Some Aspects of $G_H$-Rings

By

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Abstract. In this paper we introduce and study a class of hyperstructure called $G_H$-ring. We show that every commutative group admits a $G_H$-ring structure. Here we obtain necessary and sufficient conditions for a $G_H$-ring to be a division $G_H$-ring (and a strong division $G_H$-ring). The notion of ideals in a $G_H$-ring is also introduced and studied here. We show that every $G_H$-ring with an identity set (i.e., an $i$-set, in short) always contains a maximal ideal. This is obtained that the maximality of an ideal $I$ of a $G_H$-ring $R$ with condition $(R)$ (i.e., a multiplicative hyperring with absorbing zero) having an $i$-set, is a necessary and sufficient condition for the quotient $G_H$-ring $R/I$ of $R$ to be a $G_H$-field. We establish an isomorphism theorem on $G_H$-rings in analogy to the first isomorphism theorem on rings. We construct, over any ring $R$, a $G_H$-ring structure $R_A$, induced by each $A \subseteq P(R)$ with $|A| \geq 2$. We study such $G_H$-ring $R_A$ over a ring $R$, in accordance with the nature of the set $A$ chosen.

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1. Introduction

Krasner’s hyperring [7], introduced and studied by Krasner is a hypercompositional structure $(S, +, \cdot)$ where $(S, +)$ is a canonical hypergroup, $(S, \cdot)$ is a semigroup [4] in which the zero element is absorbing and the operation $\cdot$ is a two-sided distributive one over the hypercomposition $\cdot$. Marty initiated the study of hypergroups in 1934. A hypergroup [5] (or multigroup) $(G, \circ)$ is a semi-hypergroup (i.e., a hypercompositional structure with single associative hyperoperation) such that for all $a \in G, G \circ a = a \circ G = G$. An $H_V$-semigroup [17] is a hypercompositional structure $(G, \ast)$ with a single weak associative hyperoperation $\ast$ (in the sense that $a \ast (b \ast c) \cap (a \ast b) \ast c \neq \phi, \forall a, b, c \in G$). A canonical hypergroup [5] $(G, \circ)$ is a commutative $(a \circ b =...
hypergroup with a scalar identity $e$ (in the sense that $e \circ a = \{a\}, \forall a \in G$), of which every element has a unique inverse $a' \in G$ (in the sense that $e \in a \circ a'$) and is also invertible (in the sense that $b \in a \circ x \Rightarrow x \in a' \circ b, \forall a, b, x \in G$). Mitsui hyperstructures are a generalization of superring [10] (in which both the addition and multiplication are hyperoperations) introduced by Mittas in 1973. In [3] Chaopraknoi and Kemprasit introduced in 2005 the notion of semihypergising $(S, +, \cdot)$ where $(S, +)$ is a semihypergroup, $(S, \cdot)$ is a semigroup and the operation $\cdot$ is both left and right distributive across the hyperoperation $\circ$. Interchanging the modes of the operations involved in the hyperstructure semihyperring, we define in [16] another class of hyperstructure called hypersemiring which is an (additive) commutative semigroup $(S, +)$ endowed with a hyperoperation $\circ : S \times S \to P(S)$ such that for all $x, y, z \in S$, (i) $x \circ (y \circ z) = (x \circ y) \circ z$ and (ii) $(x + y) \circ z = x \circ z + y \circ z, x \circ (y + z) = x \circ y + x \circ z$ (where for any $A, B \in P(S), A \oplus B = \{a + b : a \in A, b \in B\}$). In contrary to Krasner’s hyperring, another kind of hyper-ring is introduced in [16]. This hyper-ring is a hypersemiring $(S, +, \circ)$ where $(S, +)$ is a commutative group whose identity element $0_S$ is absorbing in the hypersemiring $(S, +, \circ)$ (in the sense that $0_S \circ x = x \circ 0_S, \forall x \in S$). $G_H$-ring $(R, +, \circ)$ is a generalized hyper-ring [16] which being assumed to have absorbing zero $0_R$ (in the sense, $0_R \circ x = x \circ 0_R = \{0_R\}$, instead of $0_R \in 0_R \circ x = x \circ 0_R$) is defined by replacing ”set equality” type distributive axioms of hyper-ring by ”one-sided containment (⊆)” type distributive axioms $(x + y) \circ z \subseteq x \circ z + y \circ z$, and $x \circ (y + z) \subseteq x \circ y + x \circ z, \forall x, y, z \in R$.

In [15], Rota initiates the study of multiplicative hyperring which is an additive commutative group $(R, +)$ endowed with a hyperoperation $\cdot$ such that

(i) $(R, \cdot)$ is semihypergroup;

(ii) $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$, and $x \cdot (y + z) \subseteq x \cdot y + x \cdot z, \forall x, y, z \in R$ and

(iii) $(-x) \cdot y = x \cdot (-y) = -(x \cdot y), \forall x, y, z \in R$.

The $G_H$-rings and the multiplicative hyperrings form two distinct classes of hyperalgebras, none of which being contained in other (see examples 2.3 (a), (b), (c)).

Γ-ring is the motivating algebraic structure to conceptualise the idea of $G_H$-ring. The notion of Γ-ring was introduced by Nobusawa in [11] and subsequently, was studied and developed by Barnes, Booth, Kyuno, Luh, Ravisankar and many others (see [1], [2], [8], [9], [14]). Barnes
in [1] defined the Γ-ring by restricting some axioms of Nobusawa’s Γ-ring. In Barnes sense, an additive commutative group $M$ is called a Γ-ring if for another additive commutative group $\Gamma$, there exists a mapping $M \times \Gamma \times M \to M$ described by $(x, \alpha, y) \mapsto x\alpha y$ such that

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$;
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$;
(iii) $x(\alpha + \beta)y = x\alpha y + a\beta y$;
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

for any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

The identity element $0_M$ of the group $M$ is absorbing in the Γ-ring $M$ in the sense that, for any $x, y \in M$ and for any $\alpha \in \Gamma$, $x0_M = 0_Mx = x0_y = y0_M$. For any Γ-ring $M$, we define a hyperoperation $\circ$ on $M$ as follows: $a \circ b = \{aob : \alpha \in \Gamma\}$ for any $a, b \in M$. Then, $(M, +, \circ)$ is a $G_H$-ring which we call as the associated $G_H$-ring of the Γ-ring $M$ and we denote it by $M_H$.

Corresponding to every subset $A \in P^*(R) = P(R) \setminus \{0_R\}$ of a ring $(R, +, \cdot)$, there exists a $G_H$-ring $(R_A, +, \circ)$ where $R_A = R$ and for any $x, y \in R_A, x \circ y = \{x \cdot a \cdot y : a \in A\}$. This $G_H$-ring is called the $G_H$-ring over the ring $R$ induced by $A$, which we study in section 4.

2. Properties of a $G_H$-ring

Let $(M_{2 \times 3}(\mathbb{R}), +)$ be the additive group of $2 \times 3$ real matrices and $A$ (with $|A| \geq 2$) be any non-empty subset of the set $M_{2 \times 3}(\mathbb{R})$ of $2 \times 3$ real matrices. For any $a, b \in M_{2 \times 3}(\mathbb{R})$, let $a \circ b = \{a \cdot \alpha \cdot b : \alpha \in A\}$, where $\cdot$ is the usual multiplication of matrices. Then, $(M_{2 \times 3}(\mathbb{R}), \circ)$ is a semi-hypergroup. Also, this is to be observed that, $p \in a \circ (b + c) \Rightarrow p = a \cdot \alpha \cdot (b + c)$ (for some $\alpha \in A$) = $a \cdot \alpha \cdot b + a \cdot \alpha \cdot c \in a \circ b + a \circ c$. Thus, $a \circ (b + c) \subseteq a \circ b + a \circ c$. Similarly, $(b + c) \circ a \subseteq b \circ a + c \circ a$. Moreover, corresponding to $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, we see that, for $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, b = c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $a \circ (b + c) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$; while, $a \circ b = a \circ c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $a \circ (b + c) \neq a \circ b + a \circ c$. Similarly, $(b + c) \circ a \neq b \circ a + c \circ a$, for $b = c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, here the hyperoperation $\circ$ is
not in general exactly distributive over the operation $+$, but is "inclusively" distributive in the sense that, $a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(b + c) \circ a \subseteq b \circ a + c \circ a$. This evokes the idea to perceive formally and study generally a new class of hyperstructure (called the $G_H$-ring) defined as follows.

**Definition 2.1.** A non-empty set $R$ endowed with a binary operation $+$ and a hyperoperation $\circ$ is called a **generalized hyperring** or simply a $G_H$-ring if

(i) $(R, +)$ is a commutative group (with identity $0_R$),
(ii) $(R, \circ)$ is a semihypergroup,
(iii) $(x + y) \circ z \subseteq x \circ z + y \circ z$ and $x \circ (y + z) \subseteq x \circ y + x \circ z$, $\forall x, y, z \in R$,
(iv) $0_R \circ x = x \circ 0_R = \{0_R\}$, $\forall x \in R$ (absorbing property of $0_R$).

