REFINEMENT OF HARDY’S INEQUALITIES INVOLVING MANY FUNCTIONS VIA SUPERQUADRATIC FUNCTIONS

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Abstract. Some new refined Hardy type integral inequalities involving \( n \) functions \((n \in \mathbb{Z}_+)\) via superquadratic functions are established for \( p \geq 2 \) and their dual inequalities are also derived. In particular, the results obtained complement and improve some recent results of Oguntuase and Persson.

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1. Introduction

In 1920, Hardy [3] announced and proved in [4] the following integral inequality (see also [5, Theorem 330, p. 245])

\[
\int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^p \, dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(x) \, dx, \quad p > 1,
\]

where \( f \) is a nonnegative measurable function and the constant \( \left( \frac{p}{p-1} \right)^p \) is the best possible. The prehistory of (1.1) up to the time when Hardy finally proved (1.1) in 1925 can be found in [5], [6] and [7]. Concerning the history and development of inequality (1.1) we refer interested reader to the books [7], [8] and [10] devoted to this subject and also the recent historical article [6] and the references given therein.

A well-known simple fact is that (1.1) can equivalently (via the substitution \( f(x) = h(x^{(1-\frac{1}{p})} x^{\frac{1}{p}}) \), be rewritten in the form

\[
\int_0^\infty \left( \frac{1}{x} \int_0^x h(t) \, dt \right)^p \, dx \leq \int_0^\infty h^p(x) \frac{dx}{x}
\]
and in this form it even holds with equality when \( p = 1 \). In this form we see that Hardy’s inequality is a simple consequence of Jensen’s inequality but this was not discovered in the dramatic period when Hardy discovered and finally proved this inequality in [4].

In a recent paper, Oguntuase and Persson [9] using mainly the notion of superquadratic and subquadratic functions proved for \( p \geq 2 \) the refined form of Hardy’s inequality (1.1), namely

\[
\begin{align*}
\int_0^b x^{-m} \left( \int_0^x f(t) \, dt \right)^p \, dx &+ \frac{m - 1}{p} \int_0^b \int_t^b \left| \frac{p}{m - 1} \left( \frac{t}{x} \right)^{1 - \frac{m-1}{p}} f(t) - \frac{1}{x} \int_0^x f(t) \, dt \right|^p \, dx \times x^{p-m-\frac{m}{p}} \, dt \\
&\leq \left( \frac{p}{m - 1} \right)^p \int_0^b \left[ 1 - \left( \frac{x}{b} \right)^{\frac{m-1}{p}} \right] x^{p-m} f^p(x) \, dx
\end{align*}
\]

for \( m > 1 \), \( 0 < b \leq \infty \), and

\[
\begin{align*}
\int_b^\infty x^{-m} \left( \int_x^\infty f(t) \, dt \right)^p \, dx &+ \frac{1 - m}{p} \int_b^\infty \int_b^t \left| \frac{p}{1 - m} \left( \frac{t}{x} \right)^{\frac{1-m}{p}+1} f(t) - \frac{1}{x} \int_x^\infty f(t) \, dt \right|^p \, dx \times x^{1-p+pm} \, dt \\
&\leq \left( \frac{p}{1 - m} \right)^p \int_b^\infty \left[ 1 - \left( \frac{b}{x} \right)^{\frac{1-m}{p}} \right] x^{p-m} f^p(x) \, dx
\end{align*}
\]

for \( m < 1 \), \( 0 \leq b < \infty \).

Furthermore, it was shown in [9] that (1.3)-(1.4) hold in the reversed direction if \( 1 < p \leq 2 \). In particular, this means that for \( p = 2 \) we have equality in (1.3)-(1.4), respectively.

In this paper we generalize and unify some of these results by proving some new refined Hardy type integral inequalities involving \( n \) functions \( n \in \mathbb{N}_+ \) via superquadratic functions. Moreover the corresponding dual inequalities are also derived. In particular, our results further complement and improve some recent results in [9].
Our main tool in the proofs is to use the notion of superquadratic functions introduced by Abramovich, Jameson and Sinnamon in [1] (see also [2]).

**Definition 1.1** ([1, Definition 2.1]). A function \( \varphi : [0, \infty) \to \mathbb{R} \) is superquadratic provided that for all \( x \geq 0 \) there exists a constant \( C_x \in \mathbb{R} \) such that
\[
\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C_x (y - x)
\]
for all \( y \geq 0 \). We say that \( f \) is subquadratic if \( -f \) is superquadratic.

