DIVISIBLE AND SEMI-DIVISIBLE RESIDUATED LATTICES

BY

D. BUŞNEAG, D. PICIU and J. PARALESCU

Abstract. The main purpose of this paper is to present new results in divisible and semi-divisible residuated lattices.

It also presents a new characterization for boolean elements of a residuated lattice (Theorem 9).

Mathematics Subject Classification 2010: 03G10, 06D35, 06D72, 03G05, 06B20.

Key words: divisible and semi-divisible residuated lattices, Boolean algebra, semi-G-algebra, MTL-algebra, BL-algebra, MV-algebra, regular element, dense element, reflective subcategory, deductive system, maximal deductive system, radical.

1. Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull [18], Dilworth [9], Ward and Dilworth [26], Ward [25], Balbes and Dwinger [1] and Pavelka [20].

In [16], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: BCK- latices in [15], full BCK- algebras in [18], FLw- algebras in [19], and integral, residuated, commutative l-monoids in [3].

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [2], [5], [9], [12], [25], [26]).

MV-algebras are known to be special residuated lattices with a proper additive operation ⊕. MV-algebras fulfill a double negation law \( x^{**} = x \), too. For a residuated lattice \( L \) we denote by \( MV(L) \) the set of all elements
$x^* = x \to 0$ with $x \in L$. $(MV(L), \oplus^*, 0)$ is an MV-algebra iff for all $x, y \in L, (x^* \to y^*) \to y^* = (y^* \to x^*) \to x^*$ (see [22]), where for $x, y \in MV(L), x \oplus y = x^* \to y$. In particular, for any semi-divisible residuated lattice $L$, the subset $(MV(L), \oplus^*, 0)$ is an MV-algebra (see [22]).

The paper is organized as follows.

In Section 2 we recall the basic definitions of residuated lattices. Also we present $MV$-center of a semi-divisible residuated lattice, defined by Turunen and Mertanen in [22]. This is a very important construction, which associates an $MV$-algebra to every semi-divisible residuated lattice. In this way, many properties can be transferred from $MV$-algebras to residuated lattices and backwards.

In Section 3 we prove that the category $MV$ of $MV$-algebras is a reflective subcategory of the category $RL_d$ of divisible residuated lattices and as consequence we obtain some informations relative to injective divisible residuated lattices (Corollary 3).

In Section 4, for a residuated lattice $L$ we denote by $R(L)$ the set of regular elements of $L$, by $D(L)$ the set of dense elements of $L$ and by $B(L)$ the set of boolean elements of $L$. We present new characterizations for these elements. We prove that in general, $B(L) \subseteq R(L) \subseteq MV(L)$. If $L$ is a semi-divisible $MTL$-algebra, then $B(L) = B(MV(L))$.

Theorem 10 characterize the residuated lattices which are Boolean algebras.

In Section 5, we present new results relative to lattice of maximal deductive systems of a residuated lattice. We introduce the notion of semi-$G$-algebra and we obtain that if $L$ is a semi-$G$-algebra, then there is a bijection between $Max(L)$ and $RL(L, \{0, 1\}) = \{ f : L \to \{0, 1\} \mid f$ is a morphism of residuated lattices$\}$ (Theorem 14). We prove that if $L$ is semi-divisible, then there is a bijective correspondence between maximal deductive systems of $L$ and maximal deductive systems of $MV(L)$.

In Section 6, we characterize $Rad(MV(L))$, for a semi-divisible residuated lattice $L$.

2. Definitions and preliminaries

We review the basic definitions of residuated lattices. Also we present $MV$-center of a semi-divisible residuated lattice, defined by Turunen and Mertanen in [22].
Definition 1. A residuated lattice ([2], [23]) is an algebra \((L; \lor, \land, \circ, \to, 0, 1)\) of type \((2; 2, 2; 2; 0; 0)\) equipped with an order \(\leq\) satisfying the following:

\((LR_1)\) \((L; \lor, \land, 0, 1)\) is a bounded lattice;
\((LR_2)\) \((L; \circ, 1)\) is a commutative ordered monoid;
\((LR_3)\) \(\circ\) and \(\to\) form an adjoint pair, i.e. \(a \leq x \to y\) iff \(a \circ x \leq y\) for all \(a, x, y \in L\).

The relations between the pair of operations \(\circ\) and \(\to\) expressed by \((LR_3)\), is a particular case of the law of residuation ([2]). The class \(\mathcal{RL}\) of residuated lattices is equational (see [16]).

Example 1 ([23]). Let \(p\) be a fixed natural number and \(I = [0, 1]\) the real unit interval. We define for \(x, y \in I\), \(x \circ y = (\max\{0, x^p + y^p - 1\})^{1/p}\) and \(x \to y = \min\{1, (1 - x^p + y^p)^{1/p}\}\), then \((I; \max, \min, \circ, \to, 0, 1)\) become a residuated lattice called *generalized Lukasiewicz structure*. For \(p = 1\) we obtain the notion of *Lukasiewicz structure* \((x \circ y = \max\{0, x + y - 1\}, x \to y = \min\{1, 1 - x + y\})\).

Example 2 ([23]). If consider on \(I = [0, 1]\), \(\circ\) to be the usual multiplication of real numbers and for \(x, y \in I\), \(x \to y = 1\) if \(x \leq y\) and \(y/x\) otherwise, then \((I; \max, \min, \circ, \to, 0, 1)\) is a residuated lattice (called *Products structure* or *Gaines structure*).