A $G_H$-ring $(R, +, \circ)$ is called commutative if $x \circ y = y \circ x$, $\forall x, y \in R$. In case of a ring $(R, +, \cdot)$, the identity $0_R$ of the group $(R, +)$ is inherently absorbing within the structure of the ring. The insertion of the axiom (iv) in the definition of $G_H$-ring in addition to first three axioms, is simply due to make the absorbing property of its zero element inherent within its hypercompositional structure. In fact, the following two examples show that the fourth axiom is not in general a consequence of the first three in the definition of $G_H$-ring.

**Example 2.2.** (a) On the additive commutative group of real numbers $(\mathbb{R}, +)$, the hyperoperation $\circ$ defined as $x \circ y = \{r \in \mathbb{R} : 0 < r < m\}$, $\forall (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ where, $m = \max\{|x|, |y|\}$, and $0 \circ 0 = \{0\}$, is commutative and satisfies first three axioms in the definition of $G_H$-ring, whereas $0 \notin 0 \circ x$, $\forall x \in \mathbb{R} \setminus \{0\}$.

(b) On the additive commutative group of integers $(\mathbb{Z}, +)$, the hyperoperation $\circ$ defined as $x \circ y = \{0, x, y\}$ is commutative and satisfies first three axioms in the definition 2.1, but it does not satisfy axiom (iv) as we see that, $0 \in 0 \circ x = x \circ 0 \neq \{0\}$, $\forall x \in \mathbb{R} \setminus \{0\}$. So, $(\mathbb{Z}, +, \circ)$ is not a $G_H$-ring. However, every commutative group admits a $G_H$-ring structures and it is evident from the following construction of a $G_H$-ring in the example 2.3 (a).

**Example 2.3.** (a) For any commutative group $(G, +)$, the hyperstructure $(G, +, \circ)$ is a commutative $G_H$-ring when $x \circ y = \{0_G, x, y\}$, $\forall x, y \in G \setminus \{0_G\}$, and $x \circ 0_G = 0_G \circ x = \{0_G\}$, $\forall x \in G$. Clearly, this $G_H$-ring is not a multiplicative hyperring.
(b) For any commutative group \((G, +)\), the hyperstructure \((G, +, \circ)\) is a commutative multiplicative hyperring when \(x \circ y = \{0_G, x, -x, y, -y\}\), \(\forall x, y \in G\). Clearly, this multiplicative hyperring is not a \(G_H\)-ring.

(c) To deal with complementary multiplicative hyperrings, Procesi and Rota construct in [13] a multiplicative hyperring \((A, +, \circ)\) which is not a \(G_H\)-ring, where \(A = \mathbb{Z}_6\) and \(\forall a, b \in A, a \cdot b = ab + 1, I = \{0, 2, 4\}\).

In case of multiplicative hyperrings in the examples 2.3 (b), (c), though 0 is not absorbing, it is seen that \(0 \in a \circ 0 \cap 0 \circ a\) for every element \(a\). Procesi and Rota show that there are some multiplicative hyperrings \(R\) in which even \(0 \notin a \circ 0\) or \(0 \notin 0 \circ a\) for some \(a \in R\). In fact, let \(R\) be a unitary domain, \(S\) be one of its unitary subrings and \(T = S \setminus \{0\}\). On \(R, \circ\) is a hyperoperation defined by \(a \circ b = aT + bT\) for all \(a, b \in R\). Then, \((R, +, \circ)\) is a multiplicative hyperring in which \(0 \notin aT\), for all \(a \in R \setminus \{0\}\). In [12], Olson and Ward obtain that “In any multiplicative hyperring \((A, +, \cdot)\), if there are elements \(a, b \in A\) such that \(|a \cdot b| = 1\), then \(0 \cdot 0 = \{0\}\)”. Adopting the same arguments applied to prove the aforesaid result, we establish the following.

**Proposition 2.4.** A multiplicative hyperring \((A, +, \cdot)\) is a \(G_H\)-ring if and only if for each \(a \in A\) there exists \((b, c) \in A^2\) such that \(|a \cdot b| = |c \cdot a| = 1\).

**Proof.** If the multiplicative hyperring \((A, +, \cdot)\) is a \(G_H\)-ring, then for each \(a \in A\) we have \((0, 0) \in A^2\) such that \(|a \cdot 0| = |0 \cdot a| = 1\).

Conversely let \((A, +, \cdot)\) be a multiplicative hyperring such that for each \(a \in A\) there exists \((b, c) \in A^2\) such that \(|a \cdot b| = |c \cdot a| = 1\). Then, \(a \cdot 0 = a \cdot (b - b) \subseteq a \cdot b - a \cdot b = \{0\}\) (since, \(|a \cdot b| = 1\) \(\Rightarrow a \cdot 0 = \{0\}\)). Similarly, \(|c \cdot a| = 1 \Rightarrow 0 \cdot a = \{0\}\). Thus, \((A, +, \cdot)\) is a \(G_H\)-ring. \(\Box\)

Throughout this section wherever will be mentioned, \((R, +, \circ)\) will denote a \(G_H\)-ring.

**Proposition 2.5.** If \(x, y\) in \((R, +, \circ)\) and for any \(A \in P^*(R), -A = \{-a : a \in A\}\), then, (i) \((-x) \circ y \cap (-x \circ y) \neq \phi\) and (ii) \(x \circ (-y) \cap (-x \circ y) \neq \phi\).

**Proof.** (i) For any \(x, y\) in \((R, +, \circ)\), \(0_R \in 0_R \circ y = (x + (-x)) \circ y \subseteq x \circ y + (-x) \circ y \Rightarrow \exists a \in x \circ y\) such that \(-a \in (-x) \circ y \cap (-x \circ y)\).

(ii) can be proved similarly.

Let \(M\) be a \(\Gamma\)-ring and \(F\) be the free abelian group generated by the set of all ordered pairs in \(\Gamma \times M\). Then, \(A = \{\sum_{i=1}^n m_i(\gamma_i, x_i) : \sum_{i=1}^n m_i x_i = \sum_{i=1}^n m_i \gamma_i x_i = \sum_{i=1}^n m_i \gamma_i x_i\}\)
0_M, \forall x \in M} is a group. Let the factor group \( F/A \) be denoted by \( R \) and the coset \( A + (\gamma, x) \) by \( [\gamma, x] \), then, every element of \( R \) can be expressed as a finite sum \( \sum_{i=1}^{n}[\gamma_i, x_i] \) and thus \( R \) forms a ring if the multiplication is defined by \( \sum_{i=1}^{n}[\alpha_i, x_i]\sum_{j=1}^{m}[\beta_j, y_j] = \sum_{i=1}^{n}\sum_{j=1}^{m}[\alpha_i, x_i][\beta_j, y_j] \) (in short). This ring \( R \) is called the right operator ring \( [2] \) of the \( \Gamma \)-ring \( M \).

Analogously, the left operator ring \( L \) short). This ring as stated in the example 2.3(a) is an \( R \)-set of the associated \( G \)-element \( A \) is said to have the right identity \( \sum_i [\delta_i, e_i] \), [14]. Similarly, if there exists an element \( \sum_i [f_i, \gamma_i] \in L \) such that \( \sum_i f_i \gamma_i x = x, \forall x \in M \), then \( M \) is said to have the left identity \( \sum_i [f_i, \gamma_i] \), [12].

Let \( M \) be a \( \Gamma \)-ring with right identity \( \sum_j [\delta_j, e_j] \). Then, the non-empty finite subset \( A = \{e_1, e_2, \ldots, e_n\} \) of \( M \) has the property that, at least one \( e_i \neq 0_M \) and for any \( x \) of the associated \( G_H \)-ring \( M_T \) of the \( \Gamma \)-ring \( M, x \in \sum_i x \odot e_i \). Similarly, if \( M \) has the left identity \( \sum_i [f_i, \gamma_i] \) then the set \( B = \{f_1, f_2, \ldots, f_n\} \) has the property that, at least one \( f_i \neq 0_M \) and for any \( x \) of the associated \( G_H \)-ring \( M_T \) of the \( \Gamma \)-ring \( M, x \in \sum_i f_i \odot x \). The existence of such properties of sets in the associated \( G_H \)-ring of a \( \Gamma \)-ring motivates us to formally conceptualise these properties over the sets of any \( G_H \)-ring.