The paper is organized as follows: In Section 2 we present some preliminaries, including some inequalities involving superquadratic and subquadratic functions involving \( n \) functions \((n \in \mathbb{Z}_+)\) of independent interest. In Section 3 we present some new refined Hardy type inequalities involving \( n \) functions and their proofs. Our final Section 4 is devoted to some concluding remarks and examples.

**Notations.** Throughout this paper, all functions are assumed to be measurable and expressions of the form \( 0 \cdot \infty, \infty \cdot \infty \), and \( \frac{0}{0} \) are taken to be equal to zero. In addition, by a weight function \( u \) we mean a nonnegative measurable function on the actual interval.

### 2. Preliminaries

First, we state the following refinement of Jensen’s inequality in [1], which is very useful in the proofs of our results.

**Lemma 2.1** ([1, Theorem 2.3]). Let \( (\Omega, \mu) \) be a probability measure space. The inequality
\[
\varphi \left( \int_{\Omega} f(s)d\mu(s) \right) \leq \int_{\Omega} \varphi(f(s))d\mu(s) \leq \int_{\Omega} \varphi \left( |f(s) - \int_{\Omega} f(s)d\mu(s)| \right) d\mu(s)
\]
holds for all probability measures \( \mu \) and all nonnegative \( \mu \)-integrable functions \( f \) if and only if \( \varphi \) is superquadratic. Moreover, (2.1) holds in the reversed direction if and only if \( \varphi \) is subquadratic.
Proposition 2.1. Let $f_1, f_2, ..., f_n$ be nonnegative integrable functions and define

\begin{equation}
F_k(x) := \frac{1}{x} \int_0^x f_k(t) \, dt, \quad k = 1, 2, ..., n.
\end{equation}

Suppose that $0 < b \leq \infty$, let $u : (0, \infty) \to \mathbb{R}$ be a nonnegative weight function such that the function $x \to \frac{u(x)}{x^2}$ is locally integrable on $(0, \infty)$, and define the weight function $v$ by

\begin{equation}
v(t) = t \int_t^b \frac{u(x)}{x^2} \, dx, \quad t \in (0, b).
\end{equation}

If the real-valued function $\varphi$ is superquadratic and nondecreasing on $(a, c)$, $0 \leq a < c \leq \infty$, then

\begin{equation}
\int_0^b u(x) \varphi \left( \frac{1}{n} \sum_{k=1}^n F_k(x) \right) \frac{dx}{x} \leq \int_0^b v(x) \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(x) \right) \frac{dx}{x}
\end{equation}

holds for all $f_k$ with $a < f_k(x) < c$, $0 < x \leq b$, $k \in \{1, 2, ..., n\}$.

Proof. First we apply the Arithmetic-Geometric Mean inequality

\begin{equation}
\left( \prod_{k=1}^n F_k(x) \right)^\frac{1}{n} \leq \frac{1}{n} \sum_{k=1}^n F_k(x),
\end{equation}

to the first term on the left hand side of inequality (2.3) and thereafter, by using Lemma 2.1 and Fubini’s theorem we have that

\begin{equation}
\int_0^b u(x) \varphi \left( \frac{1}{n} \prod_{k=1}^n F_k(x) \right) \frac{dx}{x} \leq \int_0^b u(x) \varphi \left( \frac{1}{n} \sum_{k=1}^n F_k(x) \right) \frac{dx}{x}
\end{equation}
\[
\begin{align*}
&= \int_0^b u(x) \varphi \left( \frac{1}{x} \int_0^x \left( \frac{1}{n} \sum_{k=1}^n f_k(t) \right) dt \right) \frac{dx}{x} \\
&\leq \int_0^b \frac{u(x)}{x^2} \int_0^x \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) \right) dt dx \\
&\quad - \int_0^b \frac{u(x)}{x^2} \int_0^x \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) - \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt \right) dt dx \\
&= \int_0^b \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) \right) \int_t^b \frac{u(x)}{x^2} dx dt \\
&\quad - \int_0^b \int_t^b \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) - \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt \right) \frac{u(x)}{x^2} dx dt \\
&= \int_0^b v(t) \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) \right) \frac{dt}{t} \\
&\quad - \int_0^b \int_t^b \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) - \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt \right) \frac{u(x)}{x^2} dx dt.
\end{align*}
\]

from which (2.3) follows and the proof is complete. \(\square\)

