SOME LOCALLY SYMMETRIC ANTI-HERMITIAN STRUCTURES ON THE TANGENT BUNDLE∗

BY

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Dedicated to Professor Vasile Oproiu on his 65th birthday

Abstract. We consider certain classes of almost anti-Hermitian structures \((G, J)\) on the tangent bundle \(TM\) of a Riemannian manifold \((M, g)\). The semi-Riemannian metric \(G\) is a natural lift to \(TM\) of the metric \(g\) such that the vertical and horizontal distributions \(VTM, HTM\) are maximally isotropic and the almost complex structure \(J\) is a natural lift of diagonal type of \(g\). The characterization of the considered almost anti-Hermitian structures have been obtained in [8]. In fact, we obtain the necessary and sufficient conditions under which the considered structures are locally symmetric.

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1. Introduction. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and denote by \(\tau : TM \rightarrow M\) its tangent bundle. There are several Riemannian and semi-Riemannian metrics induced on \(TM\) from the Riemannian metric \(g\) on \(M\). Among them, we may quote the Sasaki metric and the complete lift of the metric \(g\). On the other hand, there are the lifts of \(g\) of natural type, leading to several new geometric structures with many nice geometric properties (see [3]).

In [8] OPROIU and the present author have considered a natural almost anti-Hermitian structure \((G, J)\), defined on \(TM\) by using some natural lifts

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of the Riemannian metric \( g \). The vertical distribution \( \mathcal{V TM} \) and the horizontal distribution \( \mathcal{H TM} \) are interchanged by the considered almost complex structure \( J \), while they are maximally isotropic with respect to the semi-Riemannian metric \( G \). Using the classification of the almost anti-Hermitian structures in eight classes, given in [1], in [8] have been obtained the necessary and sufficient conditions under which the considered almost anti-Hermitian structure \( (TM, G, J) \) belongs to each of these eight classes.

In this paper we consider the classes of anti-Kählerian, quasi-anti-Kählerian, conformally anti-Kählerian, complex anti-Hermitian and special complex anti-Hermitian structures \( (G, J) \) on \( TM \) obtained in [8] and we obtain necessary and sufficient conditions under which these structures are locally symmetric.

The manifolds, tensor fields and other geometric objects we consider in this paper are assumed to be differentiable of class \( C^\infty \) (i.e. smooth). We use the computations in local coordinates in a fixed local chart but many results may be expressed in an invariant form by using the vertical and horizontal lifts. The well known summation convention is used throughout this paper, the range of the indices \( h, i, j, k, l \) being always \( \{1, \ldots, n\} \).

2. Classes of almost anti-Hermitian structure on \( TM \). Let \( (M, g) \) be a smooth \( n \)-dimensional Riemannian manifold and denote its tangent bundle by \( \tau : TM \to M \). Recall that \( TM \) has a structure of a 2\( n \)-dimensional smooth manifold, induced from the structure of smooth \( n \)-dimensional manifold of \( M \). From every local chart \( (U, \varphi) = (U, x^1, \ldots, x^n) \) on \( M \) it is induced a local chart \( (\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n) \), on \( TM \), as follows. For a tangent vector \( y \in \tau^{-1}(U) \subset TM \), the first local coordinates \( x^1, \ldots, x^n \) are the local coordinates \( x^1, \ldots, x^n \) of its base point \( x = \tau(y) \) in the local chart \( (U, \varphi) \) (in fact we made an abuse of notation, identifying \( x^i \) with \( \tau^* x^i = x^i \circ \tau, i = 1, \ldots, n \)). The last \( n \) local coordinates \( y^1, \ldots, y^n \) of \( y \in \tau^{-1}(U) \) are the vector space coordinates of \( y \) with respect to the natural basis \( ((\frac{\partial}{\partial x^1})_{\tau(y)}, \ldots, (\frac{\partial}{\partial x^n})_{\tau(y)}) \), defined by the local chart \( (U, \varphi) \), i.e. \( y = y^i (\frac{\partial}{\partial x^i})_{\tau(y)} \). Due to this special structure of differentiable manifold for \( TM \), it is possible to introduce the concept of \( M \)-tensor field on it (see [8] for details).