**Definition 2.6.** A non-empty finite subset \( E_l \) (resp. \( E_r \)) \( = \{e_1, e_2, \ldots, e_n\} \) of a \( G_H \)-ring \( (R, +, \odot) \) is called a left (resp. right) identity set (in short \( i \)-set) of \( R \) if

1. \( e_i \neq 0_R \) for at least one \( i = 1, 2, \ldots, n \), and
2. for any \( a \in R \), \( a \in \sum_{i=1}^{n} e_i \odot a \) (resp. \( a \in \sum_{i=1}^{n} a \odot e_i \)).

A non-empty finite subset \( E \) of a \( G_H \)-ring \( (R, +, \odot) \) is called an \( i \)-set of \( R \) if it is both a left \( i \)-set and a right \( i \)-set of \( R \). An element \( e \) of a \( G_H \)-ring \( (R, +, \odot) \) is called a (left, right) hyperidentity of \( R \) if the set \( \{e\} \) is an (resp. left, right) \( i \)-set of \( R \).

**Example 2.7.** (a) Every non-empty finite subset \( (\neq \{0_G\}) \) of the \( G_H \)-ring as stated in the example 2.3(a) is an \( i \)-set of the same.

(b) Let \( R = \{a\sqrt{2} + b\sqrt{3} : a, b \in \mathbb{Q}\} \) and \( A = \{\sqrt{2}, \sqrt{3}\} \). Then, with respect to usual addition + of reals, \( (R, +) \) is a commutative group, with identity 0. On \( R, \odot \) is a hyperoperation defined by

\[
(a\sqrt{2} + b\sqrt{3}) \odot (c\sqrt{2} + d\sqrt{3}) = \{(a\sqrt{2} + b\sqrt{3}) \cdot (c\sqrt{2} + d\sqrt{3}) : t \in A\} = \{p\sqrt{2} + 2q\sqrt{3}, 3q\sqrt{2} + p\sqrt{3}\},
\]
where $p = 2ac + 3bd$ and $q = bc + ad$. Then, $(R, +, \circ)$ is a commutative $G_H$-ring, in which $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{5}}$ are two hyperidentities. This is also to be noted that $\{\sqrt{2} + \sqrt{3}, -\sqrt{2} - \sqrt{3}\}$ is an $i$-set of $(R, +, \circ)$. In fact, for any $x \in R$, we see that,

$$x = x \cdot (\sqrt{3} - \sqrt{2}) \cdot (\sqrt{2} + \sqrt{3}) = x \cdot \sqrt{3} \cdot (\sqrt{2} + \sqrt{3}) + x \cdot \sqrt{2} \cdot (-\sqrt{2} - \sqrt{3})$$

$$\in x \circ (\sqrt{2} + \sqrt{3}) + x \circ (-\sqrt{2} - \sqrt{3}).$$

(c) On a non-trivial commutative group $(G, +)$, a hyperoperation $\circ$ is defined as $x \circ y = G \setminus \{0_G\} = G^*$, $\forall x, y \in G^*$ and $0_G \circ x = x \circ 0_G = \{0_G\}$, $\forall x \in G$. Then, $(G, +, \circ)$ is a commutative $G_H$-ring in which every non-empty finite subset $x$ is an $i$-set and thus in particular, every non-zero element of $G_H$-ring $(G, +, \circ)$ is its hyperidentity.

(d) The commutative (additive) group of integers $(\mathbb{Z}, +)$ endowed with the hyperoperation $x \circ y = \{xy, -xy\}, \forall x, y \in \mathbb{Z}$ is a commutative $G_H$-ring in which a non-empty finite subset $E = \{e_1, e_2, \ldots, e_n\}$ is an $i$-set if and only if 1 or −1 is an element of $\sum_{i=1}^n A_i$, where $A_i = \{e_i, -e_i\}$. So, $G_H$-ring $(\mathbb{Z}, +, \circ)$ has exactly two hyperidentities which are 1 and −1. In fact, for any $x \in \mathbb{Z}$, $x \circ e_1 = \{xf_i : f_i \in A_i\}$ and also $1 = \sum_{i=1}^n f_i \in \sum_{i=1}^n A_i \iff -1 = \sum_{i=1}^n (-f_i) \in \sum_{i=1}^n A_i$. Thus here, the subset $E = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{Z}$ is an $i$-set $\iff x \in \sum_{i=1}^n x \circ e_i$ (for any $x \in \mathbb{Z}$) $\iff x = \sum_{i=1}^n x f_i$ (for some $f_i \in A_i$) $= x \sum_{i=1}^n f_i \iff 1 = \sum_{i=1}^n f_i \in \sum_{i=1}^n A_i$ (taking in ($\iff$) implication, only a particular $x \neq 0$ in $\mathbb{Z}$).

(e) The additive group of $2 \times 2$ real matrices $(M_2(\mathbb{R}), +)$ endowed with hyperoperation $\circ$ defined as

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \circ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ \begin{bmatrix} ay + cx & by + dx \\ at + cz & bt + dz \end{bmatrix}, \begin{bmatrix} ay - cx & by - dx \\ at - cz & bt - dz \end{bmatrix} \right\}$$

is a non-commutative $G_H$-ring with only two hyperidentities $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(f) On the (additive) commutative group of integers $(\mathbb{Z}, +)\circ$ is a hyperoperation defined by $x \circ y = \{2xy, 3xy\}$ for all $x, y \in \mathbb{Z}$. Then, $(\mathbb{Z}, +, \circ)$ is a commutative $G_H$-ring without having any hyperidentity. For any $x \in \mathbb{Z}$, $x = 5 \cdot 2 \cdot x + (-3) \cdot 3 \cdot x \in 5 \circ x + (-3) \circ x$ and hence, $(\mathbb{Z}, +, \circ)$ is a commutative $G_H$-ring with an $i$-set $E = \{5, -3\}$.
(g) On the (additive) commutative group of integers \((\mathbb{Z}, +)\), \(\circ\) is a hyperoperation defined by \(x \circ y = \{6xy, 9xy\}\) for all \(x, y \in \mathbb{Z}\). Then, \((\mathbb{Z}, +, \circ)\) is a commutative \(G_H\)-ring without having any \(i\)-set (and thus having no hyperidentity), since, the equation \(6x + 9y = 1\) does not have any integral solution.

**Definition 2.8.** A non-zero element \(a\) of a \(G_H\)-ring \((R, +, \circ)\) with an \(i\)-set \(E = \{e_1, e_2, \ldots, e_n\}\) is said to be left (right) invertible with respect to \(E\) if for each \(i = 1, 2, \ldots, n, (n \in \mathbb{N})\), there exist \(r_{ij} \in R\) such that \(e_i \in \sum_{j=1}^{n} r_{ij} \circ a\) (resp. \(e_i \in \sum_{j=1}^{n} a \circ r_{ij}\)). A non-zero element \(a\) of a \(G_H\)-ring with an \(i\)-set \(E\) is said to be invertible or a unit with respect to \(E\) if it is both left and right invertible with respect to \(E\).

A non-zero element \(a\) of a \(G_H\)-ring \((R, +, \circ)\) with a hyperidentity \(e\) is said to be left (right) hyperinvertible with respect to \(e\) if there exist \(r \in R\) such that \(e \in r \circ a\) (resp. \(e \in a \circ r\)). A non-zero element \(a\) of a \(G_H\)-ring with a hyperidentity \(e\) is said to be hyperinvertible or a hyperunit with respect to \(e\) if it is both left and right invertible with respect to \(e\).

**Example 2.9.** (a) The \(G_H\)-ring in the example 2.7(f), having no hyperidentity does not contain any hyperunit. But it has two units \(1\) and \(-1\) with respect to the \(i\)-set \(E = \{5, -3\}\).

(b) Every non-zero element of the \(G_H\)-ring in the example 2.7(b) is a hyperunit with respect to the hyperidentity \(\frac{1}{\sqrt{2}}\). In fact, for any \(x \in R\), we see that \(\frac{1}{\sqrt{2}} x = x \cdot \sqrt{2} \cdot r \in x \circ r\) where \(r = \frac{1}{\sqrt{2}} \in R\).

(c) Every non-singular matrix in the \(G_H\)-ring stated in example 2.7 (e) is a hyperunit with respect to each of the hyperidentities \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

and \[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}.
\]

**Proposition 2.10.** Let \(E = \{e_1, e_2, \ldots, e_n\}\) and \(E' = \{e'_1, e'_2, \ldots, e'_m\}\) be two \(i\)-sets of a \(G_H\)-ring \((R, +, \circ)\) and \(a \in R\) be a unit with respect to \(E\). Then, \(a\) is also a unit with respect to \(E'\).