**Remark 2.1.** For the case \(n = 1\) Proposition 2.1 coincides with Proposition 2.1 (a) in [9].

**Proposition 2.2.** Let \(0 \leq b < \infty, u : (b, \infty) \to \mathbb{R}\) be a nonnegative locally integrable function on \((b, \infty)\), and define the function \(v\) by

\[
v(t) = \frac{1}{t} \int_b^t u(x) dx, \quad t \in (b, \infty).
\]

If the real-valued function \(\varphi\) is superquadratic and nondecreasing on \((a, c)\), \(0 \leq a < c \leq \infty\), then the inequality

\[
\int_b^\infty u(x) \varphi \left( \left[ \prod_{k=1}^n G_k(x) \right] \frac{1}{n} \right) \frac{dx}{x}
\]
\[ (2.5) \quad + \int_{b}^{\infty} \int_{b}^{t} \varphi \left( \frac{1}{n} \sum_{k=1}^{n} f_k(t) - x \int_{x}^{\infty} \frac{1}{n} \sum_{k=1}^{n} f_k(t) \frac{dt}{t^2} \right) u(x) dx \frac{dt}{t^2} \]

\[ \leq \int_{b}^{\infty} v(x) \varphi \left( \frac{1}{n} \sum_{k=1}^{n} f_k(x) \right) \frac{dx}{x} \]

holds for all \( f_k \) with \( a < f_k(x) < c, \ 0 < x \leq b, \ k \in \{1, 2, ..., n\} \), where

\[ G_k(x) := x \int_{x}^{\infty} f_k(t) \frac{dt}{t^2}. \]

**Proof.** The proof is similar to that of Proposition 2.1 so we omit the details. \( \square \)

**Remark 2.2.** For the case \( n = 1 \) Proposition 2.2 coincides with Proposition 2.2 (a) in [9].

### 3. Refinement of Hardy-type inequalities involving many functions

Our first result reads:

**Theorem 3.1.** Let \( p > 1, \ m > 1, \ 0 < b \leq \infty, \ n \in \mathbb{Z}_+ \) and let the functions \( f_k, \ k \in \{1, 2, ..., n\} \), be nonnegative and locally integrable on \((0, b)\) such that

\[ 0 < \int_{0}^{b} x^{p-m} \left( \sum_{k=1}^{n} f_k(x) \right)^{p} dx < \infty. \]

(i) If \( p \geq 2 \), then

\[ \int_{0}^{b} x^{-m} \left( \prod_{k=1}^{n} \int_{0}^{x} f_k(t) dt \right)^{\frac{p}{n}} dx \]

\[ + \frac{m-1}{p} \int_{0}^{b} \int_{t}^{b} \left| \frac{p}{m-1} \sum_{k=1}^{n} f_k(t) \left( \frac{t}{x} \right)^{1-\frac{m-1}{p}} \right| \]

\[ - \frac{1}{x} \int_{0}^{x} \sum_{k=1}^{n} f_k(t) \frac{dt}{n} \left| x^{p-m-\frac{m-1}{p}} dx \right| \frac{m-1}{p} \frac{dt}{m-1} \]

\[ \leq \left( \frac{p}{nm-n} \right)^{p} \int_{0}^{b} \left[ 1 - \left( \frac{x}{b} \right)^{\frac{m-1}{p}} \right] x^{p-m} \left( \sum_{k=1}^{n} f_k(x) \right)^{p} dx. \]

(ii) If \( 1 < p \leq 2 \), then inequality (3.1) holds in the reversed direction.
**Proof.** This follows directly from the proof of Theorem 3.1 in [9] by using the substitution \( f(x) = \sum_{k=1}^{n} \frac{f_k(x)}{n} \) and the arithmetic-geometric inequality (2.3).

**Remark 3.1.** In the special case \( n = 1 \) Theorem 3.1 coincides with Theorem 3.1 in [9]. For the case \( p = 2 \) and \( n = 1 \) we even have equality in (3.1).

Next, we formulate the dual version of Theorem 3.1.

