RIEMANNIAN METRICS ON THE TANGENT BUNDLE OF A FINSLER SUBMANIFOLD

BY

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Abstract. Let $IF^n = (M, F)$ be a Finsler submanifold of a Finsler manifold $IF^{n+p} = (\tilde{M}, \tilde{F})$. Then the induced non-linear connection $HTM^o$ and the canonical non-linear connection $GM^o$ define two Riemannian metrics $G$ and $G^*$ on $TM^o = TM \setminus \{0\}$, both of Sasaki-Finsler type. On the other hand, the Sasaki-Finsler metric $Gg$ on $TfM^o = TM \setminus \{0\}$ induces a Riemannian metric $G_{ind}$ on $TM^o$. We prove that $IF^n$ is totally geodesic immersed in $IF^{n+p}$ if and only if $G = G^* = G_{ind}$ on $TM^o$.

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1. Preliminaries

Let $IF^{n+p} = (\tilde{M}, \tilde{F})$ be an $(n+p)$-dimensional Finsler manifold, where $\tilde{F}$ is the fundamental function of $IF^{n+p}$ that is of class $C^\infty$ on the slit tangent bundle $T\tilde{M}^o = T\tilde{M} \setminus \{0\}$ (see Bao-Chern-Shen [1], Bejancu-Farran [3], Matsumoto [4], Rund [5], for basic results on Finsler geometry). Denote by $(x^i, y^i)$, $i = \{1, \ldots, n+p\}$ the local coordinates on $T\tilde{M}$, where $(x^i)$ are the local coordinates of a point $x \in \tilde{M}$ and $(y^i)$ are the coordinates of a vector $y \in T_x\tilde{M}$. The vertical vector bundle $VT\tilde{M}^o$ on $T\tilde{M}^o$ is endowed with a Riemannian metric whose local components are given by

$$\tilde{g}_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \tilde{F}^2}{\partial y^i \partial y^j}.$$

Denote by $\tilde{g}^{ij}(x, y)$ the entries of the inverse of the $(n+p) \times (n+p)$-matrix $[\tilde{g}_{ij}(x, y)]$. Then the geodesics of $IF^{n+p}$ are given by the solutions of the
system of differential equations
\[
\frac{d^2 x^i}{ds^2} + \tilde{G}^i \left(x^j(s), \frac{dx^j}{ds}\right) = 0,
\]
where $\tilde{G}^i$ are given by
\[
\tilde{G}^i(x, y) = \frac{1}{4} g^{ih} \left\{ \frac{\partial^2 \tilde{F}^2}{\partial y^i \partial x^j} y^j - \frac{\partial \tilde{F}^2}{\partial x^h} \right\}.
\]
We note that $VT\tilde{M}^o$ is an integrable distribution on $T\tilde{M}^o$ which is locally spanned by $\left\{ \frac{\partial}{\partial y^i} \right\}$, $i \in \{1, \ldots, n+p\}$. Also, on $T\tilde{M}^o$ there exists a distribution $GT\tilde{M}^o$ (in general, non-integrable) which is given by the non-holonomic frame field:
\[
\frac{\delta^*}{\delta^* x^i} = \frac{\partial}{\partial x^i} - \tilde{G}^i \frac{\partial}{\partial y^i}, \quad i \in \{1, \ldots, n+p\},
\]
where $\tilde{G}^i$ are given by
\[
\tilde{G}^i = \frac{\partial \tilde{G}^i}{\partial y^i}.
\]
It is important to note that we have
\[
TT\tilde{M}^o = GT\tilde{M}^o \oplus VT\tilde{M}^o,
\]
that is, $GT\tilde{M}^o$ and $VT\tilde{M}^o$ are complementary distributions on $T\tilde{M}^o$. We call $GT\tilde{M}^o$ the canonical non-linear connection on $T\tilde{M}^o$. The above decomposition of $TT\tilde{M}^o$ enables us to define the Sasaki-Finsler metric $\bar{G}$ on $T\tilde{M}^o$ given locally as follows (cf. Bao-Chern-Shen [1], p. 48, Bejancu-Farran [3], p. 35, Matsumoto [4], p. 136)

\[
(1) \quad \bar{G} \left( \frac{\delta^*}{\delta^* x^i}, \frac{\delta^*}{\delta^* x^j} \right) = \bar{G} \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = \bar{g}_{ij}(x, y), \quad \bar{G} \left( \frac{\delta^*}{\delta^* x^i}, \frac{\partial}{\partial y^i} \right) = 0.
\]