Example 3 ([23]). If \((B; \lor, \land, \lor, 0, 1)\) is a Boolean algebra, then if we define for every \(x, y \in B\), \(x \circ y = x \land y\) and \(x \to y = x' \lor y\), then \((B; \lor, \land, \circ, \to, 0, 1)\) becomes a residuated lattice.

In \(L\) we consider the following identities:

\((BL_1)\) \(x \circ (x \to y) = x \land y\) (divisibility);
\((BL_2)\) \((x \to y) \lor (y \to x) = 1\) (preliminarity);
\((BL_3)\) \([x^* \circ (x^* \to y^*)]^* = (x^* \land y^*)^*\) (semi-divisibility).

Definition 2. The residuated lattice \(L\) is called:

(i) divisible if \(L\) verify \((BL_1)\);
(ii) \textit{MTL-algebra} if \( L \) verify \((\text{BL}_2)\);

(iii) \textit{BL-algebra} if \( L \) verify \((\text{BL}_1)\) and \((\text{BL}_2)\) (that is, \( L \) is a divisible \textit{MTL-algebra});

(iv) \textit{semi-divisible} if \( L \) verify \((\text{BL}_1^s)\).

We denote by \( \mathcal{RL} \) (\( \mathcal{RL}_d, \mathcal{MTL}, \mathcal{BL}, \mathcal{RL}_{sd} \)) the class of residuated lattices (divisible residuated lattices, \textit{MTL}-algebras, \textit{BL}-algebras, semi-divisible residuated lattices).

\textbf{Proposition 1} \([22]\). For a residuated lattice \( L \), the following conditions are equivalent:

(i) \( L \in \mathcal{RL}_d \);

(ii) For every \( x, y \in L \) with \( x \leq y \) there exists \( z \in L \) such that \( x = y \circ z \);

(iii) For every \( x, y, z \in L \), \( x \to (y \land z) = (x \to y) \circ [(x \land y) \to z] \).

We recall \([7], [23]\) that an \textit{MV-algebra} is an algebra \((M; \oplus^*, 0)\) of type \((2, 1, 0)\) such that:

\begin{itemize}
  \item \((\text{MV}_1)\) \((M; \oplus, 0)\) is a commutative monoid;
  \item \((\text{MV}_2)\) \(x^{**} = x\), for every \( x \in M \);
  \item \((\text{MV}_3)\) \((x \to y) \to y = (y \to x) \to x\), for every \( x, y \in M \) (where \( x \to y = x^* \oplus y \)).
\end{itemize}

We denote by \( \mathcal{MV} \) the class of \textit{MV-algebras}.

\textbf{Remark 1} \([14], [23]\). 1. It is not hard to see that an equivalent presentation of \textit{MV}-algebras can be given as \textit{BL}-algebras plus condition \((\text{MV}_2)\).

2. Let \((L; \lor, \land, \circ, \to, 0, 1)\) be a residuated lattice. If for \( x, y \in L \) we denote \( x \oplus y = x^* \to y \), then \((L; \oplus^*, 0)\) is an \textit{MV-algebra} iff \((x \to y) \to y = (y \to x) \to x\), for every \( x, y \in L \).

\textbf{Remark 2}. Lukasiewicz structures and Boolean algebras are \textit{BL}-algebras; not every residuated lattice is a \textit{BL}-algebra (see [23], p.16).
Example 4 ([17]). We give another example of a finite residuated lattice, which is not a BL-algebra. Let \( L = \{0, a, b, c, 1\} \) with \( 0 < a, b < c < 1 \), but \( a, b \) are incomparable. \( L \) become a residuated lattice relative to the following operations:

\[
\begin{array}{cccccc}
\rightarrow & 0 & a & b & c & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
a & b & 1 & b & 1 & 1 \\
b & a & 1 & 1 & 1 & 1 \\
c & 0 & a & b & 1 & 1 \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\odot & 0 & a & b & c & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & a & a \\
b & 0 & 0 & b & b & b \\
c & 0 & a & b & c & c \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

In what follows by \( L \) we denote the universe of a residuated lattice; for \( x \in L \) and a natural number \( n \), we define \( x^* = x \rightarrow 0 \), \( x^{**} = (x^*)^* \), \( x^0 = 1 \) and \( x^n = x^{n-1} \odot x \) for \( n \geq 1 \).

In residuated lattices we have the following rules of calculus:

**Theorem 1** ([5], [6], [8], [12], [23]). Let \( x, x_1, x_2, y, y_1, y_2, z \in L \). Then we have:

\((c_1)\) \( 1 \rightarrow x = x, x \rightarrow 1 = 1, 0 \rightarrow x = 1, x \odot 0 = 0; \)

\((c_2)\) \( x \leq y \) iff \( x \rightarrow y = 1; \)

\((c_3)\) \( x \leq y \rightarrow x, x \leq (x \rightarrow y) \rightarrow y, ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y; \)

\((c_4)\) \( x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y); \)

\((c_5)\) \( x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z); \)

\((c_6)\) \( (x \rightarrow y) \rightarrow (x \rightarrow z) \leq x \rightarrow (y \rightarrow z); \)

\((c_7)\) \( x \rightarrow (y \land z) = (x \rightarrow y) \land (x \rightarrow z), (y \land z) \rightarrow x \geq (y \rightarrow x) \land (z \rightarrow x); \)

\((c_8)\) \( (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z); \)

\((c_9)\) \( x \leq y \) implies \( z \rightarrow x \leq z \rightarrow y \) and \( y \rightarrow z \leq x \rightarrow z; \)