**Theorem 3.2.** Let \( p > 1, m < 1, 0 \leq b < \infty, n \in \mathbb{Z}_+ \) and let the functions \( f_k, k \in \{1, 2, \ldots, n\}, \) be nonnegative and locally integrable on \((b, \infty)\) such that \( 0 < \int_b^\infty x^{p-m} (\sum_{k=1}^{n} f_k(x))^p \, dx < \infty. \)

(iii) If \( p \geq 2, \) then

\[
\int_b^\infty x^{-m} \left( \prod_{k=1}^{n} \int_x^\infty f_k(t) \, dt \right)^{\frac{p}{n}} \, dx
\]

\[
+ \frac{1-m}{p} \int_b^\infty \int_b^t \frac{p}{1-m} \sum_{k=1}^{n} f_k(t) (\frac{t}{x})^{\frac{1-m}{p}+1} \, dt
\]

\[
- \frac{1}{x} \int_x^\infty \sum_{k=1}^{n} \frac{f_k(t)}{n} \, dt \left[ 1 - \left( \frac{b}{x} \right)^{\frac{1-m}{p}} \right] x^{p-m} \left( \sum_{k=1}^{n} f_k(x) \right)^p \, dx.
\]

(iv) If \( 1 < p \leq 2, \) then inequality (3.2) holds in the reversed direction.

**Proof.** This follows directly from the proof of Theorem 3.2 in [9] by using the substitution \( f(x) = \sum_{k=1}^{n} \frac{f_k(x)}{n} \) and the arithmetic-geometric inequality (2.3).

**Remark 3.2.** If we set \( n = 1, \) then Theorem 3.2 reduces to inequality Theorem 3.2 in [9]. For the case \( p = 2 \) and \( n = 1 \) we even have equality in (3.2).

4. Concluding remarks and examples

Next, we obtain the following generalizations of our results in Theorems 3.1 and 3.2 respectively.
Theorem 4.1. Let \( p > 1, m > 1, 0 < b \leq \infty, \) and \( n \in \mathbb{Z}_+ \). Let \( \{\alpha_k\}_{k=1}^n \) be a positive sequence such that \( \sum_{k=1}^n \alpha_k = 1 \) and \( \{f_k\}_{k=1}^n \) be a sequence of integrable functions on \((0, b)\) such that

\[
0 < \int_0^b x^{p-m} \left( \sum_{k=1}^n f_k(x) \right)^p dx < \infty.
\]

\((v)\) If \( p \geq 2 \), then

\[
\int_0^b x^{(1-m)\alpha_k - 1} \left( \prod_{k=1}^n \int_0^x f_k(t) dt \right)^{p\alpha_k} dx
\]

\[
+ \left( \frac{m-1}{p} \right)^{p+1-n\alpha_k} \int_0^b \int_t^b \frac{1}{m-1} \sum_{k=1}^n f_k(t) \left( \frac{t}{x} \right)^{1-\frac{m-1}{p}} dx
\]

\[
- \frac{1}{x} \int_0^b \sum_{k=1}^n \frac{f_k(t)}{n} \left( \frac{p}{m-1} \right)^{p} dx^{m-1} \cdot x^{m-1} \cdot dx^{m-1} \cdot dt
\]

\[
\leq \alpha_k^p \left( \frac{p}{m-1} \right)^{n\alpha_k} \int_0^b \left[ 1 - \left( \frac{x}{b} \right)^{\frac{m-1}{p}} \right] x^{p-m} \left( \sum_{k=1}^n f_k(x) \right)^p dx.
\]

\((vi)\) If \( 1 < p \leq 2 \), then inequality (4.1) holds in the reversed direction.

**Proof.** Let \( p \geq 2 \). Then, by the more general Arithmetic-Geometric Mean inequality

\[
\prod_{k=1}^n g_k^{\alpha_k}(x) \leq \sum_{k=1}^n \alpha_k g_k(x),
\]

we have that

\[
(4.2) \quad \left( \prod_{k=1}^n F_k^{\alpha_k}(x) \right)^p \leq \left( \sum_{k=1}^n \alpha_k F_k(x) \right)^p = \left( \frac{1}{x} \int_0^x \sum_{k=1}^n \alpha_k f_k(t) dt \right)^p.
\]

The rest of the proof is just formal modification of the proof of Theorem 3.1 by using inequality (4.2) together with the substitution \( f(x) = \sum_{k=1}^n \alpha_k f_k(x) \). So we omit the details. \( \square \)

**Remark 4.1.** If \( \alpha_k = \frac{1}{k}, k \in \{1, 2, ..., n\}, \) and \( m > 1 \), then Theorem 4.1 reduces to Theorem 3.1.
Theorem 4.2. Let \( p > 1, m < 1, 0 \leq b < \infty, \) and \( n \in \mathbb{Z}_+ \). Let \( \{\alpha_k\}_{k=1}^n \) be a positive sequence such that \( \sum_{k=1}^n \alpha_k = 1 \) and \( \{f_k\}_{k=1}^n \) be a sequence of integrable functions on \((b, \infty)\) such that

\[
0 < \int_b^\infty x^{p-m} \left( \sum_{k=1}^n f_k(x) \right)^p \, dx < \infty.
\]

