RICCI FLOW ON QUASI EINSTEIN MANIFOLD*

BY

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Abstract. In this paper we study the Ricci flow on a quasi Einstein manifold, obtained some results on basic control of the scalar curvature and volume of the manifold.

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1. Introduction

In 1982 HAMILTON introduced the Ricci flow [2] and after him he himself and many authors worked on it in [9], [10], [4], [5], [6]. Ricci flow is a mean of processing a smooth Reimannian metric $g$ on a smooth closed manifold $M$ by allowing it to satisfy the equation

(1.1) \[ \frac{\partial g}{\partial t} = -2Ric(g), \]

where $Ric(g)$ is the Ricci curvature. The behaviour of the flow depends on the topology of the underlying manifolds. For example in an Einstein manifold, if one takes a metric $g_0$ such that

(1.2) \[ Ric(g_0) = \lambda g_0, \]

for some constant $\lambda \in \mathbb{R}$ (where $\mathbb{R}$ is set of reals), then a solution of (1.1) with $g(0) = g_0$ is given by

(1.3) \[ g(t) = (1 - 2\lambda t)g_0. \]

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In particular for unit sphere \((S^n, g_0)\), we have

\begin{equation}
Ric(g_0) = (n - 1)g_0, \tag{1.4}
\end{equation}

and so

\begin{equation}
g(t) = [1 - 2(n - 1)t]g_0, \tag{1.5}
\end{equation}

and the sphere collapses to a point at time \(T = \frac{1}{2(n-1)}\). If \(g_0\) is a hyperbolic metric, i.e., of constant sectional curvature -1, then

\begin{equation}
Ric(g_0) = -(n - 1)g_0, \tag{1.6}
\end{equation}

and the solution

\begin{equation}
g(t) = [1 + 2(n - 1)t]g_0, \tag{1.7}
\end{equation}

which implies the manifold expands homothetically for all time. The flow is called steady, expanding or shrinking depending on whether \(\lambda = 0\), \(\lambda < 0\) or \(\lambda > 0\) respectively. From [7] and [11], we have under the Ricci flow, the scalar curvature \(R\) evolves according to

\begin{equation}
\frac{\partial R}{\partial t} = \triangle R + 2|Ric|^2, \tag{1.8}
\end{equation}

and as a corollary of it we get

\begin{equation}
\frac{\partial R}{\partial t} \geq \triangle R + \frac{2}{n}R^2. \tag{1.9}
\end{equation}

Next from [11], we have weak maximum principle for scalars.

**Theorem 1.1.** For \(t \in [0, T]\), where \(0 < T < \infty\), suppose \(g(t)\) is a smooth family of metrics and \(X(t)\) is a smooth family of vector fields on a closed manifold \(M\). Let \(F : \mathbb{R} \times [0, T] \to \mathbb{R}\) be smooth. Suppose \(u \in C^\infty(M \times [0, T], \mathbb{R})\) solves

\begin{equation}
\frac{\partial u}{\partial t} \leq \triangle_{g(t)}u + \langle X(t), \nabla u \rangle + F(u, t), \tag{1.10}
\end{equation}

and suppose further that \(\varphi : [0, T] \to \mathbb{R}\) solves

\begin{equation}
\frac{d\varphi}{dt} = F(\varphi(t), t), \varphi(0) = \alpha \in \mathbb{R}, \tag{1.11}
\end{equation}

where \(\alpha\) is arbitrary.
then if \( u(\cdot, 0) \leq \alpha \), then \( u(\cdot, t) \leq \varphi(t), \forall t \in [0, T] \).

Reversing the inequalities one can obtain weak minimum principle by modifying \( F \) appropriately. From [11] we get basic control on the evolution of scalar curvature on a closed manifold \( M \).

**Theorem 1.2.** Suppose \( g(t) \) is a Ricci flow on a closed manifold \( M \) and for \( t \in [0, T] \). If \( R \geq \alpha \in \mathbb{R} \) at time \( t = 0 \), then

\[
R \geq \frac{\alpha}{1 - (\frac{2\alpha}{n})t}, \forall t \in [0, T].
\]

**Corollary 1.2.** (a) Suppose \( g(t) \) is a Ricci flow on a closed manifold \( M \), for \( t \in [0, T] \). If \( R \geq \alpha > 0 \) at time \( t = 0 \), then we must have \( T \leq \frac{n}{2\alpha} \).

(b) Suppose \( g(t) \) is a Ricci flow on a closed manifold \( M \), for \( t \in [0, T] \). Then \( R \geq -\frac{n^2}{4}, \forall t \in [0, T] \).