Next, let $M$ be an $n$-dimensional submanifold of $\tilde{M}$ that is locally given by the equations
\[
x^i = x^i(u^1, \ldots, u^n), \quad i \in \{1, \ldots, n+p\}.
\]
Throughout the paper we use the following ranges for indices: \( i, j, k, \ldots \in \{1, \ldots, n + p\} \), \( \alpha, \beta, \gamma, \ldots \in \{1, \ldots, n\} \), \( a, b, c, \ldots \in \{n + 1, \ldots, n + p\} \). Consider \((u^\alpha, v^\alpha)\) as local coordinates on \( TM^o = TM \setminus \{0\} \) and define the function \( F(u^\alpha, v^\alpha) = \tilde{F}(x^i(u), y^j(u,v)) \), where we put
\[
y^j(u,v) = B^j_\alpha v^\alpha, \quad B^j_\alpha = \frac{\partial x^i}{\partial u^\alpha}.
\]
Then \( \mathbb{F}^n = (M, F) \) is a Finsler submanifold of \( \mathbb{F}^{n+p} \).
As above, we have the decomposition
\[
TTM^o = GTM^o \oplus VTM^o,
\]
where \( VTM^o \) and \( GTM^o \) are the vertical vector bundle and the canonical non-linear connection on \( TM^o \), locally spanned by \( \left\{ \frac{\partial}{\partial v^\alpha} \right\} \) and
\[
\frac{\delta^s}{\delta^s u^\alpha} = \frac{\partial}{\partial u^\alpha} - \Gamma_{\alpha}^\beta \frac{\partial}{\partial v^\beta},
\]
respectively. Thus the Sasaki-Finsler metric \( G^o \) on \( TM^o \) defined by the decomposition (1.2) is locally given by
\[
G^o \left( \frac{\delta^s}{\delta^s u^\alpha}, \frac{\delta^s}{\delta^s v^\alpha} \right) = G^o \left( \frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha} \right) = g_{\alpha \beta}(u,v), \quad G^o \left( \frac{\delta^s}{\delta^s u^\alpha}, \frac{\partial}{\partial v^\alpha} \right) = 0,
\]
where we put \( g_{\alpha \beta}(u,v) = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^\alpha \partial v^\beta} \). By direct calculations it follows that
\[
g_{\alpha \beta} = \tilde{g}_{ij} B^i_\alpha B^j_\beta.
\]
On the other hand, \( \tilde{G} \) induces a Riemannian metric \( G_{\text{ind}} \) on \( TM^o \) as follows
\[
G_{\text{ind}}(X,Y) = \tilde{G}(X,Y), \quad \forall X,Y \in \Gamma(TTM^o).
\]
Here, and in the sequel, we denote by \( \Gamma(TTM^o) \) the \( \mathcal{F}(TM^o) \)-module of vector fields on \( TM^o \), where \( \mathcal{F}(TM^o) \) is the algebra of smooth functions on \( TM^o \). The same notation will be used for the module of sections of any vector bundle.
Now, we denote by \( VTM^o \perp \) the orthogonal complementary vector subbundle to \( VTM^o \) in \( VTM|TM^o \) with respect to \( \tilde{G} \) and call it the Finsler normal bundle of the immersion of \( \mathbb{F}^n \) in \( \mathbb{F}^{n+p} \) (cf. BEJANCU [2], p. 47). Then we consider a local field of orthonormal frames \( \{B_a = B^i_a \partial/\partial y^i \} \),
\[ a \in \{n + 1, \ldots, n + p\} \text{ in } VTM^\perp, \text{ and taking into account that } \{\partial/\partial v^\alpha = B^i_\alpha \partial/\partial y^i\} \text{ is a local field of frames in } VTM^\circ, \text{ we deduce that} \]

\[
\begin{align*}
(6) \quad (a) \quad \tilde{g}_{ij} B^i_\alpha B^j_\alpha &= 0, & (b) \quad \tilde{g}_{ij} B^i_\alpha B^j_\beta &= \delta_{\alpha\beta}.
\end{align*}
\]