(vii) If \( p \geq 2 \), then

\[
\int_b^\infty x^{(1-m)\alpha_k-1} \left( \prod_{k=1}^n \int_x^\infty f_k(t) \, dt \right)^{p\alpha_k} \, dx
\]

\[
+ \left( \frac{1 - m}{p} \right)^{p+1-np\alpha_k} \int_b^\infty \int_b^t \frac{p}{1-m} \sum_{k=1}^n f_k(t) \left( \frac{t}{x} \right)^{\frac{1-m}{p}+1} \, dt \, dx
\]

\[
- \frac{1}{x} \int_b^\infty \sum_{k=1}^n \frac{f_k(t)}{n} \, dt \left| \int_b^\infty \frac{p}{x^{p-m+p-1}} \, dx \right| \int_b^\infty \left[ 1 - \left( \frac{x}{b} \right)^{\frac{1-m}{p}} \right] x^{p-m} \left( \sum_{k=1}^n f_k(x) \right)^p \, dx.
\]

(viii) If \( 1 < p \leq 2 \), then inequality (4.3) holds in the reversed direction.

Proof. The proof of Theorem 4.2 is similar to the proof of Theorems 4.1 and so we omit the details. \( \square \)

Remark 4.2. If \( \alpha_k = \frac{1}{k}, k \in \{1, 2, ..., n\}, \) and \( m > 1 \), then Theorem 4.2 reduces to Theorem 3.2.

Example 4.1. In Proposition 2.1, by putting \( u(x) \equiv 1 \) we obtain the weight function \( v \) to be equal to

\[
v(x) = \begin{cases} 1 - \frac{x}{b}, & b < \infty, \\ 1, & b = \infty. \end{cases}
\]

Hence for \( b < \infty \), inequality (2.3) takes the form

\[
\int_0^b \varphi \left( \left[ \prod_{k=1}^n F_k(x) \right] \right) \frac{dx}{x}
\]
\[ \int_0^b \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) - \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt \right) \frac{dx}{x^2} dt \]

\[ \leq \int_0^b \left( 1 - \frac{x}{b} \right) \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) \right) \frac{dx}{x} \]

and

\[ \int_0^\infty \varphi \left( \left[ \prod_{k=1}^n F_k(x) \right]^{\frac{1}{n}} \right) \frac{dx}{x} \]

\[ + \int_0^\infty \int_t^\infty \varphi \left( \left[ \frac{1}{n} \sum_{k=1}^n f_k(t) - \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt \right] \right) \frac{dx}{x^2} dt \]

\[ \leq \int_0^\infty \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) \right) \frac{dx}{x} \]

when \( b = \infty \).

**Example 4.2.** Also by putting \( u(x) \equiv 1 \) in Proposition 2.2 yields

\[ v(x) = \begin{cases} 1 - \frac{b}{x}, & b > 0 \\ 1, & b = 0 \end{cases} \]

and so for the case \( b > 0 \), inequality (2.5) becomes

\[ \int_b^\infty \varphi \left( \left[ \prod_{k=1}^n G_k(x) \right]^{\frac{1}{n}} \right) \frac{dx}{x} \]

\[ + \int_b^\infty \int_b^t \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(t) - x \int_x^\infty \frac{1}{n} \sum_{k=1}^n f_k(t) dt \right) \frac{dx}{x^2} dt \]

\[ \leq \int_b^\infty \left( 1 - \frac{b}{x} \right) \varphi \left( \frac{1}{n} \sum_{k=1}^n f_k(x) \right) \frac{dx}{x} \]

while for \( b = 0 \) it reads

\[ \int_0^\infty \varphi \left( \left[ \prod_{k=1}^n G_k(x) \right]^{\frac{1}{n}} \right) \frac{dx}{x} \]
Remark 4.3. For the case $n = 1$ Examples 4.1 and 4.2 coincide with Examples 4.1 and 4.2 in [9].