**Proof.** Since \(a\) is a unit in \(G_H\)-ring \((R, +, \circ)\), with respect to the \(i\)-set \(E = \{e_1, e_2, \ldots, e_n\}\), so for each \(i = 1, 2, \ldots, n\), there exist \(r_{ik}, s_{il} \in R\) such that \(e_i \in (\sum_{k=1}^{p} r_{ik} \circ a) \cap (\sum_{l=1}^{q} a \circ s_{il})\). Again, for each \(j = 1, 2, \ldots, m\), \(e'_j \in \sum_{i=1}^{n} e_i \circ e'_j \cap \sum_{i=1}^{n} e'_j \circ e_i\). Now, \(e'_j \in \sum_{i=1}^{n} e_i \circ e'_j \subseteq \sum_{i=1}^{n} (\sum_{l=1}^{q} a \circ s_{il}) \circ e'_j \subseteq \sum_{i=1}^{n} (\sum_{l=1}^{q} a \circ s_{il} \circ e'_j) = \sum_{i=1}^{n} (\sum_{l=1}^{q} a \circ (s_{il} \circ e'_j)) \Rightarrow e'_j \in \sum_{i=1}^{n} (\sum_{l=1}^{q} a \circ t_{ij})\) for some \(t_{ij} \in R\). Similarly, \(e_i \in \sum_{k=1}^{p} r_{ik} \circ a\) and
\[ e'_j \in \sum_{i=1}^{n} e'_j \odot e_i \text{ imply that } e'_j \in \sum_{i=1}^{n} (\sum_{k=1}^{p} u_{ikj} \odot a) \text{ for some } u_{ikj} \in R. \]

Hence, \( a \) is a unit with respect to \( \mathcal{E}' \). \( \Box \)

**Definition 2.11.** A non-zero element \( x \) of \((R, +, \odot)\) is called a left (resp. right) zero divisor in \( R \) if \( \exists y \in R^* \) (resp. \( z \in R^* \)) such that \( 0_R \in x \odot y \) (resp. \( 0_R \in z \odot x \)). A both left and right zero divisor in a \( G_H \)-ring \( R \) is called a zero divisor of \( R \).

**Example 2.12.** Every element of the \( G_H \)-ring in the example 2.3(a) is a zero divisor, while the \( G_H \)-rings in examples 2.7. (b), (c), (d), (f), (g) have no zero divisor.

In a ring, a left (resp. right) invertible element \([6]\) can never be a left (resp. right) divisor of zero. This not true in general for a \( G_H \)-ring. In fact, on the commutative group of integers \((\mathbb{Z}, +)\), if we define a hypercomposition \( \odot \) by stating that, \( x \odot y = \{0, xy\}, \forall x, y \in \mathbb{Z} \), then \((\mathbb{Z}, +, \odot)\) is a commutative \( G_H \)-ring with a hyperidentity \( 1 \). Every non-zero element of \((\mathbb{Z}, +, \odot)\) is a zero divisor and \( 1, -1 \in \mathbb{Z} \), in particular, hyperunits of \((\mathbb{Z}, +, \odot)\). To get a parity with the ring theory in this regard, we perceive the notion of strong hyperinvertibility of an element of a \( G_H \)-ring.

**Definition 2.13.** A left (resp. right) hyperinvertible element \( x \) of a \( G_H \)-ring \((R, +, \odot)\) (with respect to a hyperidentity \( e \)) is said to be strongly left (resp. right) hyperinvertible in \( R \) if it is not a left (resp. right) zero divisor in \( R \). An element that is both strongly left and strongly right hyperinvertible with respect to \( e \) is said to be strongly hyperinvertible or to be a strong hyperunit with respect to \( e \).

**Example 2.14.** The hyperunits in the \( G_H \)-rings stated in examples 2.7 (b), (c), (d), (f) are all strong hyperunits.

**Proposition 2.15.** Let \( e \) be a hyperidentity of \((R, +, \odot)\) and \( \ast_r \) (resp. \( \ast_l \)) be a hyperoperation on the set \( V_r(R) \) (resp. \( V_l(R) \)) of all right (resp. left) hyperinvertible elements in \( R \) (with respect to \( e \)) defined by \( a \ast_r b = (a \odot b) \cap V_r(R), \forall a, b \in V_l(R) \) (resp. \( a \ast_l b = (a \odot b) \cap V_l(R), \forall a, b \in V_l(R) \)). Then, \( a \ast_r (b \ast_r c) \subseteq (a \ast_r b) \ast_r c, \forall a, b, c \in V_r(R) \) (resp. \( a \ast_l (b \ast_l c) \subseteq (a \ast_l b) \ast_l c, \forall a, b, c \in V_l(R) \)) i.e., \((V_r(R), \ast_r)\) (resp. \((V_l(R), \ast_l)\)) is an \( H_V \)-semigroup.

**Proof.** For any \( a, b \in V_r(R), \exists x, y \in R \) such that \( e \in a \odot x \) and \( e \in b \odot y \). Then \( a \in a \odot e \subseteq a \odot (b \odot y) = (a \odot b) \odot y \Rightarrow e \in a \odot x \subseteq ((a \odot b) \odot y) \odot x = \)
\[(a \circ b) \circ (y \circ x) \Rightarrow e \in c \circ u \text{ for some } c \in a \circ b \text{ and } u \in y \circ x. \] Now since \(e \neq 0_R, c \neq 0_R\). Hence, \(c \in V_e(R)\) and thus \(c \in (a \circ b) \cap V_e(R) = a \ast_r b\) i.e., \(a \ast_r b \in P^*(V_e(R)), \forall a, b \in V_e(R)\).

Now, for any \(a, b, c \in V_e(R), p \in a \ast_r (b \ast_r c) = a \ast_r ((b \circ c) \cap V_e(R)) \Rightarrow p \in a \circ s \text{ for some } s \in (b \circ c) \cap V_e(R) \Rightarrow p \in (a \circ (b \circ c)) \cap V_e(R) = ((a \circ b) \circ c) \cap V_e(R) \Rightarrow p \in (t \circ c) \cap V_e(R) \text{ for some } t \in a \circ b. \) Again, \(p \in V_e(R) \Rightarrow \exists q \in R^* \text{ such that } e \in p \circ q \Rightarrow e \in (t \circ c) \circ q = t \circ (c \circ q) \text{ (since, } p \in t \circ c \Rightarrow e \in t \circ v \text{ for some } v \in c \circ q \subseteq R \Rightarrow t \in V_e(R) \Rightarrow t \in (a \circ b) \cap V_e(R) = a \ast_r b. \) Thus, we have that \(p \in (t \circ c) \cap V_e(R) = t \ast_r c \text{ for some } t \in a \ast_r b \Rightarrow p \in (a \ast_r b) \ast_r c; \) whence, \(a \ast_r (b \ast_r c) \subseteq (a \ast_r b) \ast_r c. \)

Similarly, it can be proved that \(a \ast_l b \in P^*(V_l(R))\) and \((a \ast_l b) \ast_l c \subseteq a \ast_l (b \ast_l c), \forall a, b, c \in V_l(R). \)

**Proposition 2.16.** Let \((R, +, \circ)\) be commutative and have a hyperidentity \(e\). Then, \((V, \ast)\) is a commutative hypergroup where \(V\) is the set of all hyperunits (with respect to \(e\)) and \(a \ast b = (a \circ b) \cap V, \forall a, b \in V.\)

**Proof.** Since \((R, +, \circ)\) is commutative, \(V_e(R) = V_l(R) = V\) and hence, on \(V, \ast_r = \ast_l = \ast. \) Thus from proposition 2.15, \((V, \ast)\) is semihypergroup. Now, for any \(a \in V\), there exists \(x \in V\) (since, \((R, +, \circ)\) is commutative) such that \(e \in a \circ x\) and so, \(e \in a \ast x\) (since \(e \in V\)). Let \(b \in V\) be arbitrary. Then, \(b \in e \ast b \subseteq (a \ast x) \ast b = a \ast (x \ast b) \Rightarrow b \in a \ast r\) for some \(r \in x \ast b \Rightarrow b \in a \ast V\) (note that \(r \in x \ast b \Rightarrow r \in V\)). So, \(V \subseteq a \ast V\) whence, \(a \ast V = V\) for all \(a \in V.\) Hence, \((V, \ast)\) is a commutative hypergroup.

