THE SECOND FUNDAMENTAL FORM OF A COMPLEX DISTRIBUTION

BY

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Abstract. We define the second fundamental form of a subbundle of the complexified tangent bundle of a manifold, and write down corresponding Gauss-Codazzi-Ricci formulas. Then, we compute the second fundamental form of the $(-\sqrt{-1})$-eigendistribution of an almost Hermitian and an $f$-metric manifold, and show its relationship with the Nijenhuis tensor of these structures.

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The aim of this note is to point out an unexpected relationship between two types of geometric invariants, the second fundamental form and the Nijenhuis tensor, which are usually seen as belonging to different geometric realms. The second fundamental form of a real distribution was defined by Reinhart [7], [8]. We define the second fundamental form of a complex distribution, and write down the corresponding Gauss-Codazzi-Ricci formulas. Then, we compute the second fundamental form of the $(-\sqrt{-1})$-eigendistribution of an almost Hermitian and an $f$-metric manifold, and show that it contains the information usually derived from the Nijenhuis tensors of these structures.

1. General formulas and definitions. Let $M$ be an $m$-dimensional Riemannian manifold\footnote{"Everything" is of the class $C^\infty$ in this note.} with metric tensor $g$, and let $T^cM = TM \otimes_{\mathbb{R}} \mathbb{C}$ be its complexified tangent bundle. We denote by $h$ the Hermitian metric
defined on $T^cM$ by

$$h(X, Y) = g(X, \bar{Y}) \quad (X, Y \in \Gamma T^cM)$$  

(the bar denotes complex conjugation, and $\Gamma$ denotes spaces of global cross sections of bundles).

Let $S$ be a $p$-dimensional distribution (i.e., subbundle) of $T^cM$ and denote by $S^\perp$ the $h$-orthogonal distribution of $S$ and by

$$S : T^cM \to S, \quad S^\perp : T^cM \to S^\perp$$

the corresponding orthogonal projections.

Let $\nabla$ be the Levi-Civita connection of $g$, also extended by complex linearity to $T^cM$, and

$$\tilde{\nabla}_X Y = \mathcal{S}\nabla_X \mathcal{S}Y + S^\perp\nabla_X S^\perp Y$$

the metric product connection [8]. $\tilde{\nabla}$ satisfies the conditions

$$\tilde{\nabla}S = 0, \tilde{\nabla}S^\perp = 0, \tilde{\nabla}h = 0,$$

and has the torsion

$$T_{\tilde{\nabla}}(X, Y) = S(\nabla_Y S^\perp X - \nabla_X S^\perp Y) + S^\perp(\nabla_Y \mathcal{S}X - \nabla_X \mathcal{S}Y).$$

Let $a$ be the difference tensor of the two connections defined by

$$\nabla_X Y = \tilde{\nabla}_X Y + a(X, Y), \quad X, Y \in \Gamma T^cM.$$  

The, we may mimic the classical theory of submanifolds as follows. The equations

$$\begin{cases} \nabla_{S\mathcal{S}} \mathcal{S}Y = \tilde{\nabla}_{S\mathcal{S}} \mathcal{S}Y + a(S\mathcal{S}, \mathcal{S}Y), & a(S\mathcal{S}, \mathcal{S}Y) \in \Gamma S^\perp, \\ \nabla_{S\mathcal{S}} S^\perp Y = a(S\mathcal{S}, S^\perp Y) + \tilde{\nabla}_{S\mathcal{S}} S^\perp Y, & a(S\mathcal{S}, S^\perp Y) \in \Gamma S \end{cases}$$

will be the Gauss-Weingarten equations, and (slightly different from [8]) the tensor field

$$\alpha_{S}(X, Y) = a(S\mathcal{S}, \mathcal{S}Y) = S^\perp \nabla_{S\mathcal{S}} \mathcal{S}Y \quad (X, Y \in \Gamma T^cM)$$