By using Theorem 3.1 with $m = p$ we obtain:

**Example 4.3.** Let $0 < b \leq \infty$ and let the function $f_k, k \in \{1, 2, ..., n\}$ be locally integrable on $(0, b)$ such that $0 < \int_0^b x^{p-m} \left(\sum_{k=1}^n f_k(x)\right)^p dx < \infty$. If $p \geq 2$, then

$$
\int_0^b \left(\frac{1}{x^n} \prod_{k=1}^n \int_0^x f_k(t) dt\right) \left(\frac{1}{x^{p-1}}\right) dx + \frac{p-1}{p} \int_0^b \int_t^b \frac{1}{x^n} \prod_{k=1}^n f_k(t) \left(\frac{t}{x}\right) dx dt
$$

(4.4)

$$
- \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt \left(\frac{1}{x^{p-1}}\right) dx + \frac{p}{x} \int_0^b x^{p-1} dx + \frac{1}{x} \int_0^b \left(\sum_{k=1}^n f_k(x)\right)^p dx.
$$

Remark 4.4. For $p = 2$ and $b = \infty$ we obtain the following identity for all $f_k \in L_2(0, \infty), k \in \{1, 2, ..., n\}$:

$$
\int_0^{\infty} \left(\frac{1}{x^n} \prod_{k=1}^n \int_0^x f_k(t) dt\right) \left(\frac{1}{x^{p-1}}\right) dx
$$

(4.5)

$$
+ \frac{1}{2} \int_0^{\infty} \int_t^{\infty} \left(2 \sqrt{\frac{1}{x^n} \sum_{k=1}^n f_k(t)} - \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^n f_k(t) dt\right) \frac{dx dt}{\sqrt{x} \sqrt{t}}
$$

$$
= \left(\frac{2}{n}\right)^2 \int_0^{\infty} \left(\sum_{k=1}^n f_k(x)\right)^2 dx.
$$

In particular, if $n = 1$ inequality (4.5) coincides with Remark 4.2 in [9].
**Remark 4.5.** For the case $p = n$ in Theorem 3.1, inequality (3.1) reads:

$$
\int_0^b x^{-m} \left( \prod_{k=1}^{n} \int_0^x f_k(t) \, dt \right) \, dx
$$

$$
+ \frac{m - 1}{n} \int_0^b \int_t^b \left| \frac{1}{m - 1} \sum_{k=1}^{n} f_k(t) \left( \frac{t}{x} \right)^{1 - m - 1} \right| \, dx
$$

$$
- \frac{1}{x} \int_0^x \frac{1}{n} \sum_{k=1}^{n} f_k(t) \, dt \right|^{n} \times x^{-m - \frac{1}{n}} \, dx
$$

$$
\leq \left( \frac{1}{m - 1} \right)^{n} \int_0^b \left[ 1 - \left( \frac{x}{a} \right)^{m - 1} \right] x^{-m} \left( \sum_{k=1}^{n} f_k(x) \right)^{n} \, dx,
$$

$m > 1, n = 2, 3, 4, ...$

**Example 4.4.** For the case $n = 1$ and $b = 0$ in Theorem 3.2 we obtain that

$$
\int_0^{\infty} x^{-m} \left( \int_x^{\infty} f(t) \, dt \right)^p \, dx + \frac{1 - m}{p} \int_0^{\infty} \frac{1}{1 - m} \left( \frac{t}{x} \right)^{1 - m + 1} \, dx
$$

$$
- \frac{1}{x} \int_x^{\infty} f(t) \, dt \right|^p x^{-m + \frac{1}{p} - m} \, dx
$$

$$
\leq \left( \frac{p}{1 - m} \right)^{p} \int_0^{\infty} x^{-m} f^p(x) \, dx.
$$

**Remark 4.6.** For the case $n = p$ in Theorem 3.2 the inequality (3.2) reads:

$$
\int_b^{\infty} x^{-m} \left( \prod_{k=1}^{n} \int_x^{\infty} f_k(t) \, dt \right) \, dx
$$

$$
+ \frac{1}{n} \int_b^{\infty} \int_t^{\infty} \left| \frac{1}{m - 1} \sum_{k=1}^{n} f_k(t) \left( \frac{t}{x} \right)^{1 - m + 1} \right| \, dx
$$

$$
- \frac{1}{x} \int_x^{\infty} \frac{1}{n} \sum_{k=1}^{n} f_k(t) \, dt \right|^{n} \times x^{-m + \frac{1}{n} + \frac{1}{m} - m - 1} \, dx
$$

$$
\leq \left( \frac{1}{1 - m} \right)^{n} \int_b^{\infty} \left[ 1 - \left( \frac{b}{x} \right)^{m - 1} \right] x^{-m} \left( \sum_{k=1}^{n} f_k(x) \right)^{n} \, dx,
$$

$m < 1, n = 2, 3, 4, ...$
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REFERENCES
