ON GENERALIZATIONS OF OSTROWSKI TYPE INEQUALITIES AND SOME RELATED RESULTS

BY

B.G. PACHPATTE

Abstract. In this paper some new generalizations of Ostrowski type inequalities involving two functions and related results are given. The analysis used in the proofs is fairly elementary and our results provide new estimates on these types of inequalities.

Mathematics Subject Classification 2000: 26D15, 26D20.

Key words: Ostrowski type inequalities, second derivatives, integral identities, differentiable functions.

1. Introduction. In 1938, Ostrowski [8] (see also [6, p.468]) proved the following integral inequality:

\[ |f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) K, \]

where \( f : [a, b] \to \mathbb{R} \) is a differentiable function such that \( |f'(x)| \leq K \) for all \( x \in [a, b] \).

The inequality (1.1) gives an upper bound for the approximation of the integral average \( \frac{1}{b-a} \int_a^b f(t) \, dt \) by the value \( f(x) \) at \( x \in [a, b] \). This inequality has been generalized over the last years in a number of ways, see [2, 3, 4, 6, 11] and the references cited therein. Recently, Cerone, Dragomir and Roumeliotis [2], Dragomir and Sofo [4] and Dragomir and Barnett [3] have established some new Ostrowski type inequalities for twice differentiable functions. The main purpose of the present paper is to give some new generalizations of the Ostrowski type inequalities given in [2,
3, 4] involving two functions whose second derivatives are bounded. We also present some new inequalities related to the main results. The techniques that will be used in the proofs are elementary and based on the integral identities established in [2, 3, 4].

2. Statement of results. In what follows, $R$ and $'$ denote the set of real numbers and the derivative of a function and $[a, b]$ for $a < b$ be a given subset of $R$. First we give the following notations used to simplify the details of presentations. For a suitable function $h : [a, b] \to R$, we set

\[
L[h(x)] = h(x) - \left(x - \frac{a + b}{2}\right) h'(x),
\]

\[
M[h(x)] = h(x) + \frac{h(a) + h(b)}{2} - \left(x - \frac{a + b}{2}\right) h'(x),
\]

\[
N[h(x)] = h(x) - \frac{h(b) - h(a)}{b - a} \left(x - \frac{a + b}{2}\right).
\]

Our main results are given in the following theorems.

Theorem 1. Let $f, g : [a, b] \to R$ be twice differentiable functions on $(a, b)$ and $f'', g'' : (a, b) \to R$ are bounded, i.e., $\|f''\|_\infty = \sup_{t \in (a, b)} |f''(t)| < \infty$, $\|g''\|_\infty = \sup_{t \in (a, b)} |g''(t)| < \infty$. Then the inequalities

\[
\left| [g(x)L[f(x)] + f(x)L[g(x)] \right| = \frac{1}{b - a} \left[ g(x) \int_a^b f(t) \, dt + f(x) \int_a^b g(t) \, dt \right] \leq \frac{1}{b - a} \left[ |g(x)| \|f''\|_\infty + |f(x)| \|g''\|_\infty \right] E(x),
\]

(2.1)

and

\[
\left| L[f(x)]L[g(x)] - \frac{1}{b - a} \left[ L[g(x)] \int_a^b f(t) \, dt + L[f(x)] \int_a^b g(t) \, dt \right] \right| + \frac{1}{(b - a)^2} \left( \int_a^b f(t) \, dt \right) \left( \int_a^b g(t) \, dt \right).
\]

(2.2)
\[
\leq \frac{1}{(b-a)^2} \left\| f'' \right\|_{\infty} \left\| g'' \right\|_{\infty} E^2(x),
\]
hold for every \( x \in [a, b] \), where

\begin{equation}
E(x) = \int_a^b |k(x, t)| \, dt,
\end{equation}
in which \( k : [a, b]^2 \rightarrow \mathbb{R} \) is given by

\begin{equation}
k(x, t) = \begin{cases} 
\frac{(t-a)^2}{2} & \text{if } t \in [a, x], \\
\frac{(t-b)^2}{2} & \text{if } t \in (x, b].
\end{cases}
\end{equation}