We shall use the horizontal distribution \( \mathcal{H TM} \), defined by the Levi Civita connection \( \nabla \) of \( g \), in order to define some natural lifts to \( TM \) of the Riemannian metric \( g \) on \( M \). Denote by \( \mathcal{V TM} = \ker \tau_\ast \subset TT M \) the
vertical distribution on \( TM \). Then we have the direct sum decomposition

\[
TTM = VTM \oplus HTM.
\]

If \((\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)\) is a local chart on \( TM \), induced from the local chart \((U, \varphi)(U, x^1, \ldots, x^n)\), the local vector fields \( \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \) on \( \tau^{-1}(U) \) define a local frame for \( VTM \) over \( \tau^{-1}(U) \) and the local vector fields \( \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \) define a local frame for \( HTM \) over \( \tau^{-1}(U) \), where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i} \frac{\partial}{\partial y^h}, \quad \Gamma^h_{0i} = y^k \Gamma^h_{ki}
\]

and \( \Gamma_{ki}^h(x) \) are the Christoffel symbols of \( g \).

The set \( (\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}) \) defines a local frame on \( TM \), adapted to the direct sum decomposition (1). Remark that

\[
\frac{\partial}{\partial y^i} = \left( \frac{\partial}{\partial x^i} \right)^V, \quad \frac{\delta}{\delta x^i} = \left( \frac{\partial}{\partial x^i} \right)^H,
\]

where \( X^V \) and \( X^H \) denote the vertical and horizontal lifts of the vector field \( X \) on \( M \).

Let \( C = y^i \frac{\partial}{\partial y^i} \) be the Liouville vector field on \( TM \) and consider the horizontal vector field \( \tilde{C} = y^i \frac{\delta}{\delta x^i} \) on \( TM \), defined in a similar way.

Since we work in a fixed local chart \((U, \varphi)\) on \( M \) and in the corresponding induced local chart \((\tau^{-1}(U), \Phi)\) on \( TM \), we shall use the following simpler notations

\[
\frac{\partial}{\partial y^i} = \partial_i, \quad \frac{\delta}{\delta x^i} = \delta_i.
\]

Denote by

\[
t = \frac{1}{2} \|y\|^2 = \frac{1}{2} g_{\tau(y)}(y, y) = \frac{1}{2} g_{ik}(x)y^i y^k, \quad y \in \tau^{-1}(U)
\]

the energy density defined by \( g \) in the tangent vector \( y \). We have \( t \in [0, \infty) \) for all \( y \in TM \). Consider the real valued smooth functions \( a_1, a_2, b_1, b_2 \) defined on \([0, \infty) \subset \mathbb{R}\) and consider a 1-st order natural almost complex structure on \( TM \), by using these coefficients and the Riemannian metric \( g \), just like the 1-st order natural lifts of \( g \) to \( TM \) are obtained in [3]. If the
coefficients $a_1, a_2, b_1, b_2$ are related by some specific algebraic relations. The expression of $J$ is given by (see [10])

\[
\begin{align*}
JX_y^H &= a_1(t)X_y^V + b_1(t)g_{\tau(y)}(y, X)C_y, \\
JX_y^V &= -a_2(t)X_y^H - b_2(t)g_{\tau(y)}(y, X)\tilde{C}_y.
\end{align*}
\]

The expression of $J$ in adapted local frames is given by

\[
\begin{align*}
J\delta_i &= a_1(t)\partial_i + b_1(t)g_{0i}C, \\
J\partial_i &= -a_2(t)\delta_i - b_2(t)g_{0i}\tilde{C}.
\end{align*}
\]

