ON RICCI RIEMANNIAN MANIFOLDS

BY

S.K. SAHA

Dedicated to the memory of Professor M.C. CHAKI

Abstract. The object of this paper is to study some properties of a type of Riemannian manifold called Ricci Riemannian manifold \((M^n, g), \ n > 2\). The perfect fluid space time \((M^4, g)\) of general relativity is also studied.

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Introduction

The notion of Ricci Riemannian manifold was introduced by SAHA [8] in 2008. According to him, a nonflat Riemannian manifold \((M^n, g), \ n > 2\) is called Ricci Riemannian manifold if its Ricci tensor \(L\) of type \((1,1)\) satisfies the condition:

1) \(L^2X = \{r/(n - 1)\} LX\), for every vector field \(X\),

where \(r\) is the scalar curvature of the manifold. Here \(L\) is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor \(S\) of type \((0,2)\) and is defined by

2) \(g(LX, Y) = S(X, Y)\) for any vector field \(X, Y\).

Now, 1) can be written in virtue of 2) as

3) \(S(LX, Y) = \{r/(n - 1)\} S(X, Y)\).

Such an \(n\)-dimensional manifold is denoted by the symbol \((RRM)_n\).

The object of this paper is to study \((RRM)_n\) in some fields of Riemannian manifolds. After preliminaries, in section 2, it is shown that \((RRM)_n\)
cannot be a manifold of constant Riemannian curvature. In section 3, it is shown that a conformally flat \((RRM)_n\), \(n > 3\) with the length of the Ricci tensor \(r/\sqrt{(n - 1)}\) is a \((RRM)_n\) if and only if the curvature transformation \(R(X, Y)\) and the Ricci transformation \(L\) of the manifold commute. In section 4, it is shown that a conformally flat Pseudo Ricci symmetric manifold is a Ricci Riemannian manifold. In the next section, it is shown that a \((RRM)_n\) cannot be a Pseudo projective Ricci symmetric manifold. Finally considering semi Riemannian perfect fluid space time of general relativity, whose energy-momentum tensor obeys time like convergence condition it is shown that \((RRM)_4\) cannot contain pure matter.

1. Preliminaries

In a Ricci Riemannian manifold \([8] (M^n, g), n > 2\), we have

\[ S(LX, Y) = \{r/(n - 1)\}S(X, Y). \]  

(1.1)

Putting \(X = e_i, Y = e_i\), where \(\{e_i\}, i = 1, 2, \ldots, n\) is an orthonormal basis of the tangent space at each point and summing for \(1 \leq i \leq n\) we get

\[ S(Le_i, e_i) = r^2/(n - 1). \]

(1.2)

Let \(|S|\) be the length of the Ricci tensor and is defined by \(|S| = \sqrt{S(Le_i, e_i)}\) and we get from (1.2)

\[ |S| = r/\sqrt{(n - 1)}. \]

(1.3)

Since \(g\) is positive definite, \(r = 0\) implies \(S = 0\), which is not possible. Hence \(r\) cannot be zero. i.e.

\[ r \neq 0. \]

(1.4)

These results will be used in sequel.

2. \((RRM)_n\) satisfying the condition of the constant Riemannian curvature

It is known [5] that for a manifold of constant Riemannian curvature \(k\), we have

\[ R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y], \]

(2.1)
where $R$ is the curvature tensor of the manifold. Contracting (2.1) we get

$$S(Y, Z) = k(n - 1)g(Y, Z).$$

Again putting $Y = Z = e_i$ in (2.2) where $\{e_i\}, i = 1, 2, \ldots, n$ is an orthonormal basis of the tangent space at each point and summing for $1 \leq i \leq n$ and using

$$S(e_i, e_i) = r,$$

we get

$$r = k(n - 1)n.$$

From (2.2) we get

$$LY = k(n - 1)Y.$$

Putting $LY$ for $Y$ in (2.5) we get

$$L^2Y = k(n - 1)LY.$$

From 1), (2.4), (2.5) and (2.6) it follows that

$$k^2(n - 1)Y = 0,$$

for every vector field $Y$;

i.e. $k^2(n - 1) = 0$ since, $n > 2, k = 0$, which means that the manifold is flat. But this is not admissible in $(RRM)_n$, i.e. $(RRM)_n$ cannot be a manifold of constant Riemannian curvature. Hence we can state the following theorem:

**Theorem 1.** A Ricci Riemannian manifold $(M^n, g), n > 2$ cannot be a manifold of constant Riemannian curvature.

3. Conformally flat $(M^n, g), n > 2$ whose the curvature transformation $R(X, Y)$ and the Ricci transformation $L$ commute

If is known that Weyl conformal curvature tensor \[5\] $C$ of type (1,3) is given by

$$C(X, Y)Z = R(X, Y)Z - \{1/(n - 2)\}g(Y, Z)LX - g(X, Z)LY$$

$$+ S(Y, Z)X - S(X, Z)Y$$

$$+ \{r/(n - 1)(n - 2)\}[g(Y, Z)X - g(X, Z)Y],$$

$$+ \{r/(n - 1)(n - 2)\}[g(Y, Z)X - g(X, Z)Y],$$

$$+ \{r/(n - 1)(n - 2)\}[g(Y, Z)X - g(X, Z)Y],$$
where $R, L$ and $r$ are curvature tensor of type $(1,3)$, Ricci tensor of type $(1,1)$ and scalar curvature of the manifold respectively.

In a conformally flat Riemannian manifold $(M^n, g), n > 2,$

\[(3.2) \quad C(X, Y)Z = 0.\]

Hence for a conformally flat Riemannian manifold the curvature tensor can be written as

\[(3.3) \quad R(X, Y)Z = \{1/(n - 2)\}\{g(Y, Z)LX - g(X, Z)LY
\]

\[+ S(Y, Z)X - S(X, Z)Y] - \{r/(n - 1)(n - 2)\}\{g(Y, Z)X - g(X, Z)Y\}.\]

Again, it is known that $R(X, Y)$ is a skew symmetric endomorphism of the tangent space at each point which is called curvature transformation.

We suppose that the curvature transformation $R(X, Y)$ and the Ricci transformation $L$ of the manifold commute. Then $R(X, Y).L = L.R(X, Y)$ which implies in virtue of 2)

\[(3.4) \quad R(X, Y).S = 0,\]

i.e.

\[(3.5) \quad S(R(X, Y)Z, W) + S(Z, R(R(X, Y)W) = 0,\]

for all vector fields $X, Y, Z, W.$

Using (3.3) we get from (3.5)

\[(3.6) \quad g(Y, Z)S(LX, W) - g(X, Z)S(LY, W) + g(Y, W)S(LX, Z)
\]

\[\quad - g(X, W)S(LY, Z) - \{r/(n - 1)\}\{g(Y, Z)S(X, W)
\]

\[\quad - g(X, Z)S(Y, W) + g(Y, W)S(X, Z) - g(X, W)S(Y, Z)\} = 0.\]

Putting $Y = Z = e_i$ in (3.6) where $\{e_i\}, i = 1, 2, \ldots, n$ is an orthonormal basis of the tangent space at each point and summing for $1 \leq i \leq n$ we get

\[(3.7) \quad n[S(LX, W) - \{r/(n - 1)\}S(X, W)]
\]

\[\quad - [S(Le_i, e_i) - r^2/(n - 1)]g(X, W) = 0.\]

If $S(Le_i, e_i) = r^2/(n - 1),$ then from (3.7) we get

\[(3.8) \quad S(LX, W) = \{r/(n - 1)\}S(X, W),\]
i.e. the manifold \((M^n, g)\) is a Ricci Riemannian manifold.

Hence we can state the following theorem:

**Theorem 2.** If in a conformally flat \((M^n, g), n > 3\) with the length of the Ricci tensor \(r/\sqrt{(n-1)}\), the curvature transformation and the Ricci transformation commute, then it is a \((RRM)_n\).