will be the second fundamental form of $S$. If $S$ is tangent to a real foliation, $\alpha$ restricted to the leaves provides the usual second fundamental form of
the leaves seen as submanifolds of $M$. For the same reason, the operators $W_Y \in \text{End}_{\mathbb{C}} T^c M$ defined by

$$W_Y X := -a(SX, S^\perp Y) \quad (X, Y \in \Gamma T^c M)$$

will be called the Weingarten operators for $S$, and the metric character of $\tilde{\nabla}$ implies

$$h(W_Z X, Y) = h(Z, \alpha(X, Y)).$$

Now, notice that

$$a(X, Y) - a(Y, X) = -T_{\tilde{\nabla}}(X, Y).$$

This implies that the second fundamental form of $S$ is symmetric iff $S$ is an involutive distribution, and, in this case, $\tilde{\nabla}|_S$ may be seen as a metric, torsionless (i.e., Riemannian) connection along $S$.

The curvature tensors $R$ of the connections $\nabla$ and $\tilde{\nabla}$ are related by the formula

$$R_{\nabla}(X, Y)Z = R_{\tilde{\nabla}}(X, Y)Z + (\tilde{\nabla}_X a)(Y, Z) - (\tilde{\nabla}_Y a)(X, Z)$$

$$+ a(X, a(Y, Z)) - a(Y, a(X, Z)) + a(T_{\tilde{\nabla}}(X, Y), Z),$$

where $X, Y, Z \in \Gamma T^c M$. If we also consider the covariant curvature tensors

$$R(U, Z, X, Y) = h(R(X, Y)Z, U), \quad X, Y, Z, U \in T^c M,$$

and use (9), (10), (11) we get the Gauss formula

$$R_{\nabla}(SU, SZ, SX, SY) = R_{\tilde{\nabla}}(SU, SZ, SX, SY)$$

$$+ h(\alpha(X, Z), \alpha(Y, U)) - h(\alpha(Y, Z), \alpha(X, U))$$

$$- h(T_{\nabla}(SZ, SY), \alpha(Z, U)) + h(T_{\tilde{\nabla}}(SZ, T_{\tilde{\nabla}}(SX, SY)), SU),$$

the Codazzi formula

$$R_{\nabla}(S^\perp U, SZ, SX, SY) = h((\bar{\nabla}_{SX}a)(SY, SZ), S^\perp U)$$

$$- h((\bar{\nabla}_{SY}a)(SX, SZ), S^\perp U)$$

$$+ h(T_{\nabla}(SZ, T_{\nabla}(SX, SY)), S^\perp U),$$
the Ricci formula

\[ R_{\nabla}(S^\perp U, S^\perp Z, SX, SY) = R_{\tilde{\nabla}}(S^\perp U, S^\perp Z, SX, SY) \]

\[-h(\alpha(X, W_Z Y), U) + h(\alpha(Y, W_Z X), U),\]

and the second Codazzi formula (expressed in terms of \(a\), for simplicity, and which in the general case is not the same as the first Codazzi formula written above)

\[ R_{\nabla}(SU, S^\perp Z, SX, SY) = h((\tilde{\nabla}_{SX}a)(SY, S^\perp Z), SU) \]

\[-h((\tilde{\nabla}_{SY}a)(SX, S^\perp Z), SU) + h(a(T_{\nabla}(SX, SY), S^\perp Z), SU).\]

The previous formulas are interesting in principle, because they extend classical notions to a more general situation. But, these formulas cannot be used for integrability purposes. Integrability may be discussed by the classical moving frame method. Namely, choose local \(h\)-unitary bases \((S_i), (S^\perp_i)\) of \(S, S^\perp\), respectively, and define corresponding connection coefficients \(\Gamma_\cdot_\cdot, \tilde{\Gamma}_\cdot_\cdot\) for \(\nabla, \tilde{\nabla}\) and components \(a_\cdot\) of \(a\). Then (12) yields some necessary conditions for the local functions \(\Gamma_\cdot_\cdot, \tilde{\Gamma}_\cdot_\cdot, a_\cdot\). Now, if \(\nabla\) is flat, (6) becomes a system of partial differential equations with the unknown functions \((S_i), (S^\perp_i)\), and (12) expressed in terms of \(\Gamma_\cdot_\cdot, \tilde{\Gamma}_\cdot_\cdot, a_\cdot\) gives the integrability conditions of this system. Therefore, given a system of local functions \(\Gamma_\cdot_\cdot, \tilde{\Gamma}_\cdot_\cdot, a_\cdot\) which satisfy (12), there exists a local complex distribution \(S\) with local bases \((S_i)(S^\perp_i)\), defined up to an isometry, and such that the given functions are the connection coefficients and the components of the difference tensor with respect to these bases.