In [8] Oproiu and the present author have proved that the operator $J$ defines an almost complex structure on $TM$ if and only if

\[
(2) \quad a_1 a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1,
\]

and the almost complex structure $J$ defined by (1) on $TM$ is integrable if and only if $(M, g)$ has constant sectional curvature $c$ and the function $b_1$ is given by

\[
(3) \quad b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1}.
\]

Now, we consider a particular 1-st order natural lift $G$ of $g$ to $TM$, defining a semi-Riemannian metric of signature $(n, n)$ on $TM$. This lift is defined by two real valued smooth functions $u, v : [0, \infty) \rightarrow \mathbb{R}$ and is given by

\[
\begin{align*}
G_y(X^H, Y^H) &= 0, \quad G_y(X^V, Y^V) = 0, \\
G_y(X^H, Y^V) &= G_y(Y^V, X^H) = G_y(X^V, Y^H) = G_y(Y^H, X^V) \\
&= u(t)g_{\tau(y)}(X, Y) + v(t)g_{\tau(y)}(y, X)g_{\tau(y)}(y, Y).
\end{align*}
\]

The expression of $G$ in local adapted frames is defined by the conditions

\[
\begin{align*}
G(\delta_i, \delta_j) &= 0, \quad G(\partial_i, \partial_j) = 0, \\
G(\partial_i, \delta_j) &= G(\delta_i, \partial_j)u_{ij} + v_{gij}g_{0j}.
\end{align*}
\]
Remark that $G$ is defined, essentially, by the symmetric $M$-tensor field $G_{ij} = u g_{ij} + v g_{0i} g_{0j}$ of type $(0, 2)$. The condition for $G$ to be nondegenerate is assured if

$$u \neq 0, \ u + 2tv \neq 0.$$ 

If the coefficients $a_1, b_1, a_2, b_2, u, v$ satisfy the above relations then the tensor fields $G, J$ will define an almost anti-Hermitian structure on $TM$, i.e. $G(JX, JY) = -G(X, Y)$ for all vector fields $X, Y$ on $TM$ (see [8], [6]).

**Proposition 1.** The Levi Civita connection $\nabla$ of the pseudo-Riemannian metric $G$ on $TM$ has the following expression in the local adapted frame $(\partial_1, \ldots, \partial_n, \delta_1, \ldots, \delta_n)$

$$\nabla_{\partial_i} \partial_j Q^h_{ij} \partial_h, \ \nabla_{\delta_i} \delta_j = \Gamma^h_{ij} \partial_h + P^h_{ij} \delta_h,$$

where the $M$-tensor fields $P^h_{ij}, Q^h_{ij}, S^h_{ij}$ are given by

$$P^h_{ij} = \frac{u' - v}{2u} \left( g_{0i} \delta^h_j - \frac{u}{u + 2tv} g_{ij} y^h + \frac{v}{u + 2tv} g_{0i} g_{0j} y^h \right),$$

$$Q^h_{ij} = \frac{u' + v}{2u} (g_{0i} \delta^h_j + g_{0j} \delta^h_i) + \frac{u}{u + 2tv} g_{ij} y^h + \frac{v' u - u' v - v^2}{u(u + 2tv)} g_{0i} g_{0j} y^h,$$

$$S^h_{ij} = R^h_{0j0i} + \frac{v}{u + 2tv} R_{0j0i} y^h,$$

$R_{iklj}$ denoting the local coordinate components of the Riemann-Christoffel tensor of $\nabla$ on $M$ and $R_{0ikj} = R_{likj} y^i, R_{0i0j} = R_{likj} y^i y^k$.

