SOME IDEALS OF TERNARY SEMIGROUPS

BY

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Abstract. In this paper we characterize different types of ideals of ternary semigroup and study some interesting properties of these ideals of ternary semigroup.

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1. Introduction

There is a large literature dealing with ternary operations. The notion of ternary semigroup is a natural generalization of ternary group. The notion of ideal play very important role to study the algebraic structures. In [5], Sioson studied ideal theory in ternary semigroups. He introduced the notion of prime ideal, semiprime ideal, quasi-ideal and study regular ternary semigroup by using these ideals. In [1], Dixit and Dewan studied the notions of quasi-ideal and bi-ideal in ternary semigroups. In [4], Santiago developed the theory of ternary semigroups and semiheaps. In [2,3], we study regular ternary semigroups, intra-regular ternary semigroups and congruences on ternary semigroup.

In this paper we study some interesting properties of various ideals of ternary semigroups.

2. Ideals of ternary semigroups

Definition 2.1. A non-empty set $S$ together with a ternary operation, called ternary multiplication, denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de = a(bcd)e = ab(cde)$, for all $a, b, c, d, e \in S$. 
Definition 2.2. A ternary semigroup $S$ is said to be commutative if $x_1x_2x_3 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for every permutation $\sigma$ of $\{1, 2, 3\}$ and $x_1, x_2, x_3 \in S$.

Definition 2.3. A non-empty subset $T$ of a ternary semigroup $S$ is called a ternary subsemigroup if $t_1t_2t_3 \in T$, for all $t_1, t_2, t_3 \in T$.

Definition 2.4. A non-empty subset $I$ of a ternary semigroup $S$ is called

(i) a left ideal of $S$ if $SSI \subseteq I$;
(ii) a lateral ideal of $S$ if $SIS \subseteq I$;
(iii) a right ideal of $S$ if $ISS \subseteq I$;
(iv) an ideal of $S$ if $I$ is a left, a right, a lateral ideal of $S$.

An ideal $I$ of a ternary semigroup $S$ is called a proper ideal if $I \neq S$.

In general, a lateral ideal of a ternary semigroup $S$ is not an ideal of $S$. But in particular, we have the following result:

Proposition 2.5. A minimal lateral ideal of a ternary semigroup $S$ is a minimal ideal of $S$.

Proof. Let $M$ be a minimal lateral ideal of $S$. We shall show that $M$ is a minimal ideal of $S$. Let $m \in M$. Then $SmS \cup SSmSS \subseteq SMS \cup SSMSS \subseteq M$. Since $M$ is minimal, we have $M = SmS \cup SSmSS$. Now $MSS = (SmS \cup SSmSS)SS = (SmS)SS \cup (SSmSS)SS \subseteq SmS \cup SSmSS \subseteq M$ and $SSM = SS(SmS \cup SSmSS) = SS(SmS) \cup SS(SSmSS) \subseteq SmS \cup SSmSS \subseteq M$. This implies that $M$ is a right ideal and also a left ideal of $S$. Also $M$ is a lateral ideal of $S$. Thus $M$ is an ideal of $S$. Now it remains to show that $M$ is a minimal ideal of $S$. If possible, let $M'$ be an ideal of $S$ such that $M' \subseteq M$. Since $M'$ is an ideal of $S$, it is a lateral ideal of $S$. By hypothesis, we have $M' = M$. Consequently, $M$ is a minimal ideal of $S$.

Proposition 2.6. Let $S$ be a ternary semigroup and $a \in S$. Then the principal

(i) left ideal generated by $a$ is given by $< a >_l = SSA \cup \{a\}$;
(ii) right ideal generated by $a$ is given by $< a >_r = aSS \cup \{a\}$;
(iii) lateral ideal generated by ‘a’ is given by $< a >_m = SaS \cup SSaSS \cup \{a\}$;

(iv) ideal generated by ‘a’ is given by $< a > = SSa \cup aSS \cup SaS \cup SSaSS \cup \{a\}$.