As in the classical theory of submanifolds, the second fundamental form also provides a mean curvature vector

\[ H = H(S) = \text{tr} \alpha = \sum_{i=1}^{m} \alpha(e_i, e_j) = \sum_{i=1}^{m} \alpha(f_i, \bar{f}_j) \in \Gamma S^\perp, \]

where \((e_i)_{i=1}^{m}\) is an arbitrary real, \(g\)-orthonormal, local basis of \(TM\), and \((f_i)_{i=1}^{m}\) is an arbitrary, \(h\)-unitary, local basis of \(T^c M\). If \(S\) is a real \(p\)-dimensional foliation, at each point \(x \in M\), \(H_x\) is \(p\) times the usual mean curvature vector of the leaf of \(S\) through \(x\).
The relation between $H(S)$ and the Weingarten operators is provided by the $h$-dual 1-form $\kappa$ of $H$. Namely, from (10) we get

$$(19) \quad \kappa(Z) = \sum_{i=1}^{m} h(W_Z e_i, e_i) = \sum_{i=1}^{m} h(W_Z f_i, \bar{f}_i).$$

Accordingly, we will say that the complex distribution $S \subseteq T^cM$ is \textit{totally geodesic} if its second fundamental form vanishes and it is \textit{minimal} if $H(S) = 0$. If there exists a vector field $U \in \Gamma S^\perp$ such that

$$(20) \quad \alpha(X, Y) = g(SX, SY)U,$$

the distribution $S$ is \textit{totally umbilical}. In (20) $g$ was extended to $T^cM$ by complex linearity, and the vector $U$ must be proportional to $H$. A totally umbilical distribution $S$ must be involutive.

We end this section by indicating the fact that complex distribution are equivalent to interesting real geometric structures. A nice example is that of a \textit{coisotropic} distribution $S \subseteq T^cM$ of codimension one, where by \textit{coisotropic} we understand that the $h$-orthogonal distribution is $g$-isotropic. Let $\xi$ be a local field of unit normal vectors of $S$. Then we have

$$(21) \quad h(\xi, \xi) = 1, \quad g(\xi, \xi) = 0, \quad h(X, \xi) = 0, \quad \forall X \in S.$$

It follows that, while $\xi$ does not belong to $S$, $\bar{\xi} \in \bar{S}$, and $S \neq \bar{S}$. Therefore, $S + \bar{S} = T^cM$, and there exists a real $(m - 2)$-dimensional distribution $D \subseteq TM$ such that $D^c = S \cap \bar{S}$. Moreover, the $g$-orthogonal distribution $D^\perp$ is span\{\xi, $\bar{\xi}$\}, and has a well defined $g$-compatible complex structure $J$, such that $J\xi = \sqrt{-1}\xi$, which does not depend on the choice of $\xi$. Equivalently, $D^\perp$ has a well defined orientation. It is obvious that we can reconstruct $S$ from the real distribution $D^\perp$ of oriented 2-planes. Hence, $S$ is equivalent with a cross section of the Grassmannian bundle of oriented 2-planes. Notice also that $S$ is involutive iff $D$ is a foliation.