Again a non-flat Riemannian manifold \((M^n, g)\), \((n > 2)\) is said to be a quasi Einstein manifold [1] if its Ricci tensor \( \text{Ric}(g) \) is not identically zero and satisfies the condition

\[
\text{Ric}(g) = ag + b\omega \otimes \omega, b \neq 0,
\]

where \( \omega = g(\cdot, \rho) \) is a nonzero 1-form, \( \rho \) being a unit vector field and \( a, b \) are scalars, called associated scalars. \( \omega \) is called associated 1-form and \( \rho \) is called the generator of the manifold.

We know from [3] that the Ricci tensor of a 3-dimensional pseudo symmetric semi-Riemannian manifold satisfies (1.13) and hence a 3-dimensional pseudo symmetric semi-Riemannian manifold in the sense of Deszcz is a quasi Einstein manifold. In this paper we have studied relevant results of Ricci flow on quasi Einstein manifold \((QE)_n\), and when \( n = 3 \), we can use the obtained results in a 3-dimensional pseudo symmetric semi-Riemannian manifold.

**2. Ricci flow on \((QE)_n\)**

From (1.1) and (1.13) we get by solving them for a metric \( g_0 \)

\[
g(t) = -2(\alpha g_0 + b\rho'(g_0) \rho')t + c,
\]

where \( c \) is an arbitrary constant and \( g_0' = g(X, \rho), g_0'' = g(Y, \rho). \) At \( t = 0, g(0) = g_0 \) implies \( c = g_0 \). So from (2.1) evolution of Ricci flow in \((QE)_n\) is

\[
g(t) = (-2at + 1)g_0 - 2bt \rho'g_0' \rho'.
\]
Proposition 2.1. Under the Ricci flow on a quasi Einstein manifold the scalar curvature satisfies the differential inequality

\[
\frac{\partial R}{\partial t} \geq \Delta R + 2 \left( \frac{R^2}{n} - a^2 n^2 \right) - 2b^2.
\]

**Proof.** We have from (1.13) \( \text{Ric}(g) = ag + bg_0'g_0'' \). By making orthogonal decomposition

\[
\text{Ric}(g) = \text{Ric}(g)^o + R_n g.
\]

For traceless Ricci curvature \( \text{Ric}^o \) we have

\[
|\text{Ric}|^2 \geq \left( \frac{R^2}{n} - a^2 n^2 - b^2 \right).
\]

Therefore from Prop. 2.5.5 of [11] and (2.5) we get \( \frac{\partial R}{\partial t} \geq \Delta R + 2\left( \frac{R^2}{n} - a^2 n^2 - b^2 \right) \).

**Theorem 2.1.** Suppose \( g(t) \) is a Ricci flow on a closed \((QE)_n\) manifold \( M \) for \( t \in [0, T] \). If \( R \geq \alpha \in \mathbb{R} \) at time \( t = 0 \) then for all times \( t \in [0, T] \), \( R \geq \varphi(t) \), where

\[
\varphi(t) = (-\sqrt{a^2 n^3 + b^2 n}).
\]

\[
(\alpha - \sqrt{a^2 n^3 + b^2 n}) \frac{4! \sqrt{a^2 n^3 + b^2 n}}{n} + (\alpha + \sqrt{a^2 n^3 + b^2 n})
\]

\[
(\alpha - \sqrt{a^2 n^3 + b^2 n}) \frac{4! \sqrt{a^2 n^3 + b^2 n}}{n} - (\alpha + \sqrt{a^2 n^3 + b^2 n})
\]

**Proof.** Applying weak minimum principle, and solving

\[
\frac{d\varphi}{dt} = F(\varphi(t), t) = 2 \left( \frac{\varphi^2}{n} - a^2 n^2 - b^2 \right),
\]

with \( u \equiv R, X \equiv 0 \) and \( F(r, t) = 2\left( \frac{r^2}{n} - a^2 n^2 - b^2 \right) \), we get the desired result.

**Corollary 2.1.** In a 3-dimensional pseudo symmetric semi-Riemannian manifold \( M \) if \( g(t) \) is a Ricci flow for \( t \in [0, T] \), and \( R \geq \alpha \in \mathbb{R} \) at time
t = 0, then for all times $t \in [0, T]$,

$$R \geq (-\sqrt{27a^2 + 3b^2}).$$

(2.7)

$$\left\{ \begin{array}{c}
(\alpha - \sqrt{27a^2 + 3b^2})e^{\frac{4\alpha \sqrt{27a^2 + 3b^2}}{3}} + (\alpha + \sqrt{27a^2 + 3b^2}) \\
(\alpha - \sqrt{27a^2 + 3b^2})e^{\frac{4\alpha \sqrt{27a^2 + 3b^2}}{3}} - (\alpha + \sqrt{27a^2 + 3b^2})
\end{array} \right\}.$$