\((c_{10})\) \( x_1 \rightarrow y_1 \leq (y_2 \rightarrow x_2) \rightarrow [(y_1 \rightarrow y_2) \rightarrow (x_1 \rightarrow x_2)]; \)

\((c_{11})\) \( x \leq y \) implies \( z \odot x \leq z \odot y; \)

\((c_{12})\) \( x \odot (x \rightarrow y) \leq x \land y, x \leq y \rightarrow (x \odot y); \)
\((c_{13}) \ x \odot (x \to (x \odot y)) = x \odot y;\)

\((c_{14}) \ x \to (y \to z) = (x \odot y) \to z \equiv y \to (x \to z);\)

\((c_{15}) \ x \to y \leq (x \odot z) \to (y \odot z) \text{ and } (x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2) \to (y_1 \odot y_2);\)

\((c_{16}) \ (x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \lor x_2) \to (y_1 \lor y_2);\)

\((c_{17}) \ (x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \land x_2) \to (y_1 \land y_2);\)

\((c_{18}) \ x \lor y = 1 \Rightarrow x \odot y = x \land y;\)

\((c_{19}) \ x \lor (y \odot z) \geq (x \lor y) \odot (x \lor z), \text{ so, } x^m \lor y^n \geq (x \lor y)^{mn} \text{ for any natural numbers } m, n;\)

\((c_{20}) \ x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (x \odot z);\)

\((c_{21}) \ x \odot (y \lor z) = (x \odot y) \lor (x \odot z);\)

\((c_{22}) \ (x \to y)^n \leq x^n \to y^n \text{ for every } n \geq 1;\)

\((c_{23}) \ (x \land y)^n \leq x^n \land y^n, \text{ for every } n \geq 1;\)

\((c_{24}) \ (x \lor y)^n \geq x^n \lor y^n, \text{ for every } n \geq 1.\)

**Theorem 2** ([5], [6], [8], [12], [23]). If \(x, y \in L\), then:

\((c_{25}) \ 1^* = 0, 0^* = 1;\)

\((c_{26}) \ x \to y \leq y^* \to x^*;\)

\((c_{27}) \ x \leq y \Rightarrow y^* \leq x^*;\)

\((c_{28}) \ x \odot x^* = 0 \text{ and } x \odot y = 0 \text{ iff } x \leq y^*;\)

\((c_{29}) \ x \leq x^{**}, x^{**} \leq x^* \to x, x^{***} = x^*;\)

\((c_{30}) \ (x \odot y)^* = x \to y^* = y \to x^*;\)

\((c_{31}) \ (x \lor y)^* = x^* \land y^*;\)

\((c_{32}) \ x^{**} \to y^{**} = y^* \to x^* = x \to y^{**};\)

\((c_{33}) \ (x \to y^{**})^{**} = x \to y^{**};\)
(c34) \( (x \rightarrow y)^* \leq x^* \rightarrow y^* \);

(c35) \( x^* \circ y^* \leq (x \circ y)^* \), so \( (x^*)^n \leq (x^n)^* \) for every natural number \( n \);

(c36) \( x^* \circ y^* \leq (x \circ y)^* \).

**Corollary 1.** If \( L \) is a divisible residuated lattice, then for every \( x, y \in L \) we have:

(c37) \( (x^* \rightarrow x)^* = 0 \);

(c38) \( (x \rightarrow y)^* = x^* \rightarrow y^* \);

(c39) \( (x \circ y)^* = x^* \circ (x^* \land y^*)^* \), \( (x \land y)^* = x^* \land y^* \);

(c40) \( y^* \leq x \Rightarrow x \rightarrow (x \circ y)^* = y^* \);

(c41) \( x \circ (y \land z) = (x \circ y) \land (x \circ z) \), \( x \land (y \lor z) = (x \land y) \lor (x \land z) \).

**Proof.** (c37). From \( x^* \leq x^* \rightarrow x \Rightarrow (x^* \rightarrow x)^* \leq x^* \Rightarrow (x^* \rightarrow x)^* = x^* \land (x^* \rightarrow x)^* \) \((BL1)\) \( = x^* \circ (x^* \rightarrow x)^* \) \((c30)\) \( = x^* \circ [(x^* \rightarrow x) \circ x^*] \) \((BL1)\) \( = x^* \circ (x^* \land x)^* = x^* \circ x^* = 0 \).

(c38). From (c34) we have \( (x \rightarrow y)^* \leq x^* \rightarrow y^* = x \rightarrow y^* \). On the other hand we have \( (x \rightarrow y^*) \rightarrow (x \rightarrow y)^* = (x \rightarrow y^*)^* \rightarrow (x \rightarrow y)^* \) \((c34)\) \( = [(x \rightarrow y^*) \rightarrow (x \rightarrow y)^*] = [(x \circ (x \rightarrow y^*)) \rightarrow y^*] \) \((BL1)\) \( = [(x \land y^*) \rightarrow y]^* \geq [(x \rightarrow y) \lor (y^* \rightarrow y)]^* \geq (y^* \rightarrow y)^* \) \((c37)\) \( 0^* = 1 \Rightarrow (x \rightarrow y^*) \rightarrow (x \rightarrow y)^* = 1 \Rightarrow x \rightarrow y^* \leq (x \rightarrow y)^* \Rightarrow (x \rightarrow y)^* = x \rightarrow y^* = x^* \rightarrow y^* \).