**Theorem 2.** Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be two functions whose first derivatives are absolutely continuous on \([a, b]\) and assume that the second derivatives \( f'', g'' : (a, b) \rightarrow \mathbb{R} \) are bounded on \((a, b)\). Then the inequalities

\begin{equation}
\left| [g(x) M[f(x)] + f(x) M[g(x)]] - \frac{2}{b-a} \left[ g(x) \int_a^b f(t) \, dt + f(x) \int_a^b g(t) \, dt \right] \right|
\leq \frac{1}{b-a} \left[ |g(x)| \left\| f'' \right\|_{\infty} + |f(x)| \left\| g'' \right\|_{\infty} \right] I(x),
\end{equation}

and

\begin{equation}
\left| M[f(x)] M[g(x)] - \frac{2}{b-a} \left[ M[g(x)] \int_a^b f(t) \, dt + M[f(x)] \int_a^b g(t) \, dt \right] \right|
+ \frac{4}{(b-a)^2} \left( \int_a^b f(t) \, dt \right) \left( \int_a^b g(t) \, dt \right)
\leq \frac{1}{(b-a)^2} \left\| f'' \right\|_{\infty} \left\| g'' \right\|_{\infty} f^2(x),
\end{equation}
hold for every \( x \in [a, b] \), where

\begin{equation}
I(x) = \int_a^b |p(x, t)| \left| t - \frac{a+b}{2} \right| \, dt,
\end{equation}
in which \( p : [a, b]^2 \rightarrow \mathbb{R} \) is given by

\begin{equation}
p(x, t) = \begin{cases} 
t - a & \text{if } t \in [a, x], \\
t - b & \text{if } t \in (x, b].
\end{cases}
\end{equation}
Theorem 3. Let $f, g, f''', g''$ be as in Theorem 1. Then the inequalities
\[
\left| g(x) N[f(x)] + f(x) N[g(x)] \right| \leq \frac{1}{(b-a)^2} \left[ |g(x)| \left\| f''\right\|_{\infty} + |f(x)| \left\| g''\right\|_{\infty} \right] H(x),
\]
and
\[
\left| N[f(x)] N[g(x)] - \frac{1}{b-a} \left[ N[g(x)] \int_a^b f(t) \, dt + N[f(x)] \int_a^b g(t) \, dt \right] \right|
+ \frac{1}{(b-a)^2} \left( \int_a^b f(t) \, dt \right) \left( \int_a^b g(t) \, dt \right)
\leq \frac{1}{(b-a)^3} \left\| f''\right\|_{\infty} \left\| g''\right\|_{\infty} H^2(x),
\]
hold for every $x \in [a, b]$, where
\[
H(x) = \int_a^b \int_a^b |p(x,t)| |p(t,s)| \, ds \, dt,
\]
in which $p$ is given by (2.8).

Remark 1. We note that in [2, p.35], [4, p.231], [3, p.70] the authors have evaluated the integrals in (2.3), (2.7), (2.11) and obtained
\[
E(x) = \frac{1}{24} (b-a)^2 + \frac{1}{2} \left( x - \frac{a+b}{2} \right)^2,
\]
\[
I(x) = \frac{1}{3} \left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48},
\]
\[
H(x) = \frac{1}{2} \left\{ \left[ \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2,
\]
for $x \in [a,b]$. If we take $g(x) = 1$ and hence $g'(x) = 0, g''(x) = 0$ in the inequalities (2.1), (2.5), (2.9), then we recapture the inequalities established earlier in [2], [4], [3] when $E(x), I(x), H(x)$ are given by (2.12), (2.13), (2.14) respectively.
3. Proofs of theorems 1-3. From the hypotheses of Theorem 1, the following identities hold (see [2, p.35]):

\[(3.1) \quad L[f(x)] - \frac{1}{b-a} \int_a^b f(t) \, dt = -\frac{1}{b-a} \int_a^b k(x,t) f''(t) \, dt,\]

\[(3.2) \quad L[g(x)] - \frac{1}{b-a} \int_a^b g(t) \, dt = -\frac{1}{b-a} \int_a^b k(x,t) g''(t) \, dt.\]