**Definition 2.7.** A proper ideal $P$ of a ternary semigroup $S$ is called a prime ideal of $S$ if for any three ideals $A, B, C$ of $S$; $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

The following theorem gives a useful characterization of a prime ideal in a ternary semigroup.

**Theorem 2.8.** In a ternary semigroup $S$, the following conditions are equivalent:

(i) $P$ is a prime ideal of $S$;

(ii) For $a, b, c \in S$; $aSbSc \subseteq P$, $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$ and $SaSbSSc \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$;

(iii) For $a, b, c \in S$; $< a > < b > < c > \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$.

**Theorem 2.9.** An ideal $P$ of a commutative ternary semigroup $S$ is prime if and only if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$, for all elements $a, b, c$ of $S$.

**Proof.** Let $S$ be a commutative ternary semigroup.

Suppose $P$ is a prime ideal of $S$ and $abc \in P$ for some $a, b, c \in S$. Then $(abc)SS \subseteq PSS \subseteq P$. This implies that $aSbSc \subseteq P$, since $S$ is commutative. Similarly, $(abc)SSSS \subseteq PSS \subseteq P$ and hence by commutativity of $S$, we have $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$, $SaSbSSc \subseteq P$. Since $P$ is a prime ideal of $S$, by Theorem 2.8, we get $a \in P$ or $b \in P$ or $c \in P$.

Conversely, let $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$, for all $a, b, c$ of $S$. We have to show that $P$ is a prime ideal of $S$. Suppose $ABC \subseteq P$ for any three ideals $A, B, C$ of $S$ and $B \not\subseteq P$, $C \not\subseteq P$. Then there exist $b \in B$ such that $b \not\in P$ and $c \in C$ such that $c \not\in P$. Now for each $a \in A$, $abc \in ABC \subseteq P$ implies that $a \in P$ and hence $A \subseteq P$. Consequently, $P$ is a prime ideal of $S$.

**Example 2.10.** In the commutative ternary semigroup $\mathbb{Z}^-$ of all negative integers, the ideal $P = \{3k : k \in \mathbb{Z}^-\}$ is a prime ideal.
For \( xyz \in P \) \((x, y, z \in \mathbb{Z}^-)\)
\[\iff \quad xyz \text{ is divisible by } 3\]
\[\iff \quad x \text{ is divisible by } 3 \text{ or } y \text{ is divisible by } 3 \text{ or } z \text{ is divisible by } 3\]
\[\iff \quad x = 3k_1 \text{ or } y = 3k_2 \text{ or } z = 3k_3 \text{ for } k_1, k_2, k_3 \in \mathbb{Z}^-\]
\[\iff \quad x \in P \text{ or } y \in P \text{ or } z \in P.\]

But the ideal \( Q = \{30k : k \in \mathbb{Z}^-\} \) is not a prime ideal of \( \mathbb{Z}^- \), since
\((-2)(-3)(-5) = -30 \in Q \) but \((-2) \notin Q, (-3) \notin Q \) and \((-5) \notin Q.\)

**Theorem 2.11.** If \( I \) is an ideal of a ternary semigroup \( S \) and \( P \) is a prime ideal of \( S \), then \( I \cap P \) is a prime ideal of \( I \), considering \( I \) as a ternary semigroup.

**Proof.** Clearly, \( I \cap P \) is an ideal of \( I \). Let \( a, b, c \in I \) and \( aSbSc \subseteq I \cap P \), \( aSSbSSc \subseteq I \cap P \), \( aSbScS \subseteq I \cap P \) and \( SbSSc \subseteq I \cap P \). Then \( aSbSc \subseteq P \), \( aSSbSSc \subseteq P \), \( aSbScS \subseteq P \) and \( SaSbSSc \subseteq I \cap P \), since \( I \cap P \subseteq P \). Since \( P \) is a prime ideal of \( S \), we have \( a \in P \) or \( b \in P \) or \( c \in P \). Thus \( a \in I \cap P \) or \( b \in I \cap P \) or \( c \in I \cap P \). Consequently, by Theorem 2.8, \( I \cap P \) is a prime ideal of \( I \).