(c39). From \( x \circ y \leq x \Rightarrow (x \circ y)^* \leq x^* \Rightarrow (x \circ y)^* = (x \circ y)^* \land x^* = [(x \circ y)^*] \land x^* = (y \rightarrow x)^* \land x^* = x^* \circ [(x^* \rightarrow (y \rightarrow x)^*)] = x^* \circ [(x \rightarrow y^*) \rightarrow x^*] = x^* \circ [(x \rightarrow y^*) \rightarrow x^*] = x^* \circ [(x \rightarrow y^*) \rightarrow x^*] = x^* \circ [(x \rightarrow y^*) \rightarrow x^*] = x^* \circ (x \rightarrow y)^* = x^* \circ [x \rightarrow y)^* = x^* \circ [x \rightarrow y)^* = x^* \circ (x \rightarrow y)^* = x^* \circ (x \rightarrow y)^* = x^* \circ (x \rightarrow y)^*.

Clearly, \( (x \land y)^* \leq x^*, y^* \). Consider \( t \in L \) such that \( t \leq x^*, y^* \). Then \( x^*, y^* \leq t^* \Rightarrow x^* \lor y^* \leq t^* \Rightarrow t^* \leq (x^* \lor y^*)^* \). But \( t \leq t^* \) and \((x^* \lor y)^* = x^* \land y^* \Rightarrow t \leq x^* \land y^* \Rightarrow (x \land y)^* = x^* \land y^* \).

(c40). From \( y^* \leq x \Rightarrow x \rightarrow (x \circ y)^* = (x \circ y)^* \rightarrow x^* = (x \rightarrow y)^* \rightarrow x^* = [x \circ (x \rightarrow y^*)]^* = (x \land y^*)^* = y^* \).

(c41). Clearly \( x \circ (y \land z) \leq (x \circ y) \land (x \circ z) \).
On the other hand, \((x \odot y) \land (x \odot z) = (x \odot y) \odot [(x \odot y) \rightarrow (x \odot z)] = (x \odot y) \odot [x \rightarrow (y \rightarrow (x \odot z))] = (x \odot y) \odot [y \rightarrow (x \rightarrow (x \odot z))] = x \odot [y \odot (y \rightarrow (x \rightarrow (x \odot z))) = x \odot [y \odot (y \rightarrow (x \rightarrow (x \odot z))) = x \odot [x \rightarrow (x \rightarrow (x \odot z))] \odot ((x \rightarrow (x \odot z))) \rightarrow (y))]].\]

But \(z \leq x \rightarrow (x \odot z) \Rightarrow (x \rightarrow (x \odot z)) \rightarrow y \leq z \rightarrow y\), so \((x \odot y) \land (x \odot z) \leq x \odot [x \rightarrow (x \odot z)] \odot (z \rightarrow y) = x \odot z \odot (z \rightarrow y) = x \odot (y \land z) \Rightarrow (x \odot y) \land (x \odot z) = x \odot (y \land z).

Clearly, \(x \land (y \lor z) \geq (x \land y) \lor (x \land z)\). On the other hand we have \(x \land (y \lor z) = (y \lor z) \odot [(y \lor z) \rightarrow x]\) \(= (y \lor z) \odot ((y \lor z) \rightarrow x)\) \(= (y \lor z) \land (y \lor z) \rightarrow x\) \(\leq [y \odot (y \rightarrow x)] \lor [z \odot ((y \lor z) \rightarrow x)] = (y \land x) \lor (z \land x) = (x \land y) \lor (x \land z)\). □

3. MV-center of a divisible residuated lattice

For a residuated lattice \(L\) we consider the subset \(MV(L) = \{x^* : x \in L\} = \{x \in L : x^{**} = x\}\) of \(L\). \(MV(L)\) is non-void, since \(1 = 0^* \in MV(L)\). Moreover, \(MV(L)\) is closed with respect to the operations \(\rightarrow\) (because by \((c_{30})\), \(x^* \rightarrow y^* = (x^* \odot y^*) \in MV(L)\)), \(\land\) (by \((c_{31})\)) and in particular, with respect to the operation \(*\). For \(x, y \in L\) we define \(x \oplus y = x^* \rightarrow y\).

**Theorem 3** ([22]). \((MV(L), \oplus, ^*)\) is an MV-algebra iff for all \(x, y \in L\), \((x^* \rightarrow y^*) \rightarrow y^* = (y^* \rightarrow x^*) \rightarrow x^*\).

**Theorem 4** ([22]). If \((x^* \rightarrow y^*) \rightarrow y^* = (y^* \rightarrow x^*) \rightarrow x^*\) holds in a residuated lattice \(L\), then \(L\) is semi-divisible.

**Corollary 2** ([22]). In any semi-divisible residuated lattice \(L\), \((MV(L), \oplus, ^*, 0)\) is an MV- algebra (called MV- center of \(L\)).

Recall that an algebra \((W, \rightarrow, *, 1)\) of type \((2, 1, 0)\) is a Wajsberg algebra if it satisfies the following conditions:

\((W_1)\) \(1 \rightarrow x = x\);

\((W_2)\) \((x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1\);

\((W_3)\) \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\);

\((W_4)\) \((x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1\).

Following [23], if \((W, \rightarrow, *, 1)\) is a Wajsberg algebra, then \((W, \oplus, ^*, 0)\) is an MV-algebra, where for \(x, y \in W, x \oplus y = x^* \rightarrow y\). Conversely, if
(W, \oplus, *, 0) is an MV-algebra, then (W, \to, *, 1) is a Wajsberg algebra, where for \( x, y \in W \), \( x \to y = x^* \oplus y \).