**Definition 2.17.** A \(G_H\)-ring \((R, +, \circ)\) with a hyperidentity \(e\) is called a division \(G_H\)-ring (resp. strong division \(G_H\)-ring) if every non-zero element of \(R\) is a hyperunit (resp. strong hyperunit) in \(R\) with respect to \(e.\)

**Proposition 2.18.** A \(G_H\)-ring \((R, +, \circ)\) with a hyperidentity \(e\) is a division \(G_H\)-ring if and only if \((R \setminus \{0_R\}, \ast)\) is a hypergroup, where \(a \ast b = (a \circ b) \setminus \{0_R\}, \forall a, b \in R \setminus \{0_R\}.\)

**Proof.** Let \(e\) be a hyperidentity of \((R, +, \circ)\) and \((S, \ast)\) be a hypergroup where \(S = R \setminus \{0_R\}\) and \(a \ast b = a \circ b \setminus \{0_R\}\) for all \(a, b \in S.\) Then \(a \ast S = S \ast a\) for any \(a \in S.\) So, \(e \in a \ast x \cap y \ast a\) i.e., \(e \in a \circ x \cap y \circ a\) for some \(x, y \in S.\) Thus every non-zero element of \(R\) is a hyperunit (with respect to \(e).\) i.e., \((R, +, \circ)\) is a division \(G_H\)-ring.
Conversely let \((R, +, \odot)\) be a division \(G_H\)-ring with a hyperidentity \(e\). Then, \(V_0(R) = V_1(R) = R \setminus \{0_R\}\). So, \(a \ast b = (a \odot b) \cap (R \setminus \{0_R\})\), \(\forall a, b \in R \setminus \{0_R\}\) and hence by proposition 2.14, \((R \setminus \{0_R\}, \ast)\) is a semi hypergroup. Now, let \(b \in R \setminus \{0_R\}\) be arbitrary. Then for each \(a \in R \setminus \{0_R\}\), \(e \in a \ast c \cap d \ast a\) for some \(c, d \in R \setminus \{0_R\}\) (since, \(e \in R \setminus \{0_R\}\)). Hence, \(b \in e \ast c \subseteq a \ast (c \ast b) \Rightarrow b \in a \ast t\), for some \(t \in c \ast b \subseteq R \setminus \{0_R\}\) \(\Rightarrow b \in a \ast (R \setminus \{0_R\})\); so, \((R \setminus \{0_R\}) \subseteq a \ast (R \setminus \{0_R\})\), whence, \((R \setminus \{0_R\}) = a \ast (R \setminus \{0_R\})\). Similarly, \((R \setminus \{0_R\}) \ast a = (R \setminus \{0_R\})\). Hence, \((R \setminus \{0_R\}, \ast)\) is a hypergroup. \(\square\)

**Corollary 2.19.** A \(G_H\)-ring \((R, +, \odot)\) with a hyperidentity is a strong division \(G_H\)-ring if and only if \((R \setminus \{0_R\}, \odot)\) is a hypergroup.

**Proof.** It follows from definition 2.17 and proposition 2.18. \(\square\)

**Definition 2.20.** A commutative \(G_H\)-ring \((R, +, \odot)\) with an \(i\)-set \(E\) is said to be a \(G_H\)-field if every element of \((R \setminus \{0_R\}, \odot)\) is a unit with respect to \(E\). A commutative \(G_H\)-ring \((R, +, \odot)\) with a hyperidentity \(e\) is said to be a strong \(G_H\)-field if every element of \((R \setminus \{0_R\}, \odot)\) is a strong hyperunit with respect to \(e\).

**Example 2.21.** Each of the \(G_H\)-rings in the examples 2.7(b), (c) is a strong \(G_H\)-field.

**Remark 2.22.** A \(G_H\)-ring is a strong \(G_H\)-field if and only if it is a commutative strong division \(G_H\)-ring.

### 3. Ideals in \(G_H\)-rings

Let \(I\) be a non-empty subset of a \(G_H\)-ring \((R, +, \odot)\) such that \(a + b \in I\) and \(a \odot b \subseteq I\) for all \(a, b \in I\). Then \(I\) is called a sub-\(G_H\)-ring of \((R, +, \odot)\) if \((I, +, \odot)\) is itself a \(G_H\)-ring. A sub-\(G_H\)-ring \(I\) of \((R, +, \odot)\) is called a left (resp. right) ideal of \((R, +, \odot)\) if \(x \in R, a \in I \Rightarrow x \odot a \subseteq I\) (resp. \(x \in R, a \in I \Rightarrow a \odot x \subseteq I\)) \(I\) is called an ideal of \((R, +, \odot)\) if it is both a left and a right ideal of \((R, +, \odot)\).

For a \(\Gamma\)-ring \(M\), a subgroup \(A\) of the additive group \(M\) is called a left (right) ideal of \(\Gamma\)-ring \(M\) if for any \(x \in A, \alpha \in \Gamma, y \in M, yox \in A\) (resp. \(xay \in A\)). A subgroup \(A\) of the additive group \(M\) is called an ideal \([8]\) of the \(\Gamma\)-ring \(M\) if it is both a left and a right ideal of \(M\).
Remark 3.1. (a) A non-empty subset $I$ of a $\Gamma$-ring $M$ is an ideal of $M$ if and only if $I$ is an ideal of the associated $G_H$-ring $M_\Gamma$ of $M$.

(b) For any ideals $I$ and $J$ of $(R, +, \circ), 0_R \in I$; both $I \cap J$ and $I + J = \{i + j : i \in I, j \in J\}$ are ideals of $(R, +, \circ)$.

(c) An ideal $I$ of a $G_H$-ring $(R, +, \circ)$ with an $i$-set $E$ is proper if and only if $E \not\subseteq I$.

(d) A division $G_H$-ring (resp. a strong division $G_H$-ring) and thus a $G_H$-field (resp. a strong $G_H$-field) do not contain any proper ideal.

In case of a $G_H$-ring $(R, +, \circ)$, we have seen in proposition 2.4 that $(-x) \circ y \cap (-y) \circ (-x) \neq \phi$ and $x \circ (-y) \cap (-x) \circ y \neq \phi$ for any $x, y$ in $R$, where for any $A \in P^*(R), -A = \{-a : a \in A\}$. Unlike a ring, the equality of the set-expressions $(-x) \circ y, x \circ (-y)$ and $-x \circ y$ does not hold in general, on a $G_H$-ring $(R, +, \circ)$ for any $x, y \in R$. In fact, for any two non-zero elements $x$ and $y$ of the $G_H$-ring $(G, +, \circ)$, as stated in the example 2.3(a), we see that $(-x) \circ y \neq -(x \circ y)$ and $x \circ (-y) \neq -(x \circ y)$, while $(-x) \circ y = x \circ (-y)$ if and only if $x = y$.

Definition 3.2. A $G_H$-ring $(R, +, \circ)$ is said to satisfy the condition $(\mathcal{R})$ if the set equality $(-x) \circ y = x \circ (-y) = -(x \circ y)$ (called the condition($\mathcal{R}$)) holds true for any two elements $x$ and $y$ of $R$.

This is clear from the definition that the class of $G_H$-rings with condition $(\mathcal{R})$ is precisely that of multiplicative hyperrings with absorbing zero. Following are examples of some $G_H$-rings with condition($\mathcal{R}$).

Example 3.3. (a) The associated $G_H$-ring $M_\Gamma$ of a $\Gamma$-ring $M$ is a $G_H$-ring with condition($\mathcal{R}$).

(b) Let $(G, +)$ be any (additive) commutative group and let, $a \circ b = \{0_G, a, b, -a, -b\}$ for any $a, b \in G \setminus 0_G$ and $a \circ 0_G = 0_G \circ a = \{0_G\}$ for any $a \in G$. Then, $(G, +, \circ)$ is a commutative $G_H$-ring with condition $(\mathcal{R})$.

(c) The $G_H$-ring in the example 2.7(b) is a $G_H$-ring with condition $(\mathcal{R})$.

(d) Olson and Ward consider in [12] a non-zero ring $(R, +, \cdot)$ and define on the additive group $(R, +)$ a hyperoperation $\star$ by stating that $a \star b = \{na \cdot b : n \in \mathbb{N}\}, \forall a, b \in R$. Then, $(R, +, \star)$ is a $G_H$-ring with condition $(\mathcal{R})$.

Definition 3.4. Let $A$ be a subset of a $G_H$-ring $(R, +, \circ)$. Then, the (left, right) ideal of the $G_H$-ring $R$ generated by $A$ is the smallest (resp. left, right) ideal of the $G_H$-ring $R$ containing $A$ which is denoted by (resp.
(A), (A). The principal (left, right) ideal of the $G_H$-ring $R$ generated by an element $a$ of $R$, denoted by $(a)_l, (a)_r$ (resp. $\langle a \rangle$) is the (resp. left, right) ideal (resp. $(\{a\})_l, (\{a\})_r$) of $G_H$-ring $R$.

Remark 3.5. Since $0_R$ of a $G_H$-ring $(R, +, \circ)$ is absorbing in $R$, $(0_R)_l, (0_R)_r$ is $\{0_R\}$ and thus is the smallest (resp. left, right) ideal of $R$.