**Note 2.12.** Let \( \{P_i\} \) be a collection of prime ideals of a ternary semigroup \( S \). Then \( \bigcup P_i \) and \( \bigcap P_i \) are ideals of \( S \) but these are not prime ideals of \( S \), in general.

However; in particular, we have the following result:

**Proposition 2.13.** Let \( \{P_i\} \) be a collection of prime ideals of a ternary semigroup \( S \) such that \( \{P_i\} \) forms a chain. Then \( \bigcup P_i \) and \( \bigcap P_i \) are both prime ideals of \( S \).

**Proof.** Clearly, \( \bigcap P_i \) is an ideal of \( S \). Let \( ABC \subseteq \bigcap P_i \) for any three ideals \( A, B, C \) of \( S \). If either \( A \subseteq P_i \), for all \( i \) or \( B \subseteq P_i \), for all \( i \) or \( C \subseteq P_i \), for all \( i \), then either \( A \subseteq \bigcap P_i \) or \( B \subseteq \bigcap P_i \) or \( C \subseteq \bigcap P_i \). If possible, let \( A, B, C \not\subseteq \bigcap P_i \). Then there exist \( i, j \) and \( k \) such that \( A \not\subseteq P_i \), \( B \not\subseteq P_j \) and \( C \not\subseteq P_k \). Since \( \{P_i\} \) is a chain, let \( P_i \subseteq P_j \subseteq P_k \). This implies that \( B, C \not\subseteq P_i \). Since \( ABC \subseteq P_i \) and \( P_i \) is prime, we must have either \( A \subseteq P_i \) or \( B \subseteq P_i \) or \( C \subseteq P_i \), a contradiction. Therefore, either \( A \subseteq \bigcap P_i \) or \( B \subseteq \bigcap P_i \) or \( C \subseteq \bigcap P_i \). Consequently, \( \bigcap P_i \) is a prime ideal of \( S \).

Similarly, we can prove that \( \bigcup P_i \) is a prime ideal of \( S \).

**Definition 2.14.** A proper ideal \( Q \) of a ternary semigroup \( S \) is called a semiprime ideal of \( S \) if \( I^3 \subseteq Q \) implies \( I \subseteq Q \) for any ideal \( I \) of \( S \).
Note 2.15. Every prime ideal of a ternary semigroup \( S \) is also a semiprime ideal of \( S \).

As in the case of prime ideals of a commutative ternary semigroup, we have the following result for semiprime ideals of a commutative ternary semigroup:

**Theorem 2.16.** A proper ideal \( Q \) of a commutative ternary semigroup \( S \) is semiprime if and only if \( x^3 \in Q \) implies that \( x \in Q \) for any element \( x \) of \( S \).

**Example 2.17.** In the commutative ternary semigroup \( \mathbb{Z}^- \) of all negative integers, the ideal \( Q = \{6k : k \in \mathbb{Z}^- \} \) is a semiprime ideal.

For \( x^3 \in Q \ (x \in \mathbb{Z}^-) \)
\[ \iff \] \( x^3 \) is divisible by 6
\[ \iff \] \( x \) is divisible by 6
\[ \iff \] \( x = 6k_1 \) for \( k_1 \in \mathbb{Z}^- \)
\[ \iff \] \( x \in Q. \)

**Definition 2.18.** A proper ideal \( I \) of a ternary semigroup \( S \) is said to be weakly irreducible if for ideals \( H \) and \( K \) of \( S \), \( H \cap K = I \) implies that \( I = H \) or \( I = K \).

We simply use the term irreducible to mean weakly irreducible.

**Definition 2.19.** A proper ideal \( I \) of a ternary semigroup \( S \) is said to be strongly irreducible if for ideals \( H \) and \( K \) of \( S \), \( H \cap K \subseteq I \) implies that \( H \subseteq I \) or \( K \subseteq I \).

**Note 2.20.** It is to be noted here that a strongly irreducible ideal of a ternary semigroup \( S \) is an irreducible ideal of \( S \).

**Definition 2.21.** A non-empty subset \( A \) of a ternary semigroup \( S \) is called an \( i \)-system if \( a, b \in A \) implies that \( < a > \cap < b > \cap A \neq \emptyset \).