Again, we have

\[
\begin{align*}
[R(X,Y).S](Z, W) &= -R(R(X,Y)Z, W) - R(Z, R(X,Y)W) \\
&= -\{1/(n-2)\}[g(Y, Z)S(LX, W) - g(X, Z)S(LY, W)] \\
&\quad + g(Y, W)S(LX, Z) - g(X, W)S(LY, Z) \\
&\quad + \{r/(n-1)(n-2)\}[g(Y, Z)S(X, W)] \\
&\quad - g(X, Z)S(Y, W) + g(Y, W)S(X, Z) - g(X, W)S(Y, Z]) \\
&= \{1/(n-2)\}[g(Y, Z)\{r/(n-1)\}S(X, W) \\
&\quad - g(X, Z)\{r/(n-1)\}S(Y, W) \\
&\quad + g(Y, W)\{r/(n-1)\}S(X, Z) - g(X, W)\{r/(n-1)\}S(Y, Z)] \\
&\quad + \{r/(n-1)(n-2)\}[g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\
&\quad + g(Y, W), S(X, Z) - g(X, W)S(Y, Z)] = 0, \ i.e.
\end{align*}
\]

(3.10) \(R(X,Y).S = 0\) for all \(Z, W\).

In virtue of 2) we get from (3.10)

(3.11) \(R(X,Y).L = L.R(X,Y)\).

Hence we can state the following theorem:

**Theorem 3.** In a conformally flat \((RRM)_n, n > 3\), the curvature transformation and the Ricci transformation commute.

From theorem 2 and theorem 3 we can state the following theorem:

**Theorem 4.** A conformally flat \((M^n, g), n > 3\) with the length of the Ricci tensor \(r/\sqrt{(n-1)}\) is \((RRM)_n\) if and only if the curvature transformation and the Ricci transformation commute.
4. Conformally flat pseudo Ricci symmetric manifold

The notion of pseudo Ricci symmetric manifold was introduced by Chaki in [2]. According to him a nonflat Riemannian manifold \((M^n, g), n > 3\), whose Ricci tensor \(S\) of type \((0,2)\) satisfies the condition

\[
(\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),
\]

where \(A\) is a non-zero 1-form defined by

\[
g(X, \rho) = A(X),
\]

for every vector field \(X, Y\) and \(Z\) and \(\nabla\) denotes the operator of covariant differentiation. Such an \(n\)-dimensional manifold is denoted by \((PRS)_n\).

It is known [2] that in a \((PRS)_n\)

\[
dr(X) = 2A(X)r
\]

and

\[
A(LX) = 0.
\]

It is known [5] that in a conformally flat \((M^n, g)\),

\[
(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = [1/(2(n-1))][dr(X)g(Y, Z) - dr(Z)g(X, Y)].
\]

From (4.1), (4.3) and (4.5) we get

\[
[S(Y, Z) - \{r/(n-1)\}g(Y, Z)]A(X) = [S(X, Y) - \{r/(n-1)\}g(X, Y)]A(Z).
\]

Putting \(LX\) for \(X\) in (4.6) and using (4.4) and 2) we get \(A(Z)[S(LX, Y) - \{r(n-1)\}S(X, Y)] = 0\). Since, \(A(Z) \neq 0\), we get

\[
S(LX, Y) = \{r/(n-1)\}S(X, Y)
\]

i.e. a conformally flat \((PRS)_n\) is a \((RRM)_n\). Hence we can state the following theorem:

**Theorem 5.** A conformally flat \((PRS)_n, n > 3\) is a \((RRM)_n\).
5. Pseudo projective Ricci symmetric \((RRM)_n\)

The notion of Pseudo projective Ricci symmetric manifold was introduced by Chaki and Saha in [3]. According to them, a nonflat Riemannian manifold \((M^n, g), n > 2\) whose projective Ricci tensor \(p\) [4] is not identically zero and satisfies the condition

\[(\nabla_X p)(Y, Z) = 2A(X)p(Y, Z) + A(Y)p(X, Z) + A(Z)p(X, Y),\]

where \(p\) is defined by

\[p(X, Y) = \begin{cases} n/(n-1)S(X, Y) - r/(n-1)g(X, Y), \quad r = n-1 \\ g(X, Y) \end{cases} \]

where \(A, S, r, \nabla\) have the meaning already given. Such an \(n\)-dimensional manifold is denoted by the symbol \((PWRS)_n\) where \(P, W, R\) and \(S\) denote Pseudo, Projective (Projective curvature tensor discovered by Weyl ([5]), Ricci and Symmetric respectively.

