^{n} (b_i - y_i + 1) |\nabla_1 \ldots \nabla_n v(y)|^2 \right)^{1/2}.
\]

This is the required inequality in (1) and the proof of Theorem 1 is complete. By using the elementary inequality \( c^{1/2} d^{1/2} \leq \frac{1}{2} (c + d) \) (for \( c, d \) non-negative reals) to the last two factors on the right side of (8) we get the desired inequality in (2) and the proof of Theorem 2 is complete.

In concluding, we note that the analysis used in the proof of Theorems 1 and 2 is quit elementary and the bounds obtained in (1) and (2) are new on this type of inequalities and can not be compared with the bound given in Hilbert’s inequality. For some recent results on Hilbert type inequalities, see [4], [6].

REFERENCES


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