A general complex distribution $S$ produces two real distributions, possibly with singularities, $D \subseteq \Delta \subseteq TM$ such that

$$(22) \quad S \cap \bar{S} = D^c = D \otimes_{\mathbb{R}} \mathbb{C}, \quad S + \bar{S} = \Delta^c = \Delta \otimes_{\mathbb{R}} \mathbb{C}.$$
which may be seen as consisting of the \((\mp \sqrt{-1})\)-eigenspaces of a well defined complex structure \(\Psi (\Psi^2 = -Id)\) on \(\Delta/D\). Conversely, the triple \((D, \Delta, \Psi)\) defines
\[
S = \{X = X_1 + \sqrt{-1}X_2 \mid X_1, X_2 \in \Delta, \Psi[X_1]_D = [X_2]_D\}.
\]
Thus, the real interpretation of a complex distribution \(S\) is that of a triple \((D, \Delta, \Psi)\) [9].


Let \(M\) be a \(2n\)-dimensional almost Hermitian manifold, i.e., a differentiable manifold endowed with a reduction of the structure group of its tangent bundle to the unitary group \(U(n)\). Equivalently, \(M\) is a manifold with an almost complex structure tensor \(J \in \Gamma\text{End}(TM)\), \(J^2 = -Id\), and a metric \(g\) such that
\[
g(JX, JY) = g(X, Y), \quad X, Y \in \Gamma TM.
\]
On this manifold we also have the Kähler form \(\Omega\) given by
\[
\Omega(X, Y) = g(JX, Y),
\]
and the Nijenhuis tensor
\[
\]
The almost complex structure \(J\) is integrable or complex if it is defined by local complex analytic coordinates on \(M\) and, if this happens \((M, J, g)\) is a Hermitian manifold. Integrability of \(J\) is equivalent to \(N = 0\) [5].

In \(T^c M\), \(J\) has the eigenvalues \(\mp \sqrt{-1}\), with corresponding \(n\)-dimensional eigendistributions \(S\) and \(S^\perp\), which are orthogonal with respect to the Hermitian metric \(h\) defined by (1). It is usual to refer to \(S\) as the anti-holomorphic distribution and to \(S^\perp\) as the holomorphic distribution. The orthogonal projections onto these distributions are
\[
S = \frac{1}{2}(Id + \sqrt{-1}J), \quad S^\perp = \frac{1}{2}(Id - \sqrt{-1}J).
\]
Involutivity of \(S\) or \(S^\perp\) is equivalent with the integrability of \(J\). The metric product connection \(\tilde{\nabla}\) is real, and given by
\[
\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)(JY).
\]
Proposition 2.1. The second fundamental form \( \alpha_S \) of the anti-holomorphic distribution \( S \) of the almost Hermitian manifold \((M, J, g)\) is given by the formulas

\[
\begin{align*}
\text{Im}(\alpha_S(X, Y)) &= \text{Re}(\alpha_S(X, JY)), \\
g(\text{Re}(\alpha_S(X, Y)), Z) &= \frac{1}{16} \{g(N(X, Y), Z) + g(N(Y, JZ), JX) \nonumber \nonumber \\
&\quad - g(N(JZ, X), JY)\},
\end{align*}
\]

where \( X, Y, Z \in \Gamma TM \).

Proof. From (8), and with a straightforward computation, one gets

\[
\alpha_S(X, Y) = \frac{1}{8} \{\nabla_X Y - \nabla_J X J Y + J \nabla_J X Y \\
+ J \nabla_J X Y + \sqrt{-1}(\nabla_J X Y + \nabla_X J Y - J \nabla_X Y \\
+ J \nabla_J X J Y)\} = -\frac{1}{8}([\nabla_X J](Y) + (\nabla_J X J)(Y) \\
- \sqrt{-1}[(\nabla_X J)(Y) - (\nabla_J X J)(JY)]\}.
\]

The first formula (30) is a direct consequence of (31). The second formula (30) follows by using a result of [4] namely,

\[
\mathcal{E}(J) := g(N(X, Y), JZ) - g(N(Y, Z), JX) + g(N(Z, X), JY) = 2g(\nabla_X J)(Y) - (\nabla_J X J)(JY), Z.
\]