**Theorem 2.22.** The following conditions in a ternary semigroup \( S \) are equivalent:

(i) \( I \) is a strongly irreducible ideal of \( S \);

(ii) If for \( a, b \in S \); \( < a > \cap < b > \subseteq I \) then \( a \in I \) or \( b \in I \);
(iii) The complement of $I$ i.e. $I^c$ is an $i$-system.

**Proof.** (i) $\implies$ (ii).
This is an immediate consequence of the Definition 2.19.

(ii) $\implies$ (iii).
If possible let $a, b \in I^c$ and $<a > \cap <b > \cap I^c = \emptyset$.
Then $<a > \cap <b > \cap I^c = \emptyset$ implies that $<a > \cap <b > \subseteq I$ and hence
by using (ii), we have $a \in I$ or $b \in I$, which is a contradiction.

Consequently, $<a > \cap <b > \cap I^c \neq \emptyset$ and hence $I^c$ is an $i$-system.

(iii) $\implies$ (i).
Let $H$ and $K$ be two ideals of $S$ such that $H \not\subseteq I$ and $K \not\subseteq I$.
Then there exist elements $a \in H - I$ and $b \in K - I$. Now from (iii),
it follows that $<a > \cap <b > \cap I^c \neq \emptyset$ i.e. there exists an element
c $\in (<a > \cap <b >) - I$. This implies that $c \in H \cap K$ and $c \notin I$. Hence
$H \cap K \not\subseteq I$. This shows that $I$ is strongly irreducible.

**Theorem 2.23.** A proper ideal $P$ of a ternary semigroup $S$ is prime if
and only if it is semiprime and strongly irreducible.

**Proof.** Suppose $P$ is a prime ideal of $S$. Then $P$ is a semiprime ideal
of $S$. Again, let $H$ and $K$ be two ideals of $S$ such that $H \cap K \subseteq P$.
Now $SHK \subseteq SSK \subseteq K$ and $SHK \subseteq SHS \subseteq H$. This implies that
$SHK \subseteq H \cap K \subseteq P$. Since $P$ is a prime ideal of $S$, it follows that
$H \subseteq P$ or $K \subseteq P$ and hence $P$ is strongly irreducible.

Conversely, suppose that a proper ideal $P$ of $S$ is both semiprime and
strongly irreducible. Let $ABC \subseteq P$ for any three ideals $A, B, C$ of $S$. Now
$[(A \cap B) \cap C][[(A \cap B) \cap C][(A \cap B) \cap C] \subseteq ABC \subseteq P$. Since $P$ is a semiprime
ideal of $S$, it follows that $[(A \cap B) \cap C] \subseteq P$. Again, since $P$ is a strongly
irreducible ideal, we have $A \cap B \subseteq P$ or $C \subseteq P$ which again implies that
$A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$. Consequently, $P$ is a prime ideal of $S$.

**Definition 2.24.** A subset $Q$ of a ternary semigroup $S$ is called a quasi-
ideal of $S$ if $QSS \cap SQS \cap SSQ \subseteq Q$ and $QSS \cap SSQSS \cap SSQ \subseteq Q$.

**Lemma 2.25.** Every left, right, lateral ideal of a ternary semigroup $S$
is a quasi-ideal of $S$.

**Remark 2.26.** The converse of Lemma 2.25 is not true, in general i.e.
a quasi-ideal may not be a left, a right or a lateral ideal of $S$. This follows
from the following example.
Example 2.27. Let \( S = M_2(\mathbb{Z}_0^-) \) be the ternary semigroup of the set of all \( 2 \times 2 \) square matrices over \( \mathbb{Z}_0^- \), the set of all non-positive integers. Let \( Q = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_0^- \right\} \). Then we can easily verify that \( Q \) is a quasi-ideal of \( S \) but \( Q \) is not a right ideal, a lateral ideal or a left ideal of \( S \).

Proposition 2.28. The intersection of arbitrary collection of quasi-ideals of a ternary semigroup \( S \) is a quasi-ideal of \( S \).