Denote by \([\tilde{B}^\alpha_i \tilde{B}^i_\alpha]\) the inverse of the transition matrix \([B^\alpha_i B^i_\alpha]\) from the natural field of frames \(\{\frac{\partial}{\partial y^i}\}\) on \(VTM^\circ\) to the field of frames \(\{\frac{\partial}{\partial v^i}, B^i_\alpha\}\).

Next, we consider the induced non-linear connection \(HTM^\circ\) on \(TTM^\circ\), which is defined as the complementary orthogonal distribution to \(VTM^\circ\) in \(TTM^\circ\) with respect to \(G_{\text{ind}}\). Thus, apart from (2), \(TTM^\circ\) admits the decomposition

\[
(7) \quad TTM^\circ = HTM^\circ \oplus VTM^\circ.
\]

We should note that both (2) and (7) are orthogonal decompositions, but with respect to different metrics \(G^*\) and \(G_{\text{ind}}\), respectively. Locally, \(HTM^\circ\) is given by the field of frames \(\{N^\alpha_i\}, \alpha \in \{1, \ldots, n\}\), expressed as follows (cf. Bejancu-Farran [3], pp. 74, 75)

\[
(8) \quad \frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N^\beta_\alpha \frac{\partial}{\partial v^\beta}; \quad N^\alpha_i = \tilde{B}^\alpha_i \left(B^i_\alpha + B^j_\beta \tilde{G}^i_j\right),
\]

or

\[
(9) \quad \frac{\delta}{\delta u^\alpha} = B^i_\alpha \frac{\delta^*}{\delta^* x^i} + H^\alpha_i B^i_\alpha; \quad H^\alpha_i = \tilde{B}^i_\alpha \left(B^i_\alpha + B^j_\beta \tilde{G}^i_j\right),
\]

where we put

\[ B^i_\alpha = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} v^\beta. \]

Finally, by using the decomposition (7) we can define another Sasaki-Finsler metric \(G\) on \(TM^\circ\) given by

\[
(10) \quad G\left(\frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha}\right) = G\left(\frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha}\right) = g_{\alpha\beta}, \quad G\left(\frac{\delta}{\delta u^\beta}, \frac{\partial}{\partial u^\alpha}\right) = 0.
\]

Summing up, we constructed on \(TM^\circ\) three Riemannian metrics: \(G^*\), \(G_{\text{ind}}\) and \(G\) given by (3), (5) and (10), respectively. Thus, it is natural to ask the following question. What class of Finsler submanifolds is characterized by the condition \(G^* = G_{\text{ind}} = G\)? In the second part of this paper we give the answer to this question.
2. The main result

Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of $\mathbb{F}^{n+p} = (\tilde{M}, \tilde{F})$. Then we say that $\mathbb{F}^n$ is totally geodesic immersed in $\mathbb{F}^{n+p}$ if any geodesic of $\mathbb{F}^n$ is a geodesic of $\mathbb{F}^{n+p}$. The main purpose of this section is to prove the following theorem.

**Theorem 2.1.** Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of $\mathbb{F}^{n+p} = (\tilde{M}, \tilde{F})$. Then the following assertions are equivalent:

(i) $\mathbb{F}^n$ is totally geodesic immersed in $\mathbb{F}^{n+p}$.

(ii) $G_{\text{ind}} = G^*$ on $TM^\circ$.

(iii) $G_{\text{ind}} = G$ on $TM^\circ$.

As a direct consequence of this result we state the following.

**Corollary 2.1.** $\mathbb{F}^n$ is a totally geodesic Finsler submanifold of $\mathbb{F}^{n+p}$ if and only if $G^* = G_{\text{ind}} = G$ on $TM^\circ$.