This result can be checked by inserting in (32) the expression (27) of \( N \), and by using

\[
\nabla g = 0, \ [X, Y] = \nabla_X Y - \nabla_Y X.
\]

If we apply \( J \) to the two arguments of \( g \) of the right hand side of (32), and look at (31), we see that (32) means

\[
\mathcal{E}(J) = 16g(\text{Re}(\alpha_S(X, Y), JZ).
\]

Q.e.d.
Technical computations give the following equivalent form of the second formula (30)

\[ g(\text{Re}(\alpha_S(X,Y)),Z) = -\frac{1}{8}\{\langle \nabla JX\Omega(Y,Z) + \langle \nabla X\Omega(JY,Z) \rangle \}. \]

Formula (31) shows that

\[ \alpha_S(JX,JY) = -\alpha_S(X,Y), \]

and also yields the following interpretation of the Nijenhuis tensor

\[ N = 16\text{Re}(\text{alt}(\alpha_S)). \]

Another interesting tensor that should be considered is

\[ Q = \text{Re}(\text{sym}(\alpha_S)), \quad (Q(X,Y) = Q(Y,X)). \]

The almost Hermitian manifolds which satisfy the condition \( Q = 0 \) are exactly those in the Grey-Hervella class \( G_1 \) [4].

Finally, we notice the

**Corollary 2.1.** The mean curvature vector \( H(S) \) of the anti-holomorphic distribution of any almost Hermitian manifold is zero. The second fundamental form of \( S \) vanishes iff the manifold is Hermitian.

**Proof.** If we use the definition (18) of \( H(S) \) with an orthonormal basis of the form \((e_i, Je_i)_{i=1}^n\), we get \( H(S) = 0 \), because of (36). Formulas (37) and (30) show that \( \alpha_S = 0 \) is equivalent with \( N = 0 \). Q.e.d.

3. Metric \( f \)-structures. A generalization of the situation discussed in the previous section is obtained if we consider \( m \)-dimensional manifolds \( M \) endowed with a reduction of the structure group of the tangent bundle to \( U(p) \times O(q) \ (2p + q = m) \). In [1] such manifolds were called hor-Ehresmannian manifolds, and in [10] metric \( f \)-manifolds.

In this situation the tangent bundle \( TM \) decomposes into a horizontal and a vertical part, \( TM = H^{2p} \oplus V^q \), with corresponding projectors \( \mathcal{H}, \mathcal{V} \), and there exists a complex structure \( J \) of the bundle \( H \). This allows to define

\[ F = J \oplus 0 \in \text{End}(TM), \]
and the following relations hold

\[ F^3 + F = 0, \quad F \circ H = H \circ F = F, \quad F \circ V = V \circ F = 0, \]
\[ H = -F^2, \quad V = Id + F^2. \]

A \((1,1)\)-tensor field \(F\) which satisfies the first relation (40) is called an \(f\)-structure. If such a structure is given, the last two relations (40) define complementary projectors \(H, V\), \(F\) has the eigenvalues \(0, \pm \sqrt{-1}\) with the corresponding eigendistributions \(V, S, S'\), where \(H \otimes_{\mathbb{R}} C = S \oplus S'\), and this decomposition defines a complex structure \(J\) on \(H\). The projections onto the distributions \(S, S', V\) are

\[ (40) \quad S = \frac{1}{2}(H + \sqrt{-1}F), \quad S' = \frac{1}{2}(H - \sqrt{-1}F), \quad V. \]

Now, back to the manifold \(M\) with the \(U(p) \times O(q)\)-structure, since \(U(p) \times O(q)\) is an orthogonal group, the manifold \(M\) also is endowed with a Riemannian metric \(g\) such that the distributions \(H\) and \(V\) are orthogonal, and \(g|_H\) is compatible with the complex structure \(J\) of \(H\). It is easy to see that these conditions are equivalent to

\[ (41) \quad g(X, FY) + g(FX, Y) = 0, \quad X, Y \in \Gamma TM. \]

Accordingly, a metric \(f\)-manifold is a triple \((M, F, g)\) which satisfies the first condition (40) and condition (41).

