ON SEMI-P-REDUCIBLE FINSLER METRICS

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Abstract. The class of semi-P-reducible manifolds contains the class of Randers manifolds and Landsberg manifolds as special cases. In this paper, we prove that every semi-P-reducible manifold with P-reducible metric reduces to a Landsberg manifold. Then we show that there is not exists P2-like Randers metric.

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1. Introduction

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians (see [4], [5], [12], [13]). The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg and others (see [10], [11]). In [3], MATSUMOTO introduces the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, MATSUMOTO-HOJO [7] proves that the converse is also true. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric $\alpha$ perturbed by a one form $\beta$. Randers metrics have important applications both in mathematics and physics ([14]). Then as a generalization of C-reducible metrics, Matsumoto-Shimada introduce the notion of P-reducible metrics ([8], [9]).

Let us remark some important curvatures in Finsler geometry. For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric $F$ is called a Berwald metric if $G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y \in T_xM$ for any $x \in M$. 
The second derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ is an inner product $g_y$ on $T_xM$. The third order derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ is a symmetric trilinear forms $C_y$ on $T_xM$. We call $g_y$ and $C_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of $C_y$ along geodesics is the Landsberg curvature $L_y$ on $T_xM$ for any $y \in T_xM_0$. $F$ is said to be Landsbergian if $L = 0$.

There is a weaker notion of metrics- weakly Landsberg metrics. Set $I_y := \sum_{i=1}^n C_y(e_i, e_i, \cdot)$ and $J_y := \sum_{i=1}^n L_y(e_i, e_i, \cdot)$, where $\{e_i\}$ is an orthonormal basis for $(T_xM, g_y)$. $I_y$ and $J_y$ is called the mean Cartan and mean Landsberg curvature, respectively. A Finsler metric $F$ is said to be weakly Landsbergian if $J = 0$.

In [5], Matsumoto-Shibata introduce the notion of semi-C-reducibility by considering the form of Cartan torsion of a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ with dimension $n \geq 3$. A Finsler metric is called semi-C-reducible if its Cartan tensor is given by $C_{ijk} = \frac{p}{1+n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{1+n} I_i I_j I_k$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on $TM$, $h_{ij}$ is the angular metric and $C^2 = I^2 I_k$. The function $p$ is called characteristic scalar of $F$. If $q = 0$, then $F$ is called C-reducible metric. It is remarkable that, an $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form on $M$.

As a generalization of C-reducible metrics, Matsumoto-Shimada introduce the notion of P-reducible metrics (see [6], [16]). A Finsler metric is called P-reducible if its Landsberg tensor is given by following $L_{ijk} = \frac{1}{1+n} \{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}$. In [15], Rastogi introduces a new class of Finsler spaces named by semi-P-reducible spaces, which contains the notion of P-reducible metrics, as a special case. A Finsler metric $F$ is called semi-P-reducible if its Landsberg tensor is given by

$$L_{ijk} = \lambda J_i h_{jk} + J_j h_{ki} + J_k h_{ij} + 3\mu J_i J_j J_k,$$

where $\lambda = \lambda(x, y)$ and $\mu = \mu(x, y)$ are scalar functions on $TM$. We have some special cases as follows: If $\mu = 0$, then $F$ is a P-reducible metric; if $\lambda = 0$, then $F$ is a $P^2$-like metric [15] and if $\mu = \lambda = 0$, then $F$ is a Landsberg metric. The geometric meaning of P-reducible Finsler metrics is studied in [16]. Since the class of semi-P-reducible metrics contains the class of C-reducible metrics as a special case, therefore the study of this class of Finsler spaces will enhance our understanding of the geometric meaning of Randers metrics.
In this paper, we prove that every semi-P-reducible manifold with P-reducible metric reduces to a Landsberg manifold. Then we show that there is not exists any P2-like Randers metric.

In this paper, we use the Berwald connection on Finsler manifolds. The $h$- and the $v$-covariant derivatives of a Finsler tensor field are denoted by “$|$” and “,” respectively.

2. Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_xM$ the tangent bundle of $M$. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;

(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$;

(iii) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_xM$ is positive definite, $g_y(u, v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]_{s, t = 0}$, $u, v \in T_xM$.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y+tu}(u, v) \right]_{t=0}$, $u, v, w \in T_xM$. The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if $F$ is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $I_y$ by $I_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}$. By Diecke Theorem, $F$ is Riemannian if and only if $I_y = 0$ ([17]). For $y \in T_xM_0$, define the Matsumoto torsion $M_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $M_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where $M_{ijk} := C_{ijk} - \frac{1}{n+1} \left( I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \right)$, and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2} g_{ip}g_{jp}g_{kq}y^k y^q$ is the angular metric. A Finsler metric $F$ is said to be $C$-reducible if $M_y = 0$. This quantity is introduced by MATSUMOTO [3]. Matsumoto proves that every Randers metric satisfies $M_y = 0$. Later on, MATSUMOTO-HÔJÔ [7] proves that the converse is true too.