Proposition 3.6. Let $(R, +, \circ)$ be a $G_H$-ring with condition $(\mathcal{R})$ having an i-set $\mathcal{E}$. Then, for any $A$ in $P^+(R)$, $(A)_l = \overline{R} \odot A = \cup \{\sum_{i=1}^n x_i \circ a_i : x_i \in R, a_i \in A, n \in \mathbb{N}\}$, $(A)_r = A \odot \overline{R} = \cup \{\sum_{i=1}^n a_i \circ x_i : x_i \in R, a_i \in A, n \in \mathbb{N}\}$ and $(A) = \overline{R} \odot A \odot \overline{R} = \cup \{\sum_{i=1}^n x_i \circ a_i : x_i \in R, a_i \in A, n \in \mathbb{N}\}$

Proof. Let $p, q \in \overline{R} \odot A$ and $x \in R$. Then, $p = \sum_{i=1}^n x_i \circ a_i$ and $q = \sum_{i=1}^m y_j \circ b_j$ for some $x_i, y_j \in R$ and $a_i, b_j \in A$. So, $p - q = (\sum_{i=1}^n x_i \circ a_i) - (\sum_{j=1}^m y_j \circ b_j) = \sum_{i=1}^n x_i \circ a_i + \sum_{j=1}^m -y_j \circ b_j = \sum_{i=1}^n x_i \circ a_i + (\sum_{j=1}^m y_j \circ b_j) \subseteq \overline{R} \odot A$ and also $r \in x \circ p \subseteq x \circ \sum_{i=1}^n x_i \circ a_i \subseteq \sum_{i=1}^n x_i \circ (x \circ y_i) = \sum_{i=1}^n (x \circ x_i) \circ a_i \implies r \in \sum_{i=1}^n t_i \circ a_i \subseteq \overline{R} \odot A$ for some $t_i \in (x \circ x_i) \subseteq R$. Hence, $\overline{R} \odot A$ is an ideal of $R$. Again, for any $a \in A$, and for the i-set $\mathcal{E} = \{e_1, e_1, \ldots, e_n\}$, $a \in \sum_{i=1}^n e_i \circ a \subseteq \overline{R} \odot a \Rightarrow a \in \overline{R} \odot A$. So, $A \subseteq \overline{R} \odot A$. Moreover, for any ideal $K$ containing $A$, clearly $\overline{R} \odot A \subseteq K$. Hence, $(A)_l = \overline{R} \odot A$. The cases for $(A)$ and $(A)$ can be proved similarly.

Definition 3.7. An ideal (resp. left, right) $I$ of a $G_H$-ring $(R, +, \circ)$ is said to be a maximal ideal (resp. left, right) if $I \neq R$ and for any ideal (resp. left, right) $K$ of $R$, $I \subseteq K \subseteq R \Rightarrow K = R$.

Proposition 3.8. Let $(R, +, \circ)$ be a $G_H$-ring with an i-set $\mathcal{E}$. Then, every proper ideal in $R$ is contained in a maximal ideal.

Proof. Let $I$ be an ideal of $R$ such that $I \neq R$ and $\mathcal{P}$ be the set of all ideals $J$ in $R$ such that $I \subseteq J \neq R$. Then, $\mathcal{P}$ is non-empty, since $I \in \mathcal{P}$. With respect to set theoretic inclusion, $\mathcal{P}$ is a poset. We consider any chain $\mathcal{C} = \{K_i : i \in \Lambda\}$ of ideals in the poset $\mathcal{P}$. Let $K = \{\bigcup_{i \in \Lambda} K_i\}$ and let $a, b \in K$. Then, there are $i, j \in \Lambda$ such that $a \in K_i$ and $b \in K_j$. Since $\mathcal{C}$ is a chain, we suppose that $K_j \subseteq K_i$; then $a, b \in K_i$ and hence $a - b \in K_i \subseteq K$ and $x \circ a, a \circ x \subseteq K_i \subseteq K$ for all $x \in R$, whence $K$ is an ideal of $R$. Now, $I \subseteq K_i, \forall i \in \Lambda \Rightarrow I \subseteq \{\bigcup_{i \in \Lambda} K_i\} = K$. Again,
Corollary 3.9. In a $G_H$-ring with an i-set maximal ideal always exists.

Proof. In a $G_H$-ring $(R, +, \circ)$, with an i-set $(0_R) \neq R$ and hence by proposition 3.8, there exists in $G_H$-ring $R$, a maximal ideal containing $(0_R)$.

Definition 3.10. Let $\rho$ be an equivalence relation on a $G_H$-ring $(R, +, \circ)$ and let $A, B \in P^*(R)$. Then we define that, $A\rho B$ if for each $a \in A$ there exists $b' \in B$ such that, $a\rho b'$ holds and for each $b \in B$ there exists $a' \in A$ such that, $a'\rho b$ holds. An equivalence relation $\rho$ on a $G_H$-ring $(R, +, \circ)$ is called a left (resp. right) congruence on $R$ if for any $a, b, x \in R, a\rho b$ holds. An equivalence relation $\rho$ on a $G_H$-ring $(R, +, \circ)$ is called a congruence on $R$ if it is both left and right congruence on $R$.

Proposition 3.11. Let $I$ be an ideal of a $G_H$-ring $(R, +, \circ)$ and $\rho$ be a relation on $R$ defined by $a\rho b \Leftrightarrow a - b \in I$. Then, $\rho$ is a congruence on $R$ and the set $R/I = \{a + I : a \in R\}$ of all $p$-congruence classes is a $G_H$-ring (called the quotient $G_H$-ring of $R$ induced by the ideal $I$) with respect to a binary operation $+\circ$ defined by $(a + I) + (b + I) = (a + b) + I$ and a hyperoperation $\circ$ defined by $(a + I)\circ(b + I) = (a\circ b) + I = \{p + I : p \in a\circ b\}.

Proof. Since $I$ is an ideal of the $G_H$-ring $(R, +, \circ)$, so $(I, +)$ is a subgroup of the commutative group $(R, +)$. Hence the relation $\rho$ is a congruence relation on the group $(R, +)$. Now, let $a, b \in R$ be such that $a\rho b$ holds. Then, $a = b + i$ for some $i \in I$. Thus, for any $x \in R, x\circ a = x\circ (b + i) \subseteq x\circ b + x\circ i \subseteq x\circ b + I \Rightarrow$ for any $p \in (x\circ a), \exists q' \in x\circ b$ and $j \in I$ such that $p = q' + j$ i.e., $p\rho q'$ holds. Similarly, for any $q \in (x\circ b), \exists p' \in x\circ a$ such that, $p'\rho q$ holds. Hence, $(x\circ a)\rho(x\circ b)$. So, $\rho$ is a congruence on $R$.

Now, if $a, b, c, d \in R$ be such that $a + I = c + I$ and $b + I = d + I$, then $(a\circ b) + I = (c\circ d) + I$. In fact, $p \in a\circ b = (c + i)\circ (d + j)$ (for some $i, j \in I$) $\subseteq (c\circ d) + (c \circ j) + (i \circ d) + (i \circ j) \subseteq c\circ d + I$ (since $I$
is an ideal of $R$) ⇒ for any $p ∈ a ⊙ b$, $∃q ∈ c ⊙ d$ such that $p − q ∈ I$; so, $(a ⊙ b) + I ⊆ (c ⊙ d) + I$. Similarly, $(c ⊙ d) + I ⊆ (a ⊙ b) + I$.

Clearly, here ($R/I, +$) is a commutative group with identity $0_{R/I} = 0_R + I = I$ and $(R/I, ⊙)$ is a semihypergroup. Again, for any $a, b, c ∈ R, (a + I) ⊙ ((b + I) ⊕ (c + I)) = (a + I) ⊙ ((b + c) + I) = (a ⊙ (b + c)) + I ⊆ (a ⊙ b + a ⊙ c) + I = ((a ⊙ b) + I) ⊕ ((a ⊙ c) + I) = ((a + I) ⊙ (b + I)) ⊕ ((a + I) ⊙ (c + I)).$ Moreover, for any $a ∈ R$, $0_{R/I} ⊕ (a + I) = (0_R + I) ⊕ (a + I) = (0_R ⊕ a) + I = \{0_R\} + I$ (since, $0_R ⊕ a = \{0_R\}$).

\[ \Box \]

**Proposition 3.12.** Let $(R, +, ⊙)$ be a commutative $G_H$-ring with condition $(\mathcal{R})$ and with an i-set $E$. Then, an ideal $I$ of $R$ is a maximal ideal of $R$ if and only if $R/I$ is a $G_H$-field.