The order relation on a Wajsberg algebra W is given by: \( x \leq y \) iff \( x \to y = 1 \). Following Theorems 3, 4 and Corollary 2, if L is a semi-divisible residuated lattice and for \( x, y \in L \) we define \( x^* \ominus y^* = (x^*)^* \to y^* = x^{**} \to y^* = y \to x^{***} = y \to x^* = x \to y^* \), then MV(L) has a structure of MV-algebra (hence, Wajsberg algebra) and the order relation \( \leq_W \) on the Wajsberg algebra MV(L) is defined by \( x^* \leq_W y^* \) iff \( x^* \to y^* = 1 \iff x^* \leq y^* \) (in L). Hence the order relation on MV(L) coincides with the order relation on L.

According to [22], the lattice operations least upper bound \( \{x^*, y^*\} \), denoted by \( x^* \vee_W y^* \), and greatest lower bound \( \{x^*, y^*\} \), denoted by \( x^* \wedge_W y^* \), on the Wajsberg algebra MV(L) are defined via \( x^* \vee_W y^* = (x^* \to y^*) \to y^* \) and \( x^* \wedge_W y^* = (x^{**} \vee_W y^{**})^* \). In general, the meet operation \( \wedge_W \) on MV(L) may not be the restriction of the meet operation \( \wedge \) on L and, similarly, for join operation.

**Proposition 2** ([22]). If L is a semi-divisible residuated lattice, then \( \wedge_W \) and \( \wedge \) coincide on MV(L).

**Lemma 1.** If \( x, y \in L \), then:

\[(c_{42}) \quad [x^* \odot (x^* \to y^*)]^* = (y^{**} \to x^{**}) \to x^{**};\]

\[(c_{43}) \quad x^{**} \vee_W y^{**} = (x^* \wedge y^*)^* \quad (c_{31}) \quad (x \vee y)^{**}.\]

**Proof.** \((c_{42})\). We have \([x^* \odot (x^* \to y^*)]^* = [x^* \odot (y^{**} \to x^{**})]^* = [(y^{**} \to x^{**}) \odot x^*]^* = (y^{**} \to x^{**}) \odot (x^* \to 0) = (y^{**} \to x^{**}) \to x^{**}.\]

\((c_{43})\). From \( x^* \wedge y^* \leq x^* \to x^{**} \leq (x^* \wedge y^*)^* \) and analogously \( y^* \leq (x^* \wedge y^*)^* \). To prove \( x^{**} \vee_W y^{**} = (x^* \wedge y^*)^* \) consider \( t \in MV(L) \) such that \( x^{**}, y^{**} \leq t \). Then \( t^* \leq x^*, y^* \to t^* \leq x^{**} \wedge y^{**} \to (x^* \wedge y^*)^* \leq t^{**} = t \).

We present a new characterization for semi-divisible residuated lattices:

**Proposition 3.** For a residuated lattice L, the following conditions are equivalent:

(i) \( L \in RL_{sd}; \)

(ii) \( (x^{**} \to y^{**}) \to y^{**} = (y^{**} \to x^{**}) \to x^{**} \) for every \( x, y \in L \).
Proof. $(i) \Rightarrow (ii)$. If $L \in \mathcal{RL}_{sd}$, then for $x, y \in L$, $[x^* \circ (x^* \to y^*)]^* = [y^* \circ (y^* \to x^*)]^* = (x^* \land y^*)^*$ and by (c42) we deduce that $(x^{**} \to y^{**}) \to y^{**} = (y^{**} \to x^{**}) \to x^{**}$.

$(ii) \Rightarrow (i)$. If $x, y \in L$ and $(x^{**} \to y^{**}) \to y^{**} = (y^{**} \to x^{**}) \to x^{**}$, since $(x^* \land y^*)^* = x^{**} \lor y^{**} = (x^{**} \to y^{**}) \to y^{**}$ and $[x^* \circ (x^* \to y^*)]^* = (y^{**} \to x^{**}) \to x^{**}$, then we obtain that $[x^* \circ (x^* \to y^*)]^* = (x^* \land y^*)^*$, that is, $L \in \mathcal{RL}_{sd}$. 

\[\square\]

Remark 3. 1. Every divisible residuated lattice is semi-divisible, so, every BL-algebra is semi-divisible.

2. Not every residuated lattice is semi-divisible. Consider, for example (see [22]) a fixed real number $c$, $0 < c < 1$, and define the residuated lattice $L_c = ([0,1], \lor, \land, \odot, \to, 0, 1)$ such that for all $x, y \in [0,1]$, such that $x \odot y = 0$ if $x + y \leq c$ and $\min\{x, y\}$ elsewhere, $x \to y = 1$ if $x \leq y$ and $\max\{c - x, y\}$ elsewhere. We have $MV(L_c) = [0, c) \cup \{1\}$. Let $x = \frac{3}{5}c, y = \frac{5}{6}c$. Then $x, y \in MV(L_c)$, $(y \to x) \to x = 1$, but $(x \to y) \to y = y$. Thus, the condition from Theorem 3 does not hold. So $L_c$ is not semi-divisible. Evidently, each residuated lattice $L_c$ is linear, therefore is a MTL-algebra. We deduce that MTL-algebras are not in general semi-divisible.