The tensor \(F\) has a Nijenhuis tensor

\[ (42) \quad N_F(X, Y) = H[X, Y] + F[FX, Y] + F[X, FY] - [FX, FY], \]

and \(N_F = 0\) is equivalent with the integrability of the distributions \(V\) and \(H\), and of the complex structure \(J\) along \(H\) \([10] \).

We fix our attention on the complex distribution \(S\), and compute its second fundamental form. The \(h\)-orthogonal distribution of \(S\) (recall that \(h\) is defined by (1)), is \(S^\perp = S' \oplus V\), and the required projections are \(S, S' + V\) defined by (40). The corresponding metric product connection is not real, and has the expression

\[ (43) \quad \tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}[F \circ (\nabla_X F) - V \circ (\nabla_X V)](Y) \]
\[ - \frac{\sqrt{-1}}{2}[F \circ (\nabla_X V) - (\nabla_X V) \circ F](Y). \]
The real part of $\nabla$ also is a connection which preserves the tensors $F$ and $g$.

**Proposition 3.1.** The second fundamental form $\alpha_S$ of the anti-holomorphic distribution $S$ of the metric $f$-manifold $(M, F, g)$ is given by the formulas

$$
\begin{align*}
\text{Im}(\alpha_S(X,Y)) &= \text{Re}(\alpha_S(X,FY)), \\
g(\text{Re}(\alpha_S(X,Y)), Z) &= \frac{1}{16} \{ g(N_F(HX, HY), HZ) \\
&\quad + g(N_F(HY, FZ), FX) \\
&\quad - g(N_F(FZ, HX), FY) \\
&\quad + 4g(\nabla_{HX}HY - \nabla_{FY}FX, VZ) \},
\end{align*}
$$

where $X, Y, Z \in \Gamma TM$.

**Proof.** From (8) and (40), we get

$$
\text{Re}(\alpha_S(X,Y)) = \frac{1}{8} \{(Id + V)(\nabla_{HX}HY - \nabla_{FY}FX) \\
+ F(\nabla_{FY}HX + \nabla_{HX}FY) \},
$$

and, also, the first relation (44).

From (45) we get

$$
V(\text{Re}(\alpha_S(X,Y))) = \frac{1}{4} V(\nabla_{HX}HY - \nabla_{FY}FX),
$$

and this justifies the last term of the second formula (44).

Furthermore, (32) suggests to compute the expression

$$
\mathcal{E}(F) = g(N_F(HX, HY), FZ) - g(N_F(HY, HZ), FX) \\
+ g(N_F(HZ, HX), FY).
$$

If we use the definition (42) of $N_F$, express the Lie brackets by (33), and use $\nabla g = 0$, as well as condition (41) and its consequence

$$
g(FX, FY) = g(HX, HY),
$$

and $V \perp H$, we get

$$
\mathcal{E}(F) = 16g(\text{Re}(\alpha_S(X,Y)), FZ).
$$
Using this result for $Z \mapsto -FZ$, we get the full justification of the second formula (44). Q.e.d.

From (45) it follows that

\begin{equation}
\text{alt}(\text{Re}(\alpha_S(X, Y))) = \frac{1}{16} \{ \mathcal{H}N_F(\mathcal{H}X, \mathcal{H}Y) \\
+ 2\mathcal{V}(\mathcal{H}X, \mathcal{H}Y) - [FX, FY] \},
\end{equation}

and (50) shows that the integrability of the structure $F$ i.e., $N_F = 0$, implies the symmetry of $\alpha$, therefore, the involutivity of $S$ but, the converse implication is not true.