The horizontal covariant derivatives of $C$ along geodesics give rise to the Landsberg curvature $L_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $L_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|y^s}$, $u = u^i \frac{\partial}{\partial x^i}|x$, $v = v^i \frac{\partial}{\partial x^i}|x$ and $w = w^i \frac{\partial}{\partial x^i}|x$. The family $L := \{L_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $L = 0$ ([2]).
Define $\vec{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\vec{M}_y(u, v, w) := \vec{M}_{ijk}(y)u^iv^jw^k$ where $\vec{M}_{ijk} := L_{ijk} - \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\}$. A Finsler metric $F$ is said to be $P$-reducible if $\vec{M}_y = 0$. The notion of $P$-reducibility was given by Matsumoto-Shimada [8].

3. Main results

In this section, we are going to consider semi-$P$-reducible Finsler manifold with $P$-reducible metric. Then we prove the following.

**Theorem 3.1.** Let $(M, F)$ be a semi-$P$-reducible Finsler manifold and $\mu \neq 0$. Suppose that $F$ is a $P$-reducible metric. Then $F$ reduces to a Landsberg metric.

**Proof.** Let $F$ be a $P$-reducible metric

$$L_{ijk} = \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\}. \tag{2}$$

On the other hand, $F$ is a semi-$P$-reducible metric

$$L_{ijk} = \lambda\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} + 3\mu J_iJ_jJ_k. \tag{3}$$

By (2) and (3), we get

$$\left(\frac{1}{n+1} - \lambda\right)\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} = 3\mu J_iJ_jJ_k. \tag{4}$$

Multiplying (4) with $g^{ij}$ implies that

$$\{(n+1)\lambda + 3\mu J^2 - 1\}J_k = 0. \tag{5}$$

Suppose that $J_k \neq 0$. Then we have

$$\lambda = \frac{1 - 3\mu J^2}{n+1}. \tag{6}$$

Plugging (6) into (3) implies that

$$L_{ijk} = \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} - 3\mu J^2 \left\{\frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} - \frac{1}{J^2}J_iJ_jJ_k\right\}. \tag{7}$$
Since $F$ is a P-reducible metric, thus (7) reduces to the following

\[ 3\mu J^2 \left\{ \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \} - \frac{1}{J^2} J_i J_j J_k \right\} = 0. \tag{8} \]

By (8) and our assumptions, we deduce that the following holds

\[ \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \} = \frac{1}{J^2} J_i J_j J_k. \tag{9} \]

This is impossible, since $\text{Rank}(h_{jk} J^2) = n - 1$ and $\text{Rank}(J_i J_j J_k) = 1$. Thus

\[ J^2 = J_i J_i = 0. \tag{10} \]

Since $F$ is a positive-definite metric, then $J_i = 0$. Thus by (2), we conclude that $F$ is a Landsberg metric.

**Example 3.1.** Let $(M, F)$ be a 2-dimensional Finsler manifold. We refer to the Berwald’s frame $(\ell^i, m^i)$, where $\ell^i = y^i / F(y)$, $m^i$ is the unit vector with $\ell_i m^i = 0$ and $\ell_i = g_{ij} \ell^j$. Then the Cartan tensor is given by following $C_{ijk} = C m_i m_j m_k$, where $C := m_p m^p$ (for more details see [1]). By taking a horizontal derivation of above equation, we get $L_{ijk} = F C_{0m_i m_j m_k}$, where $C_0 := C_{1y} y^s$. By multiplying of above equation with $g^{ij}$, we can deduce that every Finsler surface is P2-like.

**Theorem 3.2.** Let $(M, F)$ be a Finsler manifold of dimension $n \geq 3$. Then there is not exists any P2-like Randers metric.

**Proof.** Let $F$ be a P2-like Randers metric on a manifold $M$ of dimension $n \geq 3$. It is easy to see that $F$ is P-reducible. We have

\[ \frac{1}{n+1} \{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \} = \frac{1}{J^2} J_i J_j J_k. \tag{11} \]

We have

\[ h_{ij} J^i = (g_{ij} - \ell_i \ell_j) J^i = J_j. \tag{12} \]

Contracting (11) with $J^i$ and using (12) yields

\[ \frac{1}{n+1} \{ h_{jk} J^2 + 2J_j J_k \} = J_j J_k, \tag{13} \]

or equivalently

\[ h_{jk} J^2 = (n - 1) J_j J_k. \tag{14} \]

By the same argument used in Theorem 3.1, we conclude that there is not exists any P2-like Randers metric on a manifold $M$ of dimension $n \geq 3$. \hfill \qed
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