Multiplying (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding the resulting identities we have

\[(3.3) \quad g(x) L[f(x)] + f(x) L[g(x)] - \frac{1}{b-a} \left[ g(x) \int_a^b f(t) \, dt + f(x) \int_a^b g(t) \, dt \right] = -\frac{1}{b-a} \left[ g(x) \int_a^b k(x,t) f''(t) \, dt + f(x) \int_a^b k(x,t) g''(t) \, dt \right].\]

From (3.3) and using the properties of modulus we have

\[
|g(x) L[f(x)] + f(x) L[g(x)] - \frac{1}{b-a} \left[ g(x) \int_a^b f(t) \, dt + f(x) \int_a^b g(t) \, dt \right]|
\leq \frac{1}{b-a} \left[ |g(x)| \int_a^b |k(x,t)| \, dt + |f(x)| \int_a^b |k(x,t)| |g''(t)| \, dt \right]
\leq \frac{1}{b-a} \left[ |g(x)| \|f''\|_\infty + |f(x)| \|g''\|_\infty \right] \int_a^b |k(x,t)| \, dt
= \frac{1}{b-a} \left[ |g(x)| \|f''\|_\infty + |f(x)| \|g''\|_\infty \right] E(x).
\]

This is the required inequality in (2.1).
Multiplying the left sides and right sides of (3.1) and (3.2) we have

\[(3.4)\]

\[
\begin{align*}
L [f(x)] L [g(x)] & - \frac{1}{b-a} \left[ L [g(x)] \int_a^b f(t) \, dt + L [f(x)] \int_a^b g(t) \, dt \right] \\
& + \frac{1}{(b-a)^2} \left( \int_a^b f(t) \, dt \right) \left( \int_a^b g(t) \, dt \right) \\
& = \frac{1}{(b-a)^2} \left[ \int_a^b k(x,t) f''(t) \, dt \right] \left[ \int_a^b k(x,t) g''(t) \, dt \right].
\end{align*}
\]

From (3.4) and using the properties of moduls we have

\[
\begin{align*}
\left| L [f(x)] L [g(x)] - \frac{1}{b-a} \left[ L [g(x)] \int_a^b f(t) \, dt + L [f(x)] \int_a^b g(t) \, dt \right] \\
& + \frac{1}{(b-a)^2} \left( \int_a^b f(t) \, dt \right) \left( \int_a^b g(t) \, dt \right) \right| \\
& \leq \frac{1}{(b-a)^2} \left[ \int_a^b |k(x,t)||f''(t)| \, dt \right] \left[ \int_a^b |k(x,t)||g''(t)| \, dt \right] \\
& \leq \frac{1}{(b-a)^2} \|f''\|_{\infty} \|g''\|_{\infty} \left[ \int_a^b |k(x,t)| \right]^2 \\
& = \frac{1}{(b-a)^2} \|f''\|_{\infty} \|g''\|_{\infty} E^2(x).
\end{align*}
\]

This is the desired inequality in (2.2).

From the hypotheses of Theorem 2, we have the following identities (see [4, p.232]):

\[(3.5)\]

\[
M [f(x)] - \frac{2}{b-a} \int_a^b f(t) \, dt = - \frac{1}{b-a} \int_a^b p(x,t) \left( t - \frac{a+b}{2} \right) f''(t) \, dt,
\]

(3.6)

\[
M [g(x)] - \frac{2}{b-a} \int_a^b g(t) \, dt = - \frac{1}{b-a} \int_a^b p(x,t) \left( t - \frac{a+b}{2} \right) g''(t) \, dt.
\]

Multiplying (3.5) and (3.6) by \(g(x)\) and \(f(x)\) respectively and adding the resulting identities we obtain

\[
g(x) M [f(x)] + f(x) M [g(x)] - \frac{2}{b-a} \left[ g(x) \int_a^b f(t) \, dt + f(x) \int_a^b g(t) \, dt \right]
\]
\[ (3.7) \quad \frac{1}{b-a} \left[ g(x) \int_a^b p(x,t) \left( t - \frac{a+b}{2} \right) f''(t) \, dt + f(x) \int_a^b p(x,t) \left( t - \frac{a+b}{2} \right) g''(t) \, dt \right] . \]