3. There are residuated lattices that are semi-divisible but not divisible. Consider, for example (see [22]) the residuated lattice $L_{sd} = ([0,1], \lor, \land, \odot, \to, 0, 1)$ such that for all $x, y \in [0,1]$, $x \odot y = 0$ if $x, y \in [0, \frac{1}{2}]$ and $\min\{x, y\}$ elsewhere $x \to y = 1$ if $x \leq y; \frac{1}{2}$ if $y < x \leq \frac{1}{2}$ and $y$ if $(y < x, \frac{1}{2} < x)$. We have $MV(L_{sd}) = \{0, \frac{1}{2}, 1\}$ and the condition from Theorem 3 holds, whence, $L_{sd}$ is a semi-divisible residuated lattice. $L_{sd}$ is not divisible. Indeed, let $x = \frac{1}{3}, y = \frac{1}{2}$. Then $\frac{1}{2} \odot (\frac{1}{2} \to \frac{1}{3}) = \frac{1}{2} \odot \frac{1}{2} = 0 \neq \frac{1}{3} = \min\{\frac{1}{3}, \frac{1}{2}\}$.

We recall that if $(L_i, \lor, \land, \odot, \to, 0, 1), i = 1, 2$ are two residuated lattices then, a map $f : L_1 \to L_2$ is called morphism of residuated lattices if $f$ satisfies the following conditions, for every $x, y \in L_1$:

\[(m_1) \quad f(0) = 0;\]
\[(m_2) \quad f(1) = 1;\]
\[(m_3) \quad f(x \land y) = f(x) \land f(y);\]
\[(m_4) \quad f(x \lor y) = f(x) \lor f(y);\]
\[(m_5) \quad f(x \to y) = f(x) \to f(y);\]
$(m_6)$ $f(x \odot y) = f(x) \odot f(y)$.

We define $\text{Ker}(f) = \{x \in L_1 : f(x) = 1\}$.

**Remark 4.** 1. If $L_1, L_2$ are $BL$-algebras, then $f : L_1 \to L_2$ is morphism of residuated lattices iff $f$ verifies $(m_1), (m_5)$ and $(m_6)$.

2. If $L_1, L_2$ are divisible, then $f : L_1 \to L_2$ is morphism of residuated lattices iff $f$ verifies the conditions $(m_1), (m_4), (m_5)$ and $(m_6)$.

3. If $L_1, L_2$ are $MTL$-algebras, then $f : L_1 \to L_2$ is morphism of residuated lattices iff $f$ verifies $(m_1) - (m_3)$ and $(m_5) - (m_6)$.

So, $\mathcal{RL}$ becomes in canonical way a category such that $\mathcal{RL}_d, \mathcal{MTL}, \mathcal{RL}_{sd}$ and $\mathcal{BL}$ are subcategories of $\mathcal{RL}$.

**Remark 5** ([1, p.31]). Since the categories $\mathcal{MV}$ and $\mathcal{RL}$ are equational, then in these categories the monomorphisms are exactly the one-one morphisms.
For $x, y \in L$, let $\Phi_R(L)(x) = x^{**} = (x^{*})^* = f(x)$ for all $x, y \in L$. If $L, L'$ are divisible residuated lattices and $f : L \to L'$ is a morphism of residuated lattices, then $\mathcal{R}(f) : MV(L) \to MV(L')$ defined by $\mathcal{R}(f)(x^{*}) = f(x^{*}) = (f(x))^{*}$ for every $x \in L$ is a morphism in $\mathcal{M}V$.

Indeed, if $x, y \in L$, then $\mathcal{R}(f)(x^{*} \circ y^{*}) = \mathcal{R}(f)((x \circ y)^{*}) = (f(x \circ y))^{*} = (f(x) \circ f(y))^{*} = f(x)^{*} \circ f(y)^{*} = (\mathcal{R}(f)(x^{*})) \circ (\mathcal{R}(f)(y^{*}))$ and $\mathcal{R}(f)(x^{*})^{*} = (f(x)^{*})^{*} = (\mathcal{R}(f)(x^{*}))^{*}$. So, the assignments $L \to MV(L) = \mathcal{R}(L)$ and $f \mapsto \mathcal{R}(f)$ define a functor (covariant) $\mathcal{R} : \mathcal{RL}_d \to \mathcal{M}V$ from the category of divisible residuated lattices to the category of $MV$-algebras.

**Theorem 5.** The category $\mathcal{MV}$ of $MV$-algebras is a reflective subcategory of the category $\mathcal{RL}_d$ of divisible residuated lattices and the reflector $\mathcal{R}$ preserves monomorphisms.

**Proof.** To prove $\mathcal{R}$ is a reflector ([1]), we consider the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{f} & L' \\
\downarrow{\Phi_R(L)} & & \downarrow{\Phi_R(L')} \\
\mathcal{R}(L) & \xrightarrow{\mathcal{R}(f)} & \mathcal{R}(L')
\end{array}
$$

with $L, L' \in \mathcal{RL}_d$.

If $x \in L$, then $(\Phi_R(L') \circ f)(x) = (\Phi_R(L') \circ f)(x) = (\mathcal{R}(f)(x^{*}))^{*} = (f(x)^{*})^{*}$ and $(\mathcal{R}(f) \circ \Phi_R(L))(x) = (\mathcal{R}(f) \circ \Phi_R(L))(x) = (\mathcal{R}(f)(x^{*}))^{*} = (f(x)^{*})^{*}$, hence $\Phi_R(L') \circ f = \mathcal{R}(f) \circ \Phi_R(L)$, that is, the above diagram is commutative.