First, we prove the following lemma.

**Lemma 2.1.** Let $\mathbb{F}^n = (M, F)$ be a Finsler submanifold of $\mathbb{F}^{n+p} = (\tilde{M}, \tilde{F})$. Then we have:

\[
\begin{align*}
\text{(a)} & \quad G_{\text{ind}} \left( \frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha} \right) = g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H^a_{\alpha}H^a_{\beta}, \\
\text{(b)} & \quad G_{\text{ind}} \left( \frac{\delta}{\delta u^\beta}, \frac{\partial}{\partial v^\alpha} \right) = 0, \\
\text{(c)} & \quad G_{\text{ind}} \left( \frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\alpha} \right) = g_{\alpha\beta}.
\end{align*}
\]

**Proof.** By direct calculations using (5), (9), (1) and (4) we obtain

\[
G_{\text{ind}} \left( \frac{\delta}{\delta u^\beta}, \frac{\delta}{\delta u^\alpha} \right) = \tilde{G} \left( B^i_{\beta} \delta^* x^i + H^b_{\beta}B_b, B^i_{\alpha} \delta^* x^i + H^a_{\alpha}B_a \right)
\]

\[
= \tilde{g}_{ij} B^i_{\alpha}B^j_{\beta} + H^a_{\alpha}H^a_{\beta} \delta_{ab} = g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H^a_{\alpha}H^a_{\beta},
\]

\[
\text{(11)}
\]
which proves (11a). Then, (11b) follows from the definition of the non-linear connection $HTM^\circ$. Finally, by using (5), (1) and (4) we deduce that

$$G_{\text{ind}} \left( \frac{\partial}{\partial y^\beta}, \frac{\partial}{\partial y^\alpha} \right) = \tilde{G} \left( B_\beta^j \frac{\partial}{\partial y^j}, B_\alpha^i \frac{\partial}{\partial y^i} \right) = \tilde{g}_{ij} B_\alpha^i B_\beta^j = g_{\alpha\beta},$$

which completes the proof of the lemma.

Next, we recall the following characterization of totally geodesic Finsler submanifolds.

**Theorem 2.2** (Bejancu-Farran [3], p. 134). If $(M, F)$ is a totally geodesic Finsler submanifold of $F^{n+p}=(\tilde{M}, \tilde{F})$ if and only if we have

$$(12) \quad H_a^\alpha = 0, \quad \forall \alpha \in \{1, ..., n\}, \quad a \in \{n+1, ..., n+p\}.$$

Now, we can state the following.

**Theorem 2.3.** $F^n$ is a totally geodesic Finsler submanifold of $F^{n+p}$ if and only if $G_{\text{ind}} = G$ on $TM^\circ$.

**Proof.** By comparing (11) with (10) we deduce that $G_{\text{ind}} = G$ on $TM^\circ$ if and only if

$$\sum_{a=n+1}^{n+p} H_a^\alpha H_\beta^a = 0, \quad \forall \alpha, \beta \in \{1, ..., n\},$$

which is equivalent to (12).

In order to find another characterization of totally geodesic Finsler submanifolds we present the relationship between the induced non-linear connection $HTM^\circ$ and the canonical non-linear connection $GTM^\circ$ on $TM^\circ$. First we consider the Cartan tensor field of type $(0,3)$ on $F^{n+p}$, whose local components are given by

$$\tilde{g}_{ijk} = \frac{1}{4} \frac{\partial^3 \tilde{F}^2}{\partial y^i \partial y^j \partial y^k}.$$

Then we define on $TM^\circ$ p Finsler tensor fields of type $(1,1)$ whose local components are given by

$$g_a^\alpha = g^{\beta\gamma} \tilde{g}_{ijk} B_\alpha^i B_\beta^j B_\gamma^k, \quad a \in \{n+1, ..., n+p\}, \alpha, \beta \in \{1, ..., n\}.$$
Finally, we consider the deformation Finsler tensor field $D$ with respect to the pair $(HT^o, GT^o)$, whose local components are given by

\[
D^\beta_\alpha = g^\alpha_\beta H^\gamma_\mu u^\gamma.
\]