Multiplying the left sides and right sides of (3.5) and (3.6) we have
\[ (3.8) \quad M[f(x)] M[g(x)] - \frac{2}{b-a} \left[ M[g(x)] \int_a^b f(t) \, dt + M[f(x)] \int_a^b g(t) \, dt \right] \]
\[ + \frac{4}{(b-a)^2} \left( \int_a^b f(t) \, dt \right) \left( \int_a^b g(t) \, dt \right) \]
\[ = \frac{1}{(b-a)^2} \left[ \int_a^b p(x,t) \left( t - \frac{a+b}{2} \right) f''(t) \, dt \right] \]
\[ \times \left[ \int_a^b p(x,t) \left( t - \frac{a+b}{2} \right) g''(t) \, dt \right] . \]

From (3.7), (3.8) and following the proof of Theorem 1 we get the required inequalities in (2.5) and (2.6).

From the hypotheses of Theorem 3, we have the following identities (see [3, p.71]):
\[ (3.9) \quad N[f(x)] - \frac{1}{b-a} \int_a^b f(t) \, dt = \frac{1}{(b-a)^2} \int_a^b \int_a^b p(x,t)p(t,s) f''(s) \, ds \, dt, \]
\[ (3.10) \quad N[g(x)] - \frac{1}{b-a} \int_a^b g(t) \, dt = \frac{1}{(b-a)^2} \int_a^b \int_a^b p(x,t)p(t,s) g''(s) \, ds \, dt. \]

Multiplying (3.9) and (3.10) by \( g(x) \) and \( f(x) \) respectively and adding the resulting identities we obtain
\[ g(x) N[f(x)] + f(x) N[g(x)] - \frac{1}{b-a} \left[ g(x) \int_a^b f(t) \, dt + f(x) \int_a^b g(t) \, dt \right] \]
\[ = \frac{1}{(b-a)^2} \left[ g(x) \int_a^b \int_a^b p(x,t)p(t,s) f''(s) \, ds \, dt \right] \]
\[ + f(x) \int_a^b \int_a^b p(x,t)p(t,s) g''(s) \, ds \, dt . \]
Multiplying the left sides and right sides of (3.9) and (3.10) we have
(3.12)
\[
N [f (x)] N [g (x)] - \frac{1}{b-a} \left[ N [g (x)] \int_a^b f (t) \, dt + N [f (x)] \int_a^b g (t) \, dt \right] + \frac{1}{(b-a)^2} \left( \int_a^b f (t) \, dt \right) \left( \int_a^b g (t) \, dt \right) = \frac{1}{(b-a)^2} \left[ \int_a^b \int_a^b p (x, t) p (t, s) f'' (s) \, ds \, dt \right] \times \left[ \int_a^b \int_a^b p (x, t) p (t, s) g'' (s) \, ds \, dt \right].
\]

The proofs of (3.9) and (3.10) can be completed by closely looking at the proof of Theorem 1 with suitable modifications.

**Remark 2.** We note that, one can very easily obtain bounds on the right hand sides in the inequalities in Theorems 1-3, when \( f'' , g'' \) belongs to \( L_p [a, b] \) for \( p > 1 , \frac{1}{p} + \frac{1}{q} = 1 \) or \( L_1 [a, b] \). The precise formulations of such results are very close to those given in Theorems 1-3 with suitable modifications. We omit the details.

### 4. Some related inequalities.

In this section, we point out some new inequalities related to the inequalities given in Theorems 1-3.