**Proof.** Let $E = \{e_1, e_2, \ldots, e_n\}$ be an i-set of the $G_H$-ring $(R, +, ⊙)$ and $I$ be an ideal of $R$ such that the quotient $G_H$-ring $R/I$ is a $G_H$-field. Since, a $G_H$-field is a non-zero $G_H$-ring, so, there exists $a ∈ R$ such that $a + I \neq 0_{R/I} = I$ and hence, $a \notin I$ implying that $I$ is a proper ideal of $R$. So, there exists at least $e_i ∈ E$ such that $e_i \notin I$; hence $e_i + I \neq I = 0_{R/I}$ in $R/I$ for some $e_i ∈ E$. Now, for any $a ∈ R, a ∈ \sum_{i=1}^{n}(e_i ⊕ a) ⇒ a + I ∈ \sum_{i=1}^{n}(e_i + I) ⊕ (a + I).$ So, $E_{R/I} = \{e_1 + I, e_2 + I, \ldots, e_n + I\}$ is an i-set of the quotient $G_H$-ring $R/I$ of the $G_H$-ring $(R, +, ⊙)$. Let $K$ be an ideal of $R$ such that $I \subseteq K \subseteq R$. Then, $∃a ∈ K$ such that $a \notin I$. Consequently, $a + I \neq I = 0_{R/I}$ in $R/I$. So, for each $e_i + I ∈ E_{R/I}$, there exist $r_{ij} ∈ R$ such that $e_i + I ∈ \sum_{j=1}^{m}(r_{ij} + I) ⊕ (a + I) = \sum_{j=1}^{m}(r_{ij} ⊕ a + I) ⇒ e_i + I = \sum_{j=1}^{m}(s_{ij} + I) (for some $s_{ij} ∈ r_{ij} ⊕ a \subseteq K(\text{since $a ∈ K$ for each $j = 1, 2, \ldots, m$}) = (\sum_{j=1}^{m} s_{ij}) + I = k_i + I$ (for some $k_i = \sum_{j=1}^{m} s_{ij} ∈ K$) ⇒ $e_i − k_i ∈ I \subseteq K ⇒ e_i ∈ K$, whence, $E \subseteq K$ and thus, $K = R$. So, $I$ is a maximal ideal of $R$.

Conversely, let $I$ be a maximal ideal of the $G_H$-ring $(R, +, ⊙)$. Then, $I$ being a proper ideal of $G_H$-ring $R$, there exists $e_i ∈ E$ such that $e_i \notin I$ and hence, $e_i + I \neq I = 0_{R/I}$ in $R/I$. Thus $R/I$ is a non-zero $G_H$-ring and $E_{R/I} = \{e_1 + I, e_2 + I, \ldots, e_n + I\}$ is an i-set of $R/I$. Let, $a + I ∈ R/I \setminus \{0_{R/I}\}$. Then, $a \notin I$ (since, $0_{R/I} = I$). Suppose that, $K = (I, a)$ be the ideal of $R$ generated by $I ∪ \{a\}$. Clearly, $I \subseteq K$ and hence $K = R$ (since $I$ is a maximal ideal). So, $E \subseteq K$. Now, since $R$ is a commutative $G_H$-ring with condition $(\mathcal{R})$, so by proposition 3.6, $K = ∪(α + \sum_{j=1}^{m} x_j ⊕ a : α ∈ I, x_j ∈ R, m ∈ N)$. Thus, for each $e_i ∈ E, e_i ∈ α + \sum_{j=1}^{m} r_{ij} ⊕ a$, for some $α ∈ I$ and $r_{ij} ∈ R$ (since, $E \subseteq K$). So, $e_i + I ∈ (α + I) + \sum_{j=1}^{m}(r_{ij} ⊕ a + I) = I + \sum_{j=1}^{m}(r_{ij} ⊕ a + I)$.
Let \((R, +, \odot)\) and \((S, \oplus, \otimes)\) be two \(G_H\)-rings. A mapping \(f : R \to S\) is called a \(G_H\)-ring homomorphism if for any \(x, y \in R\),

(i) \(f(x + y) = f(x) \oplus f(y)\),

(ii) \(f(x \odot y) \subseteq f(x) \otimes f(y)\). A surjective (resp. injective) \(G_H\)-ring homomorphisms called an epimorphism (resp. monomorphism). A \(G_H\)-ring isomorphism is a bijective \(G_H\)-ring homomorphism.

**Example 3.14.** Let us consider the \(G_H\)-rings \((G, +, \odot)\) and \((G, +, \otimes)\) stated respectively in the examples 2.3(a) and 3.3(b). This is immediate to show here that the mapping \(f : (G, +, \odot) \to (G, +, \otimes)\) defined by \(f(a) = -a, \forall a \in G\) is an isomorphism from the \(G_H\)-ring \((G, +, \odot)\) to the \(G_H\)-ring \((G, +, \otimes)\).

**Proposition 3.15.** Let, \(f : R \to S\) be a \((G_H\)-ring\) homomorphism from a \(G_H\)-ring \((R, +, \odot)\) to a \(G_H\)-ring \((S, \oplus, \otimes)\). Then, the kernel of the \(G_H\)-ring homomorphism \(f\) (i.e., the set \(\ker(f) = \{x \in R : f(x) = 0_S\}\)) is an ideal of the \(G_H\)-ring \(R\).

**Proof.** Since, here \(f : R \to S\) is a \(G_H\)-ring homomorphism, so, \(f : (R, +) \to (S, \odot)\) is a group homomorphism and thus, \(f(0_R) = 0_S \Rightarrow 0_R \in \ker(f)\) and also for any \(a, b \in \ker(f)\), \(a - b \in \ker(f)\). Now, \(a \in \ker(f) \Rightarrow f(a) = 0_S \Rightarrow f(x \odot a) \subseteq f(x) \odot f(a) = 0_S \Rightarrow x \odot a \subseteq \ker(f)\). Similarly, \(a \odot x \subseteq \ker(f)\). Thus, \(\ker(f)\) is an ideal of \(R\). □

**Theorem 3.16** (Isomorphism theorem on \(G_H\)-rings). Let \(f : R \to S\) be an epimorphism from a \(G_H\)-ring \(R\) to a \(G_H\)-ring \(S\). Then, \(R / \ker(f) \simeq S\).

**Proof.** Let \(f : (R, +, \odot) \to (S, \oplus, \otimes)\) be a \(G_H\)-ring-epimorphism. Then, \(f : (R, +) \to (S, \oplus)\) is a group-epimorphism. So, the mapping \(\Psi : (R/\ker(f), +) \to (S, \oplus)\) defined by \(\Psi(a + \ker(f)) = f(a)\) is a group-isomorphism. Now, for any \(a, b \in R\), \(\Psi((a + \ker(f)) \odot (b + \ker(f))) = \Psi((a \odot b) + \ker(f)) = f(a \odot b) \subseteq f(a) \odot f(b) = \Psi(a + \ker(f)) \odot \Psi(b + \ker(f))\). Thus, \(\Psi\) is a \(G_H\)-ring-isomorphism. □
4. $G_H$-rings over a ring

Let $(R, +, \cdot)$ be a ring and $A \in P^*(R)$ with $|A| \geq 2$. On $R$, $\odot$ is a hyperoperation defined by $x \odot y = x \cdot A \cdot y = \{x \cdot a \cdot y : a \in A\}$ for all $x, y \in R$. Then, $(R, +, \odot)$ is a $G_H$-ring which is called the $G_H$-ring over the ring $(R, +, \cdot)$ induced by $A$ and is denoted by $R_A$. This is to be noted that, for any ring $(R, +, \cdot)$ and any $A \in P^*(R)$ with $|A| \geq 2$, $x \odot (-y) = \{xa(-y) : a \in A\} = \{-(xay) : a \in A\} = -(x \odot y)$ and similarly $(-x) \odot y = -(x \odot y)$, for all $x, y \in R_A$. Thus, the $G_H$-ring over a ring is a $G_H$-ring with condition $(\mathcal{R})$ and hence it is a multiplicative hyperring with absorbing zero. We shall study in this section some properties of $G_H$-rings $R_A$ over some rings $R$ induced by $A \in P^*(R)$. At the on-set of the study of $R_A$ over the ring $(R, +, \cdot)$, we restrict the symbol $L(A)$ (resp. $R(A)$) to denote the set of all those left (resp. right) zero divisors in the ring $(R, +, \cdot)$, that are taken from a non-empty subset $A$ of $R$. $L(A), R(A)$ may be empty in some cases. Throughout this section, unless otherwise stated, $(R, +, \cdot)$ will stand for a ring and for the sake of brevity, for any $x, y \in R$, the element $x \cdot y$ will be written simply by juxtaposition $xy$.

**Proposition 4.1.** The $G_H$-ring $R_A$ over $(R, +, \cdot)$ has a zero divisor if and only if at least one of the following holds true:

(i) $0_R \in A$;
(ii) $L(A) \cup R(A) \neq \phi$;
(iii) $L(xA) \neq \phi$ and $R(Ax) \neq \phi$ for some $x \in R$.

**Proof.** Let $x$ be a zero divisor of the $G_H$-ring $R_A$. Then $\exists y, z \in R^*$ such that $0_{R_A} \in x \odot y \cap z \odot x$. So, $\exists a, b \in A$ such that $0_R = xay = zbx$. Suppose that $0_R \notin A$. Then we have the following possibilities. Case-1, at least one of the elements $xa, zb, ay, bx$ of $R$ is $0_R$. If one of first two is true then $a$ or $b$ is in $L(A)$ (since, $0_R \notin A$), i.e., $L(A) \neq \phi$. If one of next two is true then, $R(A) \neq \phi$. Case-2, any of $xa, zb, ay, bx$ is not $0_R$. Then, $xa \in L(xA), zb \in L(zA), ay \in R(Ay), bx \in R(Ax)$ (since, $(xa)y = (zb)x = x(ay) = z(bx) = 0_R$). Thus for $x \in R, L(xA) \neq \phi$ and $R(Ax) \neq \phi$.