Theorem 2.29. A subset \( Q \) of a ternary semigroup \( S \) is a quasi-ideal of \( S \) if and only if \( Q \) is the intersection of a right ideal, a lateral ideal and a left ideal of \( S \).

Corollary 2.30. Every quasi-ideal of a ternary semigroup \( S \) is a ternary subsemigroup of \( S \).

Definition 2.31. A subsemigroup \( B \) of a ternary semigroup \( S \) is called a bi-ideal of \( S \) if \( BSBSB \subseteq B \).

Lemma 2.32. Every quasi-ideal of a ternary semigroup \( S \) is a bi-ideal of \( S \).

Note 2.33. The converse of Lemma 2.32 does not hold, in general i.e. a bi-ideal of a ternary semigroup \( S \) may not be a quasi-ideal of \( S \).

Remark 2.34. Since every left, right and lateral ideal of \( S \) is a quasi-ideal of \( S \), it follows that every left, right and lateral ideal of \( S \) is a bi-ideal of \( S \) but the converse is not true, in general.

In general, if \( B \) is a bi-ideal of a ternary semigroup \( S \) and \( C \) is a bi-ideal of \( B \) then \( C \) is not a bi-ideal of \( S \). But in particular, we have the following result:

Theorem 2.35. Let \( B \) be a bi-ideal of a ternary semigroup \( S \) and \( C \) a bi-ideal of \( B \) such that \( C^3 = C \). Then \( C \) is a bi-ideal of \( S \).

Proof. Since \( B \) is a bi-ideal of \( S \), \( BSBSB \subseteq B \) and since \( C \) is a bi-ideal of \( B \), \( CBCBC \subseteq C \). Therefore, \( CSCSC = (CCC)SCS(CCC) = CC(CSCSC)CC \subseteq CC(BSBSB)CC \subseteq CCBC = CCBCCCC \subseteq C(CBCBC)C \subseteq CCC = C \). Thus \( C \) is a bi-ideal of \( S \).
Definition 2.36. A ternary semigroup $S$ is called a ternary group if for all $a, b, c \in S$, the equations $abx = c$, $ayb = c$ and $zab = c$ have solutions in $S$.

Remark 2.37. In a ternary group $S$, for all $a, b, c \in S$, the equations $abx = c$, $ayb = c$ and $zab = c$ have unique solutions in $S$.

Theorem 2.38. A ternary semigroup $S$ has no proper bi-ideal if and only if $S$ is a ternary group.

Proof. Let $S$ be a ternary group. Let $B$ be a bi-ideal of $S$. Suppose that $c \in S$ and $a, b \in B$. Since $S$ is a ternary group, we have $xab = c$ has a solution in $S$. This implies that $c = yab$ for some $y \in S$. Consequently, $S = SBB$. Similarly, $S = BBS$ and $S = BSB$. Now $S = BBS = BBSBB \subseteq BB(BSBS) \subseteq BBB \subseteq B$. Consequently, $B = S$ and hence $S$ has no proper bi-ideal.

Conversely, suppose that $S$ contains no proper bi-ideals. Now for any $a, b, c \in S$, $abS$, $aSb$ and $Sab$ are all bi-ideals of $S$ and hence $abS = aSb = Sab = S$. This implies that the equations $abx = c$, $ayb = c$ and $zab = c$ have solutions in $S$. Consequently, $S$ is a ternary group.

Definition 2.39. An element $a$ in a ternary semigroup $S$ is called regular if there exists an element $x$ in $S$ such that $axa = a$.

A ternary semigroup $S$ is called regular if all of its elements are regular.

Theorem 2.40. The following conditions in a ternary semigroup $S$ are equivalent:

(i) $S$ is regular;

(ii) For any right ideal $R$, lateral ideal $M$ and left ideal $L$ of $S$, $RML = R \cap M \cap L$;

(iii) For $a, b, c \in S$, $< a >_r < b >_m < c >_l = < a >_r \cap < b >_m \cap < c >_l$;

(iv) For $a \in S$, $< a >_r < a >_m < a >_l = < a >_r \cap < a >_m \cap < a >_l$.