As a consequence of Proposition 3.1 we get

**Corollary 3.1.** The anti-holomorphic distribution $S$ of a metric $f$-manifold is totally geodesic iff

\begin{equation}
\mathcal{H}N_F(\mathcal{H}X, \mathcal{H}Y) = 0, \quad \mathcal{V}(\mathcal{H}X, \mathcal{H}Y) = 0.
\end{equation}

Furthermore, $S$ always is minimal, and it is totally umbilical iff it is totally geodesic.

**Proof.** The first assertion is an immediate consequence of (50) and of the second formula (44). Then, formula (45) and the first formula (44) show that $\alpha_S(X, Y)$ vanishes if at least one of the arguments is vertical, and that one has

\begin{equation}
\alpha_S(FX, FY) = -\alpha_S(X, Y).
\end{equation}

Finally, if we compute the mean curvature $H(S)$ using in (18) an orthonormal frame $(e_i, Fe_i, e_u)$, where $e_i \in H$, $e_u \in V$, $i = 1, \cdots, p, u = 2p + 1, \cdots, 2p + q$, we get $H(S) = 0$. This proves the second and third assertions. Q.e.d.

**Corollary 3.2.** The anti-holomorphic distribution of a metric $f$-manifold is involutive iff

\begin{equation}
\mathcal{H}N_F(\mathcal{H}X, \mathcal{H}Y) = 0, \quad \mathcal{V}(\mathcal{H}X, \mathcal{H}Y) - [FX, FY] = 0,
\end{equation}

$(X, Y \in \Gamma TM)$. If this happens, the second fundamental form of $S$ is given by the formulas

\begin{equation}
\begin{cases}
\text{Im}(\alpha_S(X, Y)) = \text{Re}(\alpha_S(X, FY)), \\
g(\text{Re}(\alpha_S(X, Y)), Z) = \frac{1}{4}g(\nabla_{\mathcal{H}X} \mathcal{H}Y - \nabla_{FX} FY, YZ).
\end{cases}
\end{equation}
Proof. The first assertion follows by asking $\alpha_S$ to be symmetric. The second assertion follows from the first and formulas (44). Q.e.d.

Formulas (54) can also be interpreted as follows. The real distribution $H$ has a second fundamental form $\alpha_H$ defined by the real version of the general formulas of Section 1. Namely (see (8)),

\[
\alpha_H(X, Y) = \nabla_{\alpha_X}H_Y, \quad X, Y \in \Gamma^cT^cM.
\]

In particular, using (40), (54) we get

\[
\begin{align*}
\alpha_H(SX, SY) &= 2(\alpha_S(X, Y) + \sqrt{-1}\alpha_S(X, FY)), \\
\alpha_H(S'X, S'Y) &= 2(\alpha_S(X, Y) - \sqrt{-1}\alpha_S(X, FY)).
\end{align*}
\]

From (56) we deduce

Proposition 3.2. If the distribution $S$ of a metric $f$-manifold is involutive, its second fundamental form is related to the second fundamental form of the distribution $H$ by the formula

\[
\alpha_S(X, Y) = \frac{1}{4}[\alpha_H(SX, SY) + \alpha_H(S'X, S'Y)].
\]

An important case of metric $f$-manifolds is that of the (almost) contact metric manifolds [2]. Another interesting particular case, where (54) and Proposition 3.2 may be used, is that of a metric compatible CR-structure $S$ on a Riemannian manifold $(M, g)$. Recall that a CR-structure is an involutive complex distribution $S$ such that $S \cup \bar{S} = \{0\}$ [3]. Then, $S \oplus \bar{S}$ is the complexification of a real distribution $H$, which has a well defined complex structure $J$, characterized by the fact that its $(-\sqrt{-1})$-eigendistribution is $S$. By the metric compatibility of $S$ we understand that $g|_H$ is $J$-compatible. (Then, one may also say that $S$ is a metric subordinated CR-structure of $(M, g)$, or that $(M, g, S)$ is a Riemannian CR-manifold). If we also consider the vertical distribution $V$ defined as the $g$-orthogonal complement of $H$, we obtain a metric $f$-structure on $M$, which may be used in the computation
of the second fundamental form of $S$.

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