Let now $L \in \mathcal{RL}_d$, $M \in \mathcal{MV}$ and $f : L \to M$ a morphism of residuated lattices

$$
\begin{array}{ccc}
L & \xrightarrow{\Phi_R(L)} & \mathcal{R}(L) \\
\downarrow{f} & & \downarrow{f'} \\
M & & 
\end{array}
$$

For $x \in L$, we define $f' : \mathcal{R}(L) \to M$ by $f'(x^{*}) = f(x)^{*}$ (that is, $f' = f|_{\mathcal{M}V(L)}$). For $x, y \in L$, we have $f'(x^{*} \circ y^{*}) = f'((x \circ y)^{*}) = (f(x \circ y)^{*})^{*} = (f(x \circ y)^{*})^{*}$.
DIVISIBLE AND SEMI-DIVISIBLE RESIDUATED LATTICES

\[ y^* = (f(x) \odot f(y))^* = f(x)^* \oplus f(y)^*, \ f((x^*)^*) = f(x)^* = f(x) = (f(x^*))^* \]
and \( f'(0) = f'(1^*) = f(1)^* = 1^* = 0 \), hence \( f' \) is a morphism in \( \mathcal{MV} \). Since for \( x \in L \), \( (f' \circ \Phi_{\mathcal{R}}(L))(x) = f'(\Phi_{\mathcal{R}}(L)(x)) = f'(x^*) = f(x)^* = f(x) \), we deduce that \( f' \circ \Phi_{\mathcal{R}}(L) = f \).

If we have again \( f'' : \mathcal{R}(L) \to M \) a morphism in \( \mathcal{MV} \) such that \( f'' \circ \Phi_{\mathcal{R}}(L) = f \), then for any \( x \in L \), \( (f'' \circ \Phi_{\mathcal{R}}(L))(x^*) = f(x^*) \), hence \( f''(x^*) = f(x^*) \) for any \( x \in L \) such that \( (n-1)x \) is a morphism in \( \mathcal{MV} \).

To prove that \( \mathcal{R} \) preserves monomorphisms, let \( f : L \to L' \) a monomorphism in \( \mathcal{RL}_d \) and \( x, y \in L \) such that \( \mathcal{R}(f)(x^*) = \mathcal{R}(f)(y^*) \). Then \( f(x^*) = f(y^*) \), hence \( x^* = y^* \), that is, \( \mathcal{R}(f) \) is a monomorphism in \( \mathcal{MV} \).

We recall that an \( MV \)-algebra is called complete if it contains the greatest lower bound and the lowest upper bound of any subset. Also, an \( MV \)-algebra \( L \) is called divisible if for any \( a \in L \) and for any natural number \( n \geq 1 \) there is \( x \in L \) such that \( nx = a \) and \( a^* \oplus [(n-1)x] = x^* \).

In [23], p.66, it is proved:

**Theorem 6.** For any \( MV \)-algebra \( L \) the following assertions are equivalent:

(i) \( L \) is injective object in the category \( \mathcal{MV} \),

(ii) \( L \) is complete and divisible \( MV \)-algebra.

So, we obtain the following:

**Corollary 3.** If \( L \) is a complete and divisible \( MV \)-algebra, then \( L \) is an injective object in the category \( \mathcal{RL}_d \).

**Proof.** By Theorem 6, \( L \) is an injective object in the category \( \mathcal{MV} \). Since \( \mathcal{MV} \) is reflective subcategory of \( \mathcal{RL}_d \) and the reflector \( R : \mathcal{RL}_d \to \mathcal{MV} \) preserves monomorphisms (by Theorem 5), then by Theorem 6 from [1] we deduce that \( L \) is injective object in the category \( \mathcal{RL}_d \).

**Remark 6.** Following Corollary 4.4 from [11] we deduce that if \( L \) is an injective object in the category \( \mathcal{RL}_d \), then \( L \) is a complete lattice.

**Open problem.** If \( L \) is an injective object in the category \( \mathcal{RL}_d \), then \( L \) is divisible?
4. Boolean center, regular and dense elements

Let \((L, \lor, \land, 0, 1)\) be a bounded lattice. Recall (see [1], [13]) that an element \(a \in L\) is called \textit{complemented} if there is an element \(b \in L\) such that \(a \lor b = 1\) and \(a \land b = 0\); if such element \(b\) exists it is called a \textit{complement} of \(a\). We will denote \(b = a'\) and the set of all complemented elements in \(L\) by \(B(L)\). Complements are generally not unique, unless the lattice is distributive.

In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

**Lemma 2** ([12]). Let \(L\) be a residuated lattice and suppose that \(a \in L\) have a complement \(b \in L\). Then, the following hold:

(i) If \(c\) is another complement of \(a\) in \(L\), then \(c = b\);
(ii) \(a' = b\) and \(b' = a\);
(iii) \(a^2 = a\).

Let \(L\) be a residuated lattice and \(B(L)\) the set of all complemented elements of the lattice \((L, \lor, \land, 0, 1)\). So, if \(e \in B(L)\), then \(e' = e^*\) and \(e^{**} = e\). Also, \(e \oplus x = e \land x\), for every \(x \in L\), see ([12]).

The set \(B(L)\) is the universe of a Boolean subalgebra of \(L\) (called the \textit{Boolean center} of \(L\) - see [12]).

**Proposition 6** ([5]). If \(L\) is a residuated lattice, then for \(e \in L\) the following are equivalent:

(i) \(e \in B(L)\);
(ii) \(e \lor e^* = 1\).