Theorem 2.41. If for every quasi-ideal $Q$ of $S$, $Q^3 = Q$ then $S$ is a regular ternary semigroup.

Proof. If $R$ is a right ideal, $M$ a lateral ideal and $L$ a left ideal of $S$, then by Theorem 2.29, it follows that $R \cap M \cap L$ is a quasi-ideal of $S$. Now by hypothesis,

$$R \cap M \cap L = (R \cap M \cap L)^3 = (R \cap M \cap L)(R \cap M \cap L)(R \cap M \cap L) \subseteq RML.$$
Again, clearly we have $RML \subseteq R \cap M \cap L$. Consequently, $R \cap M \cap L = RML$ and hence by Theorem 2.40, $S$ is a regular ternary semigroup.

The following theorem gives a characterization of a regular ternary semigroup $S$ in terms of bi-ideal and quasi-ideal of $S$.

**Theorem 2.42.** The following conditions in a ternary semigroup $S$ are equivalent:

(i) $S$ is regular;

(ii) For every bi-ideal $B$ of $S$, $BSBSB = B$;

(iii) For every quasi-ideal $Q$ of $S$, $QSQSQ = Q$.

**Proof.** (i) $\implies$ (ii)

Suppose $S$ is regular. Let $B$ be a bi-ideal of $S$. Let $b \in B$. Then there exists $x \in S$ such that $a = axa$. This implies that $a = axaxa \in BSBSB$. So we find that $B \subseteq BSBSB$. Again, since $B$ is a bi-ideal of $S$, $BSBSB \subseteq B$. Consequently, we have $BSBSB = B$.

Clearly, (ii) $\implies$ (iii), by using Lemma 2.32.

(iii) $\implies$ (i)

Suppose (iii) holds. Let $R$ be a right ideal, $M$ a lateral ideal and $L$ a left ideal of $S$. Then $Q = R \cap M \cap L$ is a quasi-ideal of $S$, by Theorem 2.29. By hypothesis, $QSQSQ = Q$. Now $R \cap M \cap L = Q = QSQSQ \subseteq RSMSL \subseteq RML$. Again, clearly $RML \subseteq R \cap M \cap L$. So $R \cap M \cap L = RML$ and hence by Theorem 2.40, $S$ is a regular ternary semigroup.

**Theorem 2.43.** A ternary subsemigroup $B$ of a regular ternary semigroup $S$ is a bi-ideal of $S$ if and only if $B = BSB$.

**Proof.** If $B = BSB$, then it is easy to see that $B$ is a bi-ideal of $S$.

Conversely, suppose that $B$ is a bi-ideal of a regular ternary semigroup $S$. Let $b \in B$. Then there exists $x \in S$ such that $b = bxb$. This implies that $b \in BSB$ and hence $B \subseteq BSB$. Again, $BSB \subseteq BSBSB \subseteq B$. Thus we find that $B = BSB$.

**Theorem 2.44.** A subsemigroup $B$ of a regular ternary semigroup $S$ is a bi-ideal of $S$ if and only if $B$ is a quasi-ideal of $S$.

**Proof.** Let $S$ be a regular ternary semigroup. If $B$ is a quasi-ideal of $S$, then from Lemma 2.32, it follows that $B$ is a bi-ideal of $S$. 
Conversely, let $B$ be a bi-ideal of $S$. From Theorem 2.40, we find that if $S$ is a regular ternary semigroup, then $R \cap M \cap L = RML$ for any right ideal $R$, any lateral ideal $M$ and any left ideal $L$. Now

\[
BSS \cap (SBS \cup SSBSS) \cap SSB = BSS(SBS \cup SSBSS)SSB \\
= B(SSS)B(SSS)B \cup B(SSS)SB(SSS)SB \subseteq BSBSB \cup BSSBSSB \\
\subseteq B \cup BSB \quad \text{(since $B$ is a bi-ideal)} \\
= B \cup B \quad \text{(by Theorem 2.42)} \\
= B.
\]

Consequently, $B$ is a quasi-ideal of $S$.