The condition under which the almost anti-Hermitian manifold $(TM, G, J)$, considered above, belongs to each from the eight classes of anti-Hermitian manifolds obtained in the classification in [1] have been obtained in [8]. We recall here five of the classes of anti-Hermitian manifolds obtained in [1], [8]. For this aim, we consider the tensor field $F$ of type $(0, 3)$ defined by

$$F(X, Y, Z) = G((\nabla_X J)Y, Z), \ X, Y, Z \in \Gamma(TM),$$

and introduce the 1-form $\phi$, associated with $F$ defined by

$$\phi(X) = G^{ij} F(E_i, E_j, X), \ X \in \Gamma(TM), \ i, j = 1, \ldots, 2n,$$
where \((E_1, \ldots, E_{2n})\) is a local frame in \(TTM\) and \(G^{ij}\) are the entries of the inverse of the matrix \((G_{ij})\) associated to \(G\) in the local frame \((E_1, \ldots, E_{2n})\).

Then the considered almost anti-Hermitian structure \((G, J)\) on \(TM\) is called:

An **anti-Kählerian structure** if

\[ F(X, Y, Z) = 0, \]

or, equivalently, \(\nabla J = 0\).

A **quasi-anti-Kählerian structure** if

\[ F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0. \]

A **conformally anti-Kählerian structure** if

\[
2nF(X, Y, Z) = G(X, Y)\phi(Z) + G(X, Z)\phi(Y) \\
+ G(X, JY)\phi(JZ) + G(X, JZ)\phi(JY).
\]

A **special complex anti-Hermitian structure** if

\[ \phi = 0, \ F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \]

A **complex anti-Hermitian structure** if

\[ F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \]

In [8], Oproiu an the present author have obtained the necessary and sufficient conditions under which the anti-Hermitian structure \((G, J)\) defined above is in one of the eight classes given in the classification in [1].

We recall here the obtained results for the five classes defined above.

**Theorem 2.** The almost anti-Hermitian manifold \((TM, G, J)\) is an anti-Kählerian manifold if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(b_1, u, v\) satisfy relations

\[
(5) \quad b_1 = \frac{a_1a_1' - c}{a_1 - 2ta_1'}, \quad u = \frac{Aa_1}{a_1^2 - 2ct}, \quad v = -\frac{Aa_1'}{a_1^2 - 2ct},
\]

where \(A\) is a nonzero real constant and \(a_1\) is an arbitrary positive function such that \(a_1^2 - 2ct \neq 0\) and \(a_1 - 2ta_1' \neq 0\).
Theorem 3. The almost anti-Hermitian manifold \((TM, G, J)\) is quasi-
anti-Kählerian if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(b_1, u, v\) satisfy relations:

\[ v = -\frac{a^2_1 a'_1 u + 2a_1 u u' + 2v u' + 2v a_1 u'}{2a_1^2 + 4u a_1}, \quad b_1 = \frac{e v - a^2_1 u'}{a_1 u + 2a_1 u}, \]

\[ u''(a^4_1 u^2 - 2a^3_1 a'_1 t u^3 + 2a^2_1 c t u^2 - 4a_1 a' c t^2 u^2) + a''_1(a^3_1 u^3 + 2a_1 c t u^3 + 2a_1^2 t u^2 u' + 4a_1 c t^2 u^2 u') \]

\[ + 4a_1 a'_1 c u^3 - 2a^3_1 a'_1 u^2 u' + 4a^2_1 c u^2 u' + 4a_1 a'_1 c t u^2 u' \]

\[ - 4a_1^2 c t u^3 - 8a_1 c t^2 u^2 u' - 4a_1^2 u' u^2 + 2a_1^2 a'_1 t u u'^2 \]

\[ + 4a_1^2 c t u u'^2 + 4a_1 a'_1 c t^2 u u'^2 - 2a_1^4 t u'^3 + 4a_1^2 c t^2 u'^3 = 0. \]

Theorem 4. The almost anti-Hermitian manifold \((TM, G, J)\) is a complex anti-Hermitian manifold if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(a_1, b_1\) are related by

\[ b_1(a_1 - 2t a'_1) = a_1 a'_1 - c. \]

