CONFORMALLY FLAT SEMI-PSEUDO SYMMETRIC MANIFOLDS

BY

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Abstract. In this paper we study a conformally flat semi pseudo symmetric manifold. Finally a special conformally flat semi pseudo symmetric manifold is studied.

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1. Introduction

In a paper [5], one of the authors, introduced a semi pseudo symmetric manifold. A non-flat Riemannian manifold \((M^n, g)\), \(n > 3\), whose curvature tensor \(R\) satisfies the condition,

\[
\]

where \(A\) is a non-zero 1-form, with associated vector field \(\rho\), that is

\[
g(X, \rho) = A(X)
\]

for every vector field \(X\) and \(\nabla\) denotes the operator of covariant differentiation with respect to the metric \(g\), is called a semi-pseudo symmetric manifold and the 1-form \(A\) is called its associated 1-form. An \(n\)-dimensional semi-pseudo symmetric manifold is denoted by \((SPS)_n\).

In the year 1995, TARAFDAR [6] had studied conformally at pseudo-symmetric manifolds. Motivated by this, the authors in the present paper
dealt with a conformally flat \((SPS)_n, n > 3\). It is shown that, in such a manifold, the vector field \(\rho\) defined by (2) is a torse forming vector field \([4]\).

Considering a conformally flat \((SPS)_n, n > 3\), of constant scalar curvature, it is shown that if in such a manifold, the associated 1-form \(A\) is not closed, then the vector field \(\rho\) is a proper torse forming vector field and if \(A\) is closed, then \(\rho\) is a proper concircular vector field. Further it is shown that if in a conformally flat \((SPS)_n, n > 3\), of constant scalar curvature, the associated 1-form is closed, then the manifold is a subprojective manifold in the sense of Kagan \([1]\).

The notion of a special conformally flat manifold was introduced by \textsc{Chen} and \textsc{Yano} \([2]\). The last section deals with a type of conformally flat \((SPS)_n, n > 3\), in which a certain scalar is constant. It is shown that such an \((SPS)_n\) is a special conformally flat manifold and by theorem \([2]\), it is shown that such an \((SPS)_n, n > 3\), can be isometrically immersed in Euclidean space \(\mathbb{R}^{n+1}\) as a hypersurface.

### 2. Preliminaries

We now consider some formulæ \([3], [5]\) which will be required in the study of conformally flat \((SPS)_n\). Let \(S\) and \(r\) denote respectively the Ricci tensor of type \((0, 2)\) and the scalar curvature and \(L\) denote the symmetric endomorphism of the tangent space at each point of the manifold, corresponding to the Ricci tensor \(S\) i.e.,

\[
S(X, Y) = g(LX, Y)
\]

for any vector fields \(X, Y\) in \(M\). Let \(B\) be the 1-form defined by,

\[
B(X) = A(LX) = g(LX, \rho)
\]

for any vector field \(X\) in \(M\). From (1) we get,

\[
\]

Contracting (5) we have,

\[
dr(X) = 2A(X)r + 3A(LX) = 2A(X)r + 3B(X).
\]

Also,

\[
\]
We shall use these formulae later on.

Let us recall:

**Definition 1.** A vector field $\rho$ in a Riemannian manifold $(M^n, g)$ is said to be torse forming [4] if it satisfies the equation $\nabla_X \rho = a X + \omega(X) \rho$, for every vector field $X$ on $M$, where $a$ is a non-zero scalar and $\omega$ is a 1-form.

**Definition 2.** A vector field $\rho$ is said to be concircular if $\omega$ is a gradient vector field. If $d\omega = 0$, then $\rho$ is said to be a proper concircular vector field [4].

### 3. Conformally flat $(SPS)_n, n > 3$

It is known that in a conformally flat $(SPS)_n, n > 3$, (on writing $Y = Z; Z = Y$ in the equation (5.5) of [5])

$$
0 = (n - 1)A(X)S(Y, Z) - (n - 1)A(Y)S(X, Z) - rA(X)g(Y, Z) + rA(Y)g(X, Z).
$$

Putting $Z = \rho$ in (8) and using (3), (4) and (2), we get,

$$
(n - 1)A(X)B(Y) - (n - 1)A(Y)B(X) = 0,
$$

or,

$$
A(X)B(Y) - A(Y)B(X) = 0, \quad \text{as } n > 3.
$$

Hence,

$$
B(X) = tA(X)
$$

holds for any vector field $X$ in $M$ where $t$ is a scalar. Using (2), (3), (4), (10) we find from (8), $(n-1)A(\rho)S(Y, Z) - (nt-t-r)A(Y)A(Z) - rA(\rho)g(Y, Z) = 0$, i.e.,

$$
S(Y, Z) = \frac{r}{n - 1} g(Y, Z) + \frac{nt - t - r}{(n - 1)A(\rho)} A(Y)A(Z).
$$