From Theorem 2.29 and Theorem 2.44, we have the following result:

**Corollary 2.45.** A subsemigroup $B$ of a regular ternary semigroup $S$ is a bi-ideal of $S$ if and only if $B$ is the intersection of a right ideal, a lateral ideal and a left ideal of $S$.

**Definition 2.46.** A subsemigroup $B_w$ of a ternary semigroup $S$ is called a weak bi-ideal of $S$ if $bSb \subseteq B_w$, for all $b \in B_w$.

**Proposition 2.47.** Every bi-ideal of a ternary semigroup $S$ is a weak bi-ideal of $S$.

**Remark 2.48.** Since every left ideal, right ideal and lateral ideal of $S$ is a bi-ideal of $S$, it follows that every left ideal, right ideal and lateral ideal of $S$ is a weak bi-ideal of $S$.

The converse of the above result is not true. This follows from the following example:

**Example 2.49.** Let $S = \mathbb{Z}^- \times \mathbb{Z}^- = \{(a, b) : a, b \in \mathbb{Z}^-, \text{ set of all negative integers}\}$.

Then $S$ is a ternary semigroup w.r.t. the ternary multiplication defined as follows: $(a, b)(c, d)(e, f) = (a, f)$.

Let $B_w = \{(b, b) : (b, b) \in S\}$. Then $B_w$ is a weak bi-ideal of $S$, since $(b, b)(u, v)(b, b)(x, y)(b, b) = (b, b) \in B_w$, for all $(u, v), (x, y) \in S$. It can be easily verified that $B_w$ is neither a bi-ideal nor a left ideal, a right ideal and a lateral ideal of $S$.

**Proposition 2.50.** The intersection of arbitrary collection of weak bi-ideals of a ternary semigroup $S$ is a weak bi-ideal of $S$. 

**Theorem 2.51.** A ternary semigroup $S$ is regular if and only if $B_w = \bigcup_{b \in B_w} bSbSb$, where $B_w$ is a weak bi-ideal of $S$.

**Proof.** Let $S$ be a regular ternary semigroup and $B_w$ be a weak bi-ideal of $S$. Clearly, $\bigcup_{b \in B_w} bSbSb \subseteq B_w$. Now let $b \in B_w$. Since $S$ is regular, we have $b = bx$ for some $x \in S$. So $b = bx = bx = \bigcup_{b \in B_w} bSbSb$. Thus $B_w \subseteq \bigcup_{b \in B_w} bSbSb$ and hence $B_w = \bigcup_{b \in B_w} bSbSb$.

Conversely, suppose that $B_w = \bigcup_{b \in B_w} bSbSb$ for any weak bi-ideal $B_w$ of $S$. Let $R$ be a right ideal, $M$ be a lateral ideal and $L$ be a left ideal of $S$. Then $R$, $M$, $L$ are weak bi-ideals of $S$. Consequently, we find that $R \cap M \cap L$ is a weak bi-ideal of $S$, by Proposition 2.50. Clearly, $RML \subseteq R \cap M \cap L$. Let $a \in R \cap M \cap L$. Then $a = bsbtb$ for some $b \in R \cap M \cap L$ and $s, t \in S$. This implies that $a = bsbtb \in RML$. So $R \cap M \cap L \subseteq RML$ and hence $RML = R \cap M \cap L$. Consequently, $S$ is a regular ternary semigroup, by Theorem 2.40.

*Diagrammatic representation of different ideals of ternary semigroups*

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Ideal
  /\        /\        /\        /\        /\   \\
Left Ideal Lateral Ideal Right Ideal
  | \      | \      | \      | \      | \   |
Quasi Ideal
  | \      | \      | \      | \      | \   |
Bi-Ideal
  | \      | \      | \      | \      | \   |
Weak Bi-Ideal
  | \      | \      | \      | \      | \   |
Ternary Subsemigroup*
```

*The reverse implication does not always hold.
Prime Ideal $\iff$ Semiprime Ideal + Strongly Irreducible Ideal.
REFERENCES


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