Dividing both sides of (3.3) and (3.4) by \( (b-a) \) and then integrating both sides with respect to \( x \) over \([a, b]\) and following closely the proof of Theorem 1, it is easy to see that the following inequalities hold:

\[
\left| \frac{1}{b-a} \int_a^b [g (x) L [f (x)] + f (x) L [g (x)] \right| dx 
\]

\[
- \frac{2}{(b-a)^2} \left( \int_a^b f (t) \, dt \right) \left( \int_a^b g (t) \, dt \right) \right| 
\]

\[
\leq \frac{1}{(b-a)^2} \int_a^b \left[ |g (x)| \left\| f'' \right\| _\infty + |f (x)| \left\| g'' \right\| _\infty \right] E (x) \, dx
\]

and

\[
\left| \frac{1}{b-a} \int_a^b L [f (x)] L [g (x)] \right| dx - \frac{1}{(b-a)^2} \left[ \left( \int_a^b L [g (x)] \right) \left( \int_a^b f (x) \right) \right]
\]
\[+ \left( \int_a^b L[f(x)] \, dx \right) \left( \int_a^b g(x) \, dx \right)\]

\[+ \frac{1}{(b-a)^2} \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right)\]

\[\leq \frac{1}{(b-a)^3} \left\| f'' \right\|_\infty \left\| g'' \right\|_\infty \int_a^b E^2(x) \, dx.\]

Similarly, dividing both sides of (3.7), (3.8), (3.11), (3.12) by \((b-a)\) and then integrating both sides with respect to \(x\) over \([a, b]\) and following similar arguments as in the proofs of Theorems 2 and 3 we get respectively

\[\left| \frac{1}{b-a} \int_a^b \left[ g(x) M[f(x)] + f(x) M[g(x)] \right] \, dx \right| \]

\[= \frac{4}{(b-a)^2} \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right)\]

\[= \frac{1}{(b-a)^2} \left\| f'' \right\|_\infty \left\| g'' \right\|_\infty \int_a^b I^2(x) \, dx,\]

\[\left| \frac{1}{b-a} \int_a^b \left[ f(x) N[g(x)] + g(x) N[f(x)] \right] \, dx \right| \]

\[= \frac{2}{(b-a)^2} \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right)\]

\[\leq \frac{1}{(b-a)^3} \left\| f'' \right\|_\infty \left\| g'' \right\|_\infty \int_a^b I^2(x) \, dx,\]

\[\left| \frac{1}{b-a} \int_a^b \left[ g(x) H[f(x)] + f(x) H[g(x)] \right] \, dx \right| \]

\[-\frac{2}{(b-a)^3} \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right)\]

\[\leq \frac{1}{(b-a)^3} \int_a^b \left| f'' \right|_\infty + \left| f(x) \right| \left\| g'' \right\|_\infty \int_a^b \, dx,\]
\[
\frac{1}{b - a} \left| \int_a^b N[f(x)] N[g(x)] \, dx \right|
\]
\[
- \frac{1}{(b - a)^2} \left[ \left( \int_a^b N[g(x)] \, dx \right) \left( \int_a^b f(x) \, dx \right) \right] 
\]
\[
+ \left( \int_a^b N[f(x)] \, dx \right) \left( \int_a^b g(x) \, dx \right)
\]
\[
+ \frac{1}{(b - a)^2} \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right)
\]
\[
\leq \frac{1}{(b - a)^5} \left\| f'' \right\|_{\infty} \left\| g'' \right\|_{\infty} \int_a^b H^2(x) \, dx.
\]

**Remark 3.** We note that the inequalities obtained in (4.1)-(4.6) are similar to the well known inequalities due to Grüss [5] and Čebyšev [1]. One can also obtain bounds on the right hand sides in the above inequalities, when \( f'' \), \( g'' \) belongs to \( L_p[a, b] \) or \( L_1[a, b] \). For various other inequalities of the above type, see [7, 9-12] and some of the references cited therein.

**REFERENCES**

5. Grüss, G. – *Über das maximum des absoluten Betrages von*
\[
\int_a^b f(x) g(x) \, dx - \frac{1}{(b - a)^2} \int_a^b f(x) \, dx \int_a^b g(x) \, dx
\]

Received: 15.XI.2005

57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431 001 (Maharashtra), INDIA,
bgpachpatte@gmail.com