Theorem 5. The almost anti-Hermitian manifold \((TM, G, J)\) is a special complex anti-Hermitian manifold if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(a_1, b_1, u, v\) satisfy the following relations:

\[ b_1 = \frac{a_1 a'_1 - c}{a_1 - 2t a'_1}, \]

\[ n(a_1 - 2t a'_1)(a_1^2 a'_1 u - 2ca_1 u + 2c a'_1 u + a^3_1 u' - 2c t a_1 u')(u + 2v) \]

\[ + 2(a^2_1 - 2c t)(a_1 a'_1 u^2 - t(a'_1)^2 u^2 + t a_1 a''_1 u^2 + a^2_1 w - 2a^2(a'_1)^2 w + 2t^2 a_1 a''_1 w \]

\[ - t a^2_1 u' v + 2a^2 a_1 a'_1 u' v + t a^2_1 w' u' - 2t^2 a_1 a'_1 a''_1 u' v) = 0. \]

Theorem 6. The almost anti-Hermitian manifold \((TM, G, J)\) is a conformally anti-Kählerian manifold if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(a_1, b_1, u, v\) satisfy relations:

\[ b_1 = \frac{a_1 a'_1 - c}{a_1 - 2t a'_1}, \quad v = -\frac{a'_1 u}{a_1}. \]
curvature tensor field \( K \) of the Levi Civita connection \( \nabla \) of the pseudo-Riemannian metric \( G \) on \( TM \) is defined by well known formula

\[
K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \Gamma(TM).
\]

The formal expression of \( K \) in the local frame \( (\frac{\delta}{\partial x^i}, \frac{\partial}{\partial y^j}) \), \( i = 1, \ldots, n \), is obtained by a straightforward computation of the form

\[
\begin{align*}
K(\delta_i, \delta_j)\delta_k &= XXX^{h}_{klj} \delta_h + y'(\hat{\nabla}_l R^h_{klj} + \frac{v}{u^{2/3}} \hat{\nabla}_l R^h_{k0lj} y^h) \frac{\partial}{\partial y^h}, \\
K(\delta_i, \delta_j)\partial_k &= XXY^{h}_{klj} \partial_h, \quad K(\partial_i, \delta_j)\partial_k = YYX^{h}_{klj} \partial_h, \\
K(\partial_i, \partial_j)\partial_k &= YYY^{h}_{klj} \partial_h, \quad K(\partial_i, \delta_j)\partial_k = YXX^{h}_{klj} \partial_h, \\
K(\partial_i, \delta_j)\delta_k &= YXY^{h}_{klj} \delta_h,
\end{align*}
\]

where \( XXX^{h}_{klj}, \ XXY^{h}_{klj}, \ YYY^{h}_{klj}, \ YXX^{h}_{klj}, \ YXY^{h}_{klj} \) are components defining \( M - \)tensor fields of type \((1,3)\) on \( TM \), whose expressions have been computed by the present author in [9]. Next, in [9], we study the conditions under which the pseudo-Riemannian manifold \((TM, G)\) is a locally symmetric space, i.e. \( \nabla K = 0 \). By computing \( (\nabla \frac{\delta}{\partial x^i} K)(\partial_i, \partial_j)\partial_k \), in [9], we get that the condition \( (\nabla \frac{\delta}{\partial x^i} K)(\partial_i, \partial_j)\partial_k = 0 \) is equivalent to the condition

\[
(9) \quad v = u'.
\]

Assuming that the condition \((9)\) is fulfilled we obtain that the other covariant derivatives of \( K \) vanishing, i.e. \( \nabla K = 0 \) if and only if \( \hat{\nabla}_l R^h_{klj} = 0 \), i.e. the base manifold \((M, g)\) is a locally symmetric space. Hence we have (see Theorem 2 in [9])

**Theorem 7.** The pseudo-Riemannian manifold \((TM, G)\), where \( G \) is given by \((4)\) is a locally symmetric space if and only if the functions \( u(t), v(t) \) satisfy the condition \((9)\) and the base manifold \((M, g)\) is a locally symmetric space.