Then it is proved in Bejancu-Farran [3], p. 112, that the local fields of frames $\{\delta^\alpha_\mu u^\alpha\}$ and $\{\delta^\alpha_\mu\}$ in $GT^o$ and $HT^o$ are related by

\[
\frac{\delta}{\delta u^\alpha} = \frac{\delta^\star}{\delta^\star u^\alpha} + D^\beta_\alpha \frac{\partial}{\partial v^\beta}.
\]

Lemma 2.2. Let $F^n = (M, F)$ be a Finsler submanifold of $F^{n+p} = (\tilde{M}, \tilde{F})$. Then we have:

\[
\begin{align*}
(a) & \quad G_{\text{ind}}(\frac{\delta^\star}{\delta^\star u^\beta}, \frac{\delta^\star}{\delta^\star u^\alpha}) = g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H^a_\alpha H^a_\beta + g_{\gamma\mu} D^\gamma_\alpha D^\mu_\beta, \\
(b) & \quad G_{\text{ind}}(\frac{\delta^\star}{\delta^\star u^\beta}, \frac{\partial}{\partial v^\alpha}) = -g_{\alpha\gamma} D^\gamma_\beta.
\end{align*}
\]

Proof. First by using (14) and (11) we obtain

\[
G_{\text{ind}}(\frac{\delta^\star}{\delta^\star u^\beta}, \frac{\delta^\star}{\delta^\star u^\alpha}) = G_{\text{ind}} \left( \frac{\delta}{\delta u^\beta} - D^\mu_\beta \frac{\partial}{\partial v^\mu}, \frac{\delta}{\delta u^\alpha} - D^\gamma_\alpha \frac{\partial}{\partial v^\gamma} \right)
\]

\[
= g_{\alpha\beta} + \sum_{a=n+1}^{n+p} H^a_\alpha H^a_\beta + g_{\gamma\mu} D^\gamma_\alpha D^\mu_\beta,
\]

and (15a) is proved. In a similar way we obtain (15b). □

Theorem 2.4. $F^n$ is a totally geodesic Finsler submanifold of $F^{n+p}$ if and only if $G_{\text{ind}} = G^*$. 

Proof. Suppose $F^n$ is totally geodesic in $F^{n+p}$. Then by (12) and (13) we have $H^\beta_\alpha = 0$ and $D^\beta_\alpha = 0$, $\forall \alpha, \beta \in \{1, \ldots, n\}$, $a \in \{n+1, \ldots, n+p\}$. Using these in (15) and comparing with (3) we deduce that $G_{\text{ind}} = G^*$. Conversely, if $G_{\text{ind}} = G^*$, from (15) and (3) we deduce that

\[
g_{\alpha\gamma} D^\gamma_\beta = 0 \quad \text{and} \quad \sum_{a=n+1}^{n+p} H^a_\alpha H^a_\beta + g_{\gamma\mu} D^\gamma_\alpha D^\mu_\beta = 0, \forall \alpha, \beta \in \{1, \ldots, n\},
\]
which imply (12). Hence $\mathbb{F}^n$ is totally geodesic.

By Theorems 2.3 and 2.4 we have a proof of Theorem 2.1. In particular, let us consider $\mathbb{F}^{n+p}=(\tilde{M}, \tilde{F})$ a Riemannian manifold, that is $\tilde{g}_{ij} = 0$, $\forall i, j, k \in \{1, ..., n+p\}$. Then by (13) we deduce that $D^\alpha_\beta = 0$ for all $\alpha, \beta \in \{1, ..., n\}$, and therefore by (14) we obtain $HTM^\circ = GTM^\circ$. As a conclusion we have $G = G^\ast$. Therefore, in the particular case of Riemannian submanifolds we obtain the following result.

**Theorem 2.5.** Let $(M, g)$ be a submanifold of a Riemannian manifold $(\tilde{M}, \tilde{g})$. Then $(M, g)$ is totally geodesic if and only if the Riemannian metric induced on $TM$ by the Sasaki metric of $TM$ coincides with the Sasaki metric induced on $TM$ by $g$.

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