It is known that [3] in a \((PWRS)_n\)

\[p(X, \rho) = 0,\]
\[S(X, \rho) = (r/n)g(X, \rho)\]

and

\[dr(X) = 0.\]

Putting \(LX\) for \(X\) in (5.4) we get

\[S(LX, \rho) = (r/n)S(X, \rho).\]

From 3) and (5.6) we get

\[rS(X, \rho) = 0.\]

Since, in a \((PWRS)_n, r \neq 0,\)

\[S(X, \rho) = 0.\]

From (5.4), (4.2) and (5.8) we get

\[A(X) = 0.\]

which is not possible in \((PWRS)_n\).

Hence we can state the following theorem:

**Theorem 6.** \(A (RRM)_n\) cannot be a \((PWRS)_n\).
6. Semi Riemannian \((RRM)_4\)

Let a semi Riemannian \((RRM)_4\) be a general relativistic space time \((M^4, g)\), where \(g\) is a Lorentz metric with signature \((+,-,+,-)\).

We know [7] that if the Ricci tensor \(S\) of the space time satisfies the condition

\[
(6.1) \quad S(X, X) \geq 0,
\]

for every time like vector field \(X\), then (6.1) is called the time like convergence condition. We consider the general relativistic perfect fluid space time \((M^4, g)\) with unit time like velocity vector field \(V\), we can write ([1],[6])

\[
(6.2) \quad g(V, V) = -1.
\]

From (1.2) for \((M^4, g)\) we have

\[
(6.3) \quad S(Le_i, e_i) = r^2/3.
\]

The sources of any gravitational field (matter and energy) are represented in relativity by a type of \((0,2)\) symmetric tensor \(T\) called the energy momentum tensor and is given by

\[
(6.4) \quad T(X, Y) = (\sigma + \rho)A(X)A(Y) + \rho g(X, Y),
\]

where \(X, Y\) are any two vector fields and \(\sigma\) and \(\rho\) denote the energy density and the isotropic pressure of the perfect fluid respectively and \(A\) is defined by

\[
(6.5) \quad g(X, V) = A(X),
\]

for all \(X\), and we suppose that \(T\) obeys time like convergence condition (6.1). Since \((RRM)_4\) is nonflat, \(T\) obeys the condition

\[
(6.6) \quad S(X, X) > 0.
\]

The Einstein’s equation without cosmological constant [6] can be written as

\[
(6.7) \quad S(X, Y) - (1/2)rg(X, Y) = KT(X, Y),
\]

where \(K\) is the gravitational constant. From (6.4) and (6.7) we get

\[
(6.8) \quad S(X, Y) - (1/2)rg(X, Y) = K[(\sigma + p)A(X)A(Y) + pg(X, Y)].
\]
Let \( \{e_i\}, i = 1, 2, 3, 4 \) be an orthonormal basis of a frame field at a point of the space time and contracting (6.8) over \( X \) and \( Y \), we obtain.

(6.9) \[ r = K(\sigma - 3p). \]

Putting \( X = Y = V \) in (6.8) and using (6.2), (6.3) and (6.9) we get

(6.10) \[ S(V, V) = (K/2)(\sigma + 3p). \]

From (6.2), (6.8) and (6.9) we get

(6.11) \[ \sigma(\sigma + 3p) = 0. \]

Since \( \sigma + 3p > 0 \), by (6.6) and (6.10), it follows from (6.11)

(6.12) \[ \sigma = 0. \]

It is known that if pure matter exists then \( \sigma > 0 \). Hence this space time \( (M^4, g) \) can not contain pure matter. From (6.9), (6.10) and (6.12) we get

(6.13) \[ r = -2S(V, V) \]

and

(6.14) \[ p = \{2S(V, V)\}/3K. \]

Since \( S(V, V) > 0 \), by (6.6); from (6.13) and (6.14) it follows that the scalar curvature of the space time is negative and pressure of the fluid is positive. Thus we can state the following theorem:

**Theorem 7.** If in a perfect fluid space time \( (RRM)_4 \), the energy momentum tensor obeys the time like convergence condition, then such a space time cannot contain pure matter. In such a space time the pressure of the fluid is positive and the scalar curvature is negative.

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REFERENCES


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Calcutta Mathematical Society,
18/348, Kumar Lane, Chinsurah,
Hooghly-712101, West Bengal,
INDIA
sksaha30@gmail.com