A TYPE OF PSEUDO PROJECTIVE $\varphi$-RECURRENT TRANS-SASAKIAN MANIFOLD

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Abstract. In this paper we have studied a special type of trans-Sasakian manifold which is pseudo projective $\varphi$-recurrent.

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1. Introduction. The notion of locally $\varphi$-symmetric Sasakian manifold was introduced by Takahashi [8] in 1977. $\varphi$-recurrent Sasakian manifold and pseudo projective curvature tensor on a Riemannian manifold were studied by the author [3] and [2] respectively.

Also J.A. Oubina in 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [1], $\alpha$-Sasakian [4], Kenmotsu [4], $\beta$-Kenmotsu [4] and cosymplectic [1] manifolds, which was called trans-Sasakian manifold [6]. After him many authors [5], [7], [9], have studied various type of properties in trans-Sasakian manifold. In this paper we have studied pseudo projective $\varphi$-recurrent trans-Sasakian manifold which satisfies $\varphi(\text{grad } \alpha) = (2n - 1) \text{ grad } \beta$, and found that the manifold is an Einstein manifold.

Further it is proved that in a pseudo projective $\varphi$-recurrent trans-Sasakian manifold $(M^{2n+1}, g)$, $n \geq 1$, the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-form are opposite directional.

2. Preliminaries. $A(2n + 1)$ dimensional, $(n \geq 1)$ almost contact metric manifold $M$ with almost contact structure $(\varphi, \xi, \eta, g)$, where $\varphi$ is a
(1,1) tensor field, \( \xi \) is a vector field, \( \eta \) is a 1-form and \( g \) is a compatible Riemannian metric such that,

\[
(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0
\]

\[
(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]

\[
(2.3) \quad g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X),
\]

for all \( X, Y \in TM \), is called trans-Sasakian manifold [6] if and only if

\[
(2.4) \quad (\nabla_X \phi)Y = \alpha (g(X, Y)\xi - \eta(Y)X) + \beta (g(\phi X, Y)\xi - \eta(Y)\phi X),
\]

for some smooth functions \( \alpha \) and \( \beta \) on \( M \).

From (2.4) it follows that

\[
(2.5) \quad \nabla_X \xi = -\alpha \phi X + \beta (X - \eta(X)\xi)
\]

\[
(2.6) \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).
\]

In [9], DE and Tripathi obtained some results which shall be useful for next sections. They are

\[
(2.7) \quad R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y)
\]

\[+ (Y \alpha)\phi X - (X \alpha)\phi Y + (Y \beta)\phi^2 X - (X \beta)\phi^2 Y
\]

\[
(2.8) \quad R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi \beta)(\eta(X)\xi - X)
\]

\[
(2.9) \quad 2\alpha\beta + \xi \alpha = 0
\]

\[
(2.10) \quad S(X, \xi) = (2n(\alpha^2 - \beta^2) - \xi \beta)\eta(X) - (2n - 1)X\beta - (\phi X)\alpha
\]

\[
(2.11) \quad Q\xi = (2n(\alpha^2 - \beta^2) - \xi \beta)\xi - (2n - 1)\text{grad} \beta + \varphi(\text{grad} \alpha).
\]

When \( \varphi(\text{grad} \alpha) = (2n - 1)\text{grad} \beta \), then (2.10) and (2.11) reduces to

\[
(2.12) \quad S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X),
\]
Again a trans-Sasakian manifold is said to be locally $\varphi$-symmetric \cite{8} if
\begin{equation}
2\varphi^2((\nabla W R)(X,Y)Z) = 0,
\end{equation}
for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

A trans-Sasakian manifold is said to be pseudo projective $\varphi$-recurrent manifold if there exist a non-zero 1-form $A$ such that
\begin{equation}
\varphi^2((\nabla W P)(X,Y)Z) = A(W)\overline{P}(X,Y)Z,
\end{equation}
for $X, Y, Z, W \in TM$, where $P$ is a pseudo projective curvature tensor given by \cite{2}
\begin{equation}
\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y]
\end{equation}
\begin{equation}
-\frac{r}{2n+1}[\frac{a}{2n} + b][g(Y,Z)X - g(X,Z)Y].
\end{equation}
Where $a, b$ are constants such that $a, b \neq 0$, $R$ is the curvature tensor, $S$ is the Ricci-tensor and $r$ is the scalar curvature.

Also,
\begin{equation}
g(QX, Y) = S(X, Y)
\end{equation}
$Q$ being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor $S$.

The 1-form $A$ is defined as
\begin{equation}
g(X, \rho) = A(X), \forall X \in TM.
\end{equation}
$\rho$ being the vector field associated to the 1-form $A$.

The above results will be useful in the next section.