Also we have

**Corollary 8.** The pseudo-Riemannian manifold \((TM, G)\), where \( G \) is given by \((4)\) is flat if and only if the functions \( u(t), v(t) \) satisfy the condition \((9)\) and the base manifold \((M, g)\) is flat.
In the following, by using Theorem 7 and the characterizations of the five classes of anti-Hermitian manifolds given by Theorems 2 - 6, we obtain the necessary and sufficient conditions under which these classes of anti-Hermitian manifolds are locally symmetric spaces.

We assume that the base manifold \((M, g)\) has constant sectional curvature \(c\), therefore \((M, g)\) is a locally symmetric space. Then, the relations (5) from Theorem 2 and the condition (9) imply that \(c = 0\), i.e. the base manifold \((M, g)\) is flat and the functions \(b_1, u, v\) are expressed by

\[
(10) \quad b_1 = \frac{a_1a_1'}{a_1 - 2ta_1'}, \quad u = \frac{A}{a_1}, \quad v = -\frac{Aa_1'}{a_1^2},
\]

where \(A\) is a nonzero real constant and \(a_1\) is an arbitrary positive function such that \(a_1 - 2ta_1' \neq 0\).

Next, we get by a straightforward but quite long computation, that the relations (6) from Theorem 3 and the conditions (9) imply also that \(c = 0\) and the functions \(b_1, u, v\) are expressed by (10).

Assuming that the second relation from Theorem 5 is fulfilled independently of the dimension \(n\) of \(M\), we get also that the relations from Theorem 5 and the condition (9) imply \(c = 0\) and the functions \(b_1, u, v\) are expressed by (10).

From the above results, we have

**Theorem 9.** The following four assertions are equivalent to each other:

(i) the anti-Kählerian manifold \((TM, G, J)\) characterized by Theorem 2 is a locally symmetric space,

(ii) the quasi-anti-Kählerian manifold \((TM, G, J)\) characterized by Theorem 3 is a locally symmetric space,

(iii) the special complex anti-Hermitian manifold \((TM, G, J)\) characterized by Theorem 5 is a locally symmetric space, independently of the dimension \(n\) of \(M\),

(iv) the base manifold \((M, g)\) is flat and the functions \(a_1, b_1, u, v\) which essentially define the almost complex structure \(J\) on \(TM\) by (1) and the pseudo-Riemannian metric \(G\) on \(TM\) by (4), are expressed by the relations (10).
Now, we use Theorem 6 which characterizes a conformally anti-Kählerian manifold \((TM, G, J)\). Thus, from the relations (8) and the condition (9), we get

\[
(11) \quad b_1 = \frac{a_1 a_1' - c}{a_1 - 2ta_1'}, \quad u = \frac{A}{a_1}, \quad v = -\frac{Aa_1'}{a_1^2},
\]

where \(A\) is a nonzero real constant and \(a_1\) is an arbitrary positive function such that \(a_1 - 2ta_1' \neq 0\). Hence we have

**Theorem 10.** The conformally anti-Kählerian manifold \((TM, G, J)\) characterized by Theorem 6 is a locally symmetric space if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(a_1, b_1, u, v\) satisfy relations (11).

Finally, from Theorems 3 and 7, we obtain

**Theorem 11.** The complex anti-Hermitian manifold \((TM, G, J)\) characterized by Theorem 3 is a locally symmetric space if and only if the base manifold \((M, g)\) has constant sectional curvature \(c\) and the functions \(a_1, b_1, u, v\) satisfy relations

\[
b_1(a_1 - 2ta_1') = a_1 a_1' - c, \quad v = u'.
\]

**REFERENCES**


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