From (11) we get,

$$
LY = \frac{r}{n - 1} Y + \frac{nt - t - r}{n - 1} A(Y) \rho.
$$
Again in a conformally flat space,
\[ R(X, Y)Z = \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)LX - g(X, Z)LY] \]
\[ + \frac{r}{(n-1)(n-2)} [g(X, Z)Y - g(Y, Z)X]. \]

Again,
\[ A(R(X, Y)\rho) = g(R(X, Y)\rho, \rho) = 0 \]
and
\[ A(R(X, \rho)Y) = \frac{t}{n-2} A(X)A(Y) - \frac{t}{n-2} g(X, Y)A(\rho). \]

Using (13) we get from (5) on using (3),(4),(10)
\[ (\nabla_X S)(Y, \rho) = 2A(X)tA(Y) + tA(Y)A(X) + A(\rho)S(X, Y) \]
\[ = A(\rho)S(X, Y) + 3tA(X)A(Y). \]

Again, \((\nabla_X B)(Y) = \nabla_X B(Y) - B(\nabla_X Y) = A(Y)\nabla_X t + t \nabla_X a(Y) - t g(\nabla_X Y, \rho), \) by (4) and (5) = \(A'(Y)(X \cdot t) + t\nabla_X g(Y, \rho) - tg(\nabla_X Y, \rho). \) Therefore,
\[ (\nabla_X B)Y = (X \cdot t)A(Y) + tg(Y, \nabla_X \rho). \]

Again, \((\nabla_X S)(Y, \rho) = \nabla_X S(Y, \rho) - S(\nabla_X Y, \rho) - S(Y, \nabla_X \rho) = (\nabla_X B)(Y) - S(Y, \nabla_X \rho), \) by (3), (4). Therefore,
\[ (\nabla_X S)(Y, \rho) = (X \cdot t)A(Y) + tg(Y, \nabla_X \rho) - S(Y, \nabla_X \rho), \] by (16).

Hence from (15) and (17), it follows that,
\[ A(\rho)S(X, Y) + 3tA(X)A(Y) = (X \cdot t)A(Y) + tg(Y, \nabla_X \rho) - S(Y, \nabla_X \rho), \]
that is,
\[ A(\rho)LX + 3tA(X)\rho = (X \cdot t)\rho + t(\nabla_X \rho) - L(\nabla_X \rho). \]

Using (12), we get
\[ \frac{r}{n-1} A(\rho)X + \left(\frac{nt - t - r}{n-1}\right) A(X)\rho \]
\[ = (X \cdot t)\rho + \left(\frac{nt - t - r}{n-1}\right) \nabla_X \rho - \left(\frac{nt - t - r}{n-1}\right) \left(\frac{A(\nabla_X \rho)}{A(\rho)}\right) \rho. \]
We note that, $A(\nabla_X \rho) = g(\nabla_X \rho, \rho)$. As, $(\nabla_X g)(\rho, \rho) = \frac{1}{2} (\nabla_X \rho) \cdot g(\rho, \rho) = \frac{1}{2} (\nabla_X \rho)$. 

Thus the above result reduces to

$$\nabla_X \rho = \left( \frac{r A(\rho)}{nt - t - r} \right) X + \left( A(X) - \frac{(n - 1)(X \cdot t)}{nt - t - r} + \frac{X \cdot A(\rho)}{2A(\rho)} \right) \rho$$

(20)  \hspace{1cm} = \sigma X + \omega(X) \rho,$$

where

(21)  \hspace{1cm} \sigma = \frac{r}{nt - t - r} A(\rho)$$

and

(22)  \hspace{1cm} \omega(X) = A(X) - \frac{n - 1}{nt - t - r} (X \cdot t) + \frac{X \cdot A(\rho)}{2A(\rho)}.$$

If $\sigma = 0$, then from (21) we get $r = 0$. Consequently from (6) we conclude that $B(X) = 0$. Using (10) this yields $tA(X) = 0$. Hence $A(X) = 0$, which is inadmissible by definition.

Hence, $\sigma \neq 0$ and we can state:

**Theorem 3.1.** In a conformally flat $(SPS)_n, n > 3$, the vector field $\rho$ given by (2) is a torse forming vector field.