3. Pseudo Projective $\varphi$-recurrent trans-Sasakian manifold.
In this section we consider a trans-Sasakian manifold which is pseudo-projective $\varphi$-recurrent. Then by virtue of (2.15) and (2.1)we have
\begin{equation}
-(\nabla W \overline{P})(X,Y)Z + \eta((\nabla W \overline{P})(X,Y)Z)\xi = A(W)\overline{P}(X,Y)Z.
\end{equation}
From (3.1) it follows that
\begin{equation}
-g((\nabla W \overline{P})(X,Y)Z, U) + \eta((\nabla W \overline{P})(X,Y)Z)\eta(U) = A(W)g(\overline{P}(X,Y)Z, U).
\end{equation}
Let \( \{ e_i \} \), \( i = 1, 2, \ldots, 2n + 1 \), be an orthonormal basis of the tangent space at any point of the manifold. Then putting \( X = U = e_i \) in (3.2) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get

\[
(3.3) \quad (\nabla_W S)(Y, Z) = A(W)[S(Y, Z) - \frac{r}{2n+1} g(Y, Z)].
\]

Replacing \( Z \) by \( \xi \) and using (2.3) and (2.12) we have

\[
(3.4) \quad (\nabla_W S)(Y, \xi) = A(W)[2n(\alpha^2 - \beta^2) - \frac{r}{2n+1}]\eta(Y).
\]

Now we know \((\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi)\). Using (2.5) and (2.12) in the above relation we get, after a brief calculation

\[
(3.5) \quad (\nabla_W S)(Y, \xi) = 2n(\alpha^2 - \beta^2)[-\alpha g(\varphi W, Y) + \beta g(\varphi X, \varphi Y)] + \alpha S(Y, \varphi W) - S(Y, \beta W) + 2n(\alpha^2 - \beta^2)\eta(Y)\eta(W).
\]

By virtue of (2.2), (3.5) reduces to

\[
(3.6) \quad (\nabla_W S)(Y, \xi) = 2n(\alpha^2 - \beta^2)[-\alpha g(Y, \varphi W) + \beta g(Y, W)] + \alpha S(Y, \varphi W) - \beta S(Y, W).
\]

From (3.4) and (3.6) we have

\[
2n(\alpha^2 - \beta^2)[-\alpha g(Y, \varphi W) + \beta g(Y, W)] + \alpha S(Y, \varphi W) - \beta S(Y, W)
= A(W)[2n(\alpha^2 - \beta^2) - \frac{r}{2n+1}]\eta(Y).
\]

Replacing \( Y \) and \( W \) by \( \varphi Y \) and \( \varphi W \) respectively in (3.7) and then using (2.1), (2.3), (2.12) and (2.17) we obtain

\[
S(Y, W) = 2n(\alpha^2 - \beta^2)g(Y, W).
\]

And

\[
S(\varphi Y, W) = 2n(\alpha^2 - \beta^2)g(\varphi Y, W).
\]

Thus we can state

**Theorem 3.1.** A pseudo projective \( \varphi \)-recurrent trans-Sasakian manifold \((M^{2n+1}, g)\) satisfying \( \varphi (\text{grad} \alpha) = (2n - 1) \text{ grad} \beta \), is an Einstein manifold.
Now from (3.1) and (2.16) we have

\[(3.10)\]
\[
a(\nabla W R)(X, Y)Z = a\eta(\nabla W R)(X, Y)Z - b(\nabla W S)(Y, Z)\eta(X) - bA(W)[S(Y, Z)X - S(X, Z)Y]
\]
\[
+ \frac{r}{2n+1} \left( \frac{a}{2n} + b \right) A(W)[g(Y, Z)X - g(X, Z)Y].
\]

Using Bianchi’s identity in (3.10) we get

\[(3.11)\]
\[
aA(W)\eta(R(X, Y)Z) + aA(X)\eta(R(Y, W)Z) + aA(Y)\eta(R(W, X)Z)
\]
\[
= bA(W)[S(X, Z)\eta(Y) - S(Y, Z)\eta(X)]
\]
\[
- \frac{r}{2n+1} \left( \frac{a}{2n} + b \right) A(W)[g(Y, Z)\eta(Y) - g(Y, Z)\eta(X)]
\]
\[
+ bA(X)[S(Y, Z)\eta(W) - S(W, Z)\eta(Y)]
\]
\[
- \frac{r}{2n+1} \left( \frac{a}{2n} + b \right) A(X)[g(Y, Z)\eta(W) - g(W, Z)\eta(Y)]
\]
\[
+ bA(Y)[S(W, Z)\eta(X) - S(X, Z)\eta(W)]
\]
\[
- \frac{r}{2n+1} \left( \frac{a}{2n} + b \right) A(Y)[g(W, Z)\eta(X) - g(X, Z)\eta(W)].
\]

Putting \(Y = Z = \{e_i\}\), where \(e_i\) be an orthonormal basis of the tangent space at any point of the manifold, in (3.11) and taking summation over \(i, 1 \leq i \leq 2n+1\), we get

\[(3.12)\]
\[
A(W) + A(X) = -\frac{1}{2n} \left( A(X)\eta(W) - A(W)\eta(X) \right)
\]

Putting again \(X = \xi\) and using (2.1) and (2.3) we obtain

\[(3.13)\]
\[
A(W) = -\frac{1}{2n-1} \eta(\rho)\eta(W),
\]

for any vector field \(W\) and \(\rho\) being the vector field associated to the 1-form \(A\), defined as (2.18).

From (3.13), we can state the following

**Theorem 3.2.** *In a pseudo projective \(\varphi\)-recurrent trans-Sasakian manifold \((M^{2n+1}, g), n \geq 1*, the characteristic vector field \(\xi\) and the vector field
\( \rho \) associated to the 1-form \( A \) are opposite directional and the 1-form \( A \) is given by

\[
(2n - 1)A(W) = -\eta(\rho)\eta(W), \quad \forall W \in TM.
\]

REFERENCES


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