Corollary 2.2. Suppose $g(t)$ is a Ricci flow on a closed $(QE)_n$ for $t \in [0, T)$. If $R \geq \alpha > 0$ at time $t = 0$, then we have

$$T \leq \frac{n}{4\sqrt{a^2n^3 + b^2n}} \log \left( \frac{\sqrt{a^2n^3 + b^2n} + \alpha}{\sqrt{a^2n^3 + b^2n} - \alpha} \right).$$

3. Ricci flow on Ricci-recurrent $(QE)_n$

A Riemannian manifold is said to be Ricci recurrent [8] if the Ricci tensor is nonzero and satisfies the condition $\nabla_X(Ric(g)) = \beta(X)Ric(g)$, where $\beta$ is a nonzero 1-form. In a Ricci-recurrent $(QE)_n$ we have,

(3.1)

$$a + b = 0.$$

So (1.13) takes the form

(3.2)

$$Ric(g) = a(g - \omega \otimes \omega).$$

Solving (1.1) and (3.2) we get for a metric $g_0$

(3.3)

$$g = -2at(g_0 - g_0'g_0'') + d,$$

where $d$ is an arbitrary constant. At $t = 0$, $g(0) = g_0$ implies $d = g_0$.

So evolution of Ricci flow in a Ricci-recurrent $(QE)_n$ will be

(3.4)

$$g = -2at(g_0 - g_0''g_0) + g_0.$$

Proposition 3.1. Under the Ricci flow on a Ricci recurrent quasi Einstein manifold the scalar curvature satisfies the differential inequality

(3.5)

$$\frac{\partial R}{\partial t} \geq \triangle R + 2 \left( \frac{R^2}{n} - a^2n \right) - 2a^2.$$
Proof. Follows from proposition (2.1) and (3.1).

Theorem 3.1. Suppose \( g(t) \) is a Ricci flow on a closed Ricci recurrent \((QE)_n\), for \( t \in [0,T] \). If \( R \geq \alpha \in \mathbb{R} \) at time \( t = 0 \) then for all times \( t \in [0,T] \)

\[
R \geq (-a\sqrt{n^3 + n}) \left\{ \left( \alpha - a\sqrt{n^3 + n} \right) e^{\frac{4at\sqrt{n}}{n}} + \left( \alpha + a\sqrt{n^3 + n} \right) \right\}.
\]

Proof. Follows from theorem 2.1 and (3.1).

Corollary 3.1. In a 3-dimensional Ricci recurrent pseudo symmetric semi-Riemannian manifold if \( g(t) \) is a Ricci flow for \( t \in [0,T] \) and \( R \geq \alpha \in \mathbb{R} \) at time \( t = 0 \), then for all time \( t \in [0,T] \)

\[
R \geq (-a\sqrt{30}) \left\{ \left( \alpha - a\sqrt{30} \right) e^{\frac{4at\sqrt{30}}{30}} + \left( \alpha + a\sqrt{30} \right) \right\}.
\]

4. Evolution of the volume under Ricci flow in \((QE)_n\)

Volume form in a manifold, \( dV = \sqrt{\det(g_{ij})}dx^1 \wedge \ldots \wedge dx^n \), evolves as the metric is deformed. We have from [11], in proposition 2.3.12

\[
\frac{\partial}{\partial t} dV = \frac{1}{2}(trh) dV,
\]

where \( h = \frac{\partial g}{\partial t} \), which gives

\[
\frac{\partial}{\partial t} dV = -RdV.
\]

In particular, when \( V(t) = vol((M, g(t))) \), one have

\[
\frac{dV}{dt} = -\int RdV.
\]

Since,

\[
R \geq -m \frac{(\alpha - m)e^{4st} + p}{(\alpha - m)e^{4st} - p},
\]
where \( m = \sqrt{a^2n^3 + b^4n}, \ s = \frac{m}{n} \) and \( p = \alpha + m \). Therefore,

\[
\frac{dV}{dt} \leq V m \frac{(\alpha - m)e^{4st} + p}{(\alpha - m)e^{4st} - p}.
\]

Integrating the above we get

\[
(4.4) \quad V \leq \frac{V_0}{(-2m)^{\frac{n}{2}}e^{nst}} \{ (\alpha - m)e^{4st} - (\alpha + m) \}^{\frac{n}{2}}.
\]

Hence we can state the following theorem

**Theorem 4.1.** If \( g(t) \) is a Ricci flow on a closed quasi Einstein manifold, for \( t \in [0, \ T] \), and at \( t = 0, \ \alpha : \inf R < 0 \), then the volume changes according to equation (4.4), which depends on the dimension of the manifold.

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