Conversely, let (i) is true. Then, every nonzero element of $R$ is a zero divisor in the $G_H$-ring $R_A$ (since, then $0_R \in A \Rightarrow 0_{R_A} = x0Rx \in x \odot x$). Suppose (ii) is true and $a \in L(A)$. Then, $\exists x \in R^*$ such that $ax = 0_R$ and so, $xax = 0_R$ which implies that $0_{R_A} \in x \odot x$ (since, $a \in L(A) \Rightarrow a \in A$) whence $x$ is a zero divisor in the $G_H$-ring $R_A$. By similar argument, $R(A) \neq \phi \Rightarrow R_A$.
has a zero divisor. If (iii) is true i.e., if \( L(xA) \neq \phi \) and \( R(Ax) \neq \phi \) for some \( x \in R \), then \( \exists a, b \in A \) and \( y, z \in R^* \) such that \( xay = 0_R \) and \( zbx = 0_R \) and so, \( 0_{RA} \in x \odot y \cap z \odot x \), i.e., \( x \) is a zero divisor in the \( G_H \)-ring \( R_A \).

\[ \Box \]

**Proposition 4.2.** Let \( I \) be a non-trivial proper ideal of \((R, +, \cdot)\). Then, the \( G_H \)-ring \( R_I \) does not have any of the left and the right \( i \)-set (and thus does not contain any hyperidentity).

**Proof.** Straightforward.

\[ \Box \]

**Proposition 4.3.** Let \((R, +, \cdot)\) be a ring with unity \( 1_R \) and \( A \in P^*(R) \) with \( |A| \geq 2 \). Then \( R_A \) is a \( G_H \)-ring with hyperidentity \( e \) if and only if \( e \) is a unit in \((R, +, \cdot)\) and \( e^{-1} \in A \).

**Proof.** \( e \) is a hyperidentity of \( R_A \) \( \Rightarrow 1_R \in e \odot 1_R \cap 1_R \odot e \Rightarrow \exists a, b \in A \) such that \( 1_R = ea = be \Rightarrow a = b = e^{-1} \Rightarrow e \) is a unit of \((R, +, \cdot)\) and \( e^{-1} \in A \). Conversely let \( e \) be a unit of \((R, +, \cdot)\) and \( e^{-1} \in A \). Then for any \( x \in R, x = e e^{-1} x = x e^{-1} e \Rightarrow x \in e \odot x \cap x \odot e \Rightarrow e \) is a hyperidentity of \( R_A \).

It follows from proposition 4.3 that, for a ring \((R, +, \cdot)\) with unity \( 1_R \), the \( G_H \)-ring \( R_A \) has \( 1_R \) as its hyperidentity if and only if \( 1_R \in A \). Proposition 4.3 also reveals the fact that the \( G_H \)-ring over the ring of integers \((\mathbb{Z}, +, \cdot)\) induced by a set \( A \in P^*(\mathbb{Z}), (|A| \geq 2) \) cannot have any hyperidentity, unless \( A \cap \{1-1\} \neq \phi \) (since, 1 and \(-1\) are the only two units in the ring of integers). However, the following theorem states that the \( G_H \)-ring \( Z_A \) over the ring of integers \((\mathbb{Z}, +, \cdot)\) may have an \( i \)-set under some conditions imposed on \( A \).

\[ \Box \]

**Proposition 4.4.** Let \((\mathbb{Z}, +, \cdot)\) be the ring of integers and \( A \) be a sub-semigroup of the group \((\mathbb{Z}, +)\). Then, \( Z_A \) is a (commutative) \( G_H \)-ring with an \( i \)-set if and only if some elements \( a_j \) \((j \in \{1, 2, \cdots, m + 2\})\), not all necessarily distinct, can be so chosen from \( A \) that the equation \( \{ \sum_{j=1}^{m} a_j x_j + x \cdot \gcd(a_{m+1}, a_{m+2}) = 1 \} \) is solvable in the ring \((\mathbb{Z}, +, \cdot)\) for \( x \) and \( x_j \).

**Proof.** Suppose, for some suitably chosen elements \( a_j \) \((j \in \{1, 2, \cdots, m + 2\})\), not all necessarily distinct, of the set \( A \), the equation \( \{ \sum_{j=1}^{m} a_j x_j \} + x \cdot \gcd(a_{m+1}, a_{m+2}) = 1 \) is solvable in the ring \((\mathbb{Z}, +, \cdot)\). Then, \( \exists (c, b_1, \cdots, b_m) \in \mathbb{Z}^m \) such that \( \{ \sum_{j=1}^{m} a_j b_j \} + c \cdot \gcd(a_{m+1}, a_{m+2}) = 1 \). Thus \( \gcd(a_{m+1}, a_{m+2}) \)
divides the number $1 - \{\sum_{j=1}^{m} a_j b_j\}$. Note that $gcd(a_{m+1}, a_{m+2})$ being here well-defined, $a_{m+1}$ and $a_{m+2}$ are not both equal to zero. So, the equation $a_{m+1}y + a_{m+2}z = 1 - \{\sum_{j=1}^{m} a_j b_j\}$ must have an integral solution. Thus, $\exists (b_{m+1}, b_{m+2}) \in \mathbb{Z}^2$ such that $a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \{\sum_{j=1}^{m} a_j b_j\} = 1 \ldots (\alpha)$. Suppose only $k(\leq m + 2)$ number of the elements $b_j (j = 1, 2, \ldots, m + 2)$ are distinct and those are $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$. Then, from $(\alpha)$, $\exists a_{i_1}, a_{i_2}, \ldots, a_{i_k} \in A$ such that $\{\sum_{p=1}^{k} a_p b_{i_p}\} = 1$ (In fact, if in $(\alpha)$, $b_l = b_l = b_{i_t}$ for some $l \neq q$ in $\{1, 2, \ldots, m + 2\}$ and for some $t \in \{1, 2, \ldots, k\}$ then, we write $a_l b_l + a_q b_q = (a_l + a_q) b_{i_t} = a_{i_t} b_{i_t}$ where $a_l + a_q = a_{i_t} \in A$, since, $A$ is a subsemigroup of the group $(\mathbb{Z}, +)$). Hence, $x = \{\sum_{p=1}^{k} x a_p b_{i_p}\}$ i.e., $x \in \{\sum_{p=1}^{k} x \odot b_{i_p}\}, \forall x \in \mathbb{Z}$. Hence, the $G_H$-ring $\mathbb{Z}_A$ has an $i$-set $B = \{b_{i_p} : p = 1, 2, \ldots, k\}$.

Conversely, let $\mathbb{Z}_A$ be a $G_H$-ring with an $i$-set $B = \{b_1, b_2, \ldots, b_{m+2}\}$. Then, $1 \in \{\sum_{p=1}^{m+2} 1 \odot b_j\} \Rightarrow \{\sum_{j=1}^{m+2} a_j b_j\} = 1$ for some $a_j \in A$ (not necessarily all distinct for different $j = 1, 2, \ldots, m + 2$). Clearly then, all of $a_j (j = 1, 2, \ldots, m + 2)$ can never be zero. Without any loss of generality, suppose that $a_{m+2} \neq 0$. Then, $a_{m+1} b_{m+1} + a_{m+2} b_{m+2} = 1 - \{\sum_{j=1}^{m} a_j b_j\} \Rightarrow$ the equation $a_{m+1} x_{m+1} + a_{m+2} x_{m+2} = 1 - \{\sum_{j=1}^{m} a_j b_j\}$ has integral solution. Thus for $a_{m+1}, a_{m+2} \in A, gcd(a_{m+1}, a_{m+2})$ divides the number $(1 - \{\sum_{j=1}^{m} a_j b_j\})$. So, $\exists c \in \mathbb{Z}$ such that $c \cdot gcd(a_{m+1}, a_{m+2}) = 1 - \{\sum_{j=1}^{m} a_j b_j\}$. Hence, the equation $\{\sum_{j=1}^{m} a_j x_j\} + x \cdot gcd(a_{m+1}, a_{m+2}) = 1$ is solvable in the ring $(\mathbb{Z}, +, \cdot)$.

**Proposition 4.5.** For the ring of integers $(\mathbb{Z}, +, \cdot)$ and for any subsemigroup $A$ of the group $(\mathbb{Z}, +)$, the $G_H$-ring $\mathbb{Z}_A$ has an $i$-set if $A$ has at least two relatively prime integers.  

**Proof.** It follows directly from the proposition 4.4.  

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