4. Conformally flat $(SPS)_n, n > 3$, with constant scalar curvature

If $r$ is a non-zero constant, then from (6) we get $2A(X)r + 3B(X) = 0$, that is,

(23)  \hspace{1cm} B(X) = -\frac{2}{3} r A(X).$$

In this case, taking $t = -\frac{2}{3} r$, from (21) and (22) we get $\sigma = -\frac{3}{2n+1} A(\rho)$ and $\omega(X) = A(X) - \frac{X \cdot A(\rho)}{2A(\rho)}$. Thus,

(24)  \hspace{1cm} \nabla_X \rho = \sigma X + \omega(X) \rho,$$

where

(25)  \hspace{1cm} \sigma = -\frac{3}{2n+1} A(\rho).$
and
\begin{equation}
\omega(X) = A(X) + \frac{X \cdot A(\rho)}{A(\rho)}.
\end{equation}
From (25), it follows that \( \sigma \) cannot be zero. Again from (26), it follows that \( \omega \) is closed if \( A \) is closed and \( \omega \) is not closed if \( A \) is not so. Hence we can state:

**Theorem 4.1.** In a conformally flat \((SPS)_n, n > 3\) with constant scalar curvature, the vector field \( \rho \) defined by (2) is a proper torse forming vector field if the associated 1-form is not closed and \( \rho \) is a proper concircular vector field if \( A \) is closed.

It is known [1] that if a conformally flat manifold admits a proper concircular vector field, then, it is a subprojective manifold in the sense of Kagan.

Thus we can state

**Theorem 4.2.** If in a conformally flat \((SPS)_n, n > 3\), with constant scalar curvature, the associated 1-form is closed, the manifold is a subprojective manifold in the sense of Kagan.

5. Conformally flat \((SPS)_n, n > 3\), with constant scalar \( t \)

It is to be noted that if in a conformally flat \((SPS)_n, n > 3\), the scalar curvature \( r \) is constant, then the scalar \( t \) defined by (10) is equal to \( -\frac{2}{3}r \) and is thus a constant. On the other hand, if the scalar \( t \) is a constant, then the scalar curvature \( r \) is not necessarily a constant. In this section, we consider a conformally flat \((SPS)_n, n > 3\), with constant scalar \( t \).

From (11) we have, \( S(Y, Z) = \frac{r}{n-1} \ g(Y, Z) + \frac{nt - t - r}{n-1} \ T(Y)T(Z) \), where
\begin{equation}
T(X) = \frac{A(X)}{\sqrt{A(\rho)}}.
\end{equation}
Let,
\begin{equation}
H(X, Y) = -\frac{1}{n-2} \ S(X, Y) + \frac{r}{2(n-1)(n-2)} \ g(X, Y).
\end{equation}
On using (26), we find from above,
\begin{equation}
H(X, Y) = -\frac{r}{2(n-1)(n-2)} \ g(X, Y) - \frac{nt - t - r}{(n-1)(n-2)} \ T(X)T(Y).
\end{equation}
We now put

\[ \alpha^2 = \frac{r}{(n-1)(n-2)}. \]

Then \[2\alpha(X \cdot \alpha) = \frac{X \cdot r}{(n-1)(n-2)}.\] Again from (6) we have \[X \cdot r = dr(X) = 2A(X)r + 3B(X) = A(X)(2r + 3t).\] Thus from above

\[ \alpha(X \cdot \alpha) = \frac{A(X)(2r + 3t)}{2(n-1)(n-2)}. \]

Hence \[\alpha(X \cdot \alpha)\alpha(Y \cdot \alpha) = \frac{A(X)A(Y)(2r + 3t)^2}{4(n-1)^2(n-2)^2} \quad \text{or,} \quad (X \cdot \alpha)(Y \cdot \alpha) \frac{r}{(n-1)(n-2)} = \frac{A(X)A(Y)}{4(n-1)^2(n-2)^2}(2r + 3t)^2 \quad \text{by (29).} \]

Thus

\[ (X \cdot \alpha)(Y \cdot \alpha) = \frac{A(\rho)(2r + 3t)^2}{4r(n-1)(n-2)} T(X)T(Y). \]

Thus (28) takes the form,

\[ H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(X \cdot \alpha)(Y \cdot \alpha) \]

where

\[ \beta = \frac{4r(r + t - nt)}{A(\rho)(2r + 3t)^2}. \]

If \(\alpha=0\), then \(r=0\) and consequently from (6), \(A(X) = 0\), which is inadmissible. Thus \(\alpha\) cannot be zero and hence we may take \(\alpha\) as positive in (32).

According to Chen and Yano [2], if in a conformally flat manifold, a \((0,2)\) tensor \(H\) defined by

\[ H(X, Y) = \frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} g(X, Y) \]

is expressible in the form

\[ H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(X \cdot \alpha)(Y \cdot \alpha) \]

where \(\alpha, \beta\) are two scalars such that \(\alpha\) is positive, then the manifold is said to be a special conformally flat manifold. Hence from (33), it follows that
(SPS)_n under consideration is a special conformally flat manifold. It is known from a theorem [2] that every simply connected special conformally flat manifold can be isometrically immersed in a Euclidean space R^{n+1} as a hypersurface. Thus we state,

**Theorem 5.1.** Every simply connected conformally flat (SPS)_n, (n > 3) with constant scalar t can be isometrically immersed in a Euclidean space R^{n+1} as a hypersurface.

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**REFERENCES**


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