**Theorem 7** ([7]). For every element \(e\) in an MV-algebra \(A\), the following conditions are equivalent:

(i) \(e \in B(A)\);
(ii) \(e \lor e^* = 1\);
(iii) \(e \land e^* = 0\);
(iv) \(e \oplus e = e\);
Proposition 7 ([5]). Let \( L \) be a residuated lattice. For \( e \in L \) we consider the following assertions:

(i) \( e \in B(L) \);
(ii) \( e^2 = e \) and \( e = e^{**} \);
(iii) \( e^2 = e \) and \( e^* \to e = e \);
(iv) \( (e \to x) \to e = e \), for every \( x \in L \);
(v) \( e \land e^* = 0 \).

Then (i) \( \Rightarrow \) (ii), (iii), (iv) and (v) but (ii) \( \not\Rightarrow \) (i), (iii) \( \not\Rightarrow \) (i), (iv) \( \not\Rightarrow \) (i), (v) \( \not\Rightarrow \) (i).

Remark 7 ([4]). If \( L \) is a BL-algebra, then all assertions from above proposition are equivalent.

Remark 8. If \( L \) is a residuated lattice, then \( B(L) \subseteq MV(L) \).

We put in evidence new rule of calculus with boolean elements:

Lemma 3. Let \( x, y \in L \) and \( e, f \in B(L) \). Then:

\( c_{42} \) \( x \odot (x \to e) = x \land e, e \odot (e \to x) = e \land x \);
\( c_{43} \) \( e \lor (x \odot y) = (e \lor x) \odot (e \lor y) \);
\( c_{44} \) \( e \land (x \odot y) = (e \land x) \odot (e \land y) \);
\( c_{45} \) \( e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)] \);
\( c_{46} \) \( x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)] \);
\( c_{47} \) \( e \to (x \to y) = (e \to x) \to (e \to y) \);
\( c_{48} \) \( e \to (e \to x) = e \to x \);
\( c_{49} \) \( (e \to x) \to e = e \);
\( c_{50} \) \( (e \to x) \to x \leq (x \to e) \to e \);
\( c_{51} \) \( e^* \to x = (e \to x) \to x = e \lor x \).
\((c_{52})\) \(e \lor (x \to y) = (e \lor x) \to (e \lor y)\);  
\((c_{53})\) \((e \lor x) \to (f \lor x) = (e \to f) \lor x\);  
\((c_{54})\) \(x \to (e \to f) = (x \to e) \to (x \to f)\);  
\((c_{55})\) \(e \land (x \lor y) = (e \land x) \lor (e \land y)\);  
\((c_{56})\) \(x \land (e \lor f) = (x \land e) \lor (x \land f)\);  
\((c_{57})\) If \(e, f \leq x\), then \(e \circ (x \to f) = f \circ (x \to e)\);  
\((c_{58})\) \((e \to x) \lor (x \to e) = 1\);  
\((c_{59})\) \(e \lor x = [(e \to x) \to x] \land [(x \to e) \to e]\).

**Proof.** For \((c_{42}) - (c_{47})\) see [5].  
\((c_{48})\). We have \(e \to (e \to x) = e^2 \to x = e \to x\).  
\((c_{49})\). See [5].  
\((c_{50})\). We have \((e \to x) \to x \leq (x \to e) \to [(e \to x) \to e] \equiv (x \to e) \to e\).  
\((c_{51})\). From \(e^* \leq e \to x \land x \leq e \to x \Rightarrow e^* \lor x \leq e \to x\). Also from \(e \leq (e \to x) \to x \leq (e \to x) \to (e^* \lor x)\) and \(e^* \leq (e \to x) \Rightarrow e^* \leq (e \to x) \to (e^* \lor x)\), since \(e \lor e^* = 1 \Rightarrow (e \to x) \to (e^* \lor x) = 1 \Rightarrow e \to x \leq e^* \lor x \Rightarrow e \to x = e^* \lor x \Rightarrow e \to x = e \lor x\).  
Since \(e, x \leq (e \to x) \to x \Rightarrow e \lor x \leq (e \to x) \to x\).  
Since \(e^* \leq e \to x \Rightarrow (e \to x) \to x \leq e^* \to x = e \lor x \Rightarrow (e \to x) \to x = e \lor x\).  
\((c_{52})\). By \((c_{51})\) we have \(e \lor (x \to y) = e^* \to (x \to y) \equiv (e^* \to x) \to (e^* \to y) = (e \lor x) \to (e \lor y)\).  
\((c_{53})\). We have \((e \lor x) \to (f \lor x) \equiv (e^* \to x) \to (f^* \to x) = f^* \to [(e^* \to x) \to x] \equiv f^* \to [(e \lor x) \to x] \equiv f^* \to (e \lor x) \equiv (f^* \circ e) \to x = (f^* \land e) \to x \land (e \to f) \lor x = (e \to f^*) \to x \equiv (e \to f^* \to x) \equiv x \equiv (e^* \lor f^*) \to x \equiv (e \land f^*) \to x \equiv (e \land f^*) \to x \equiv (f \lor x) \equiv (e \rightarrow f) \lor x\).  
\((c_{54})\). \((x \to e) \to (x \to f) \equiv [x \circ (x \to e)] \to f \equiv (x \circ e) \to f = (x \circ e) \to f \equiv (x \circ e) \to f\).  
\((c_{55})\). We have \(e \land (x \lor y) = e \circ (x \lor y) \equiv (e \circ x) \lor (e \circ y) = (e \land x) \lor (e \land y)\).  
\((c_{56})\). We have \(x \lor (e \lor f) = x \circ (e \lor f) \equiv (x \circ e) \lor (x \circ f) = (x \land e) \lor (x \land f)\).  
\((c_{57})\). We have \(e \circ (x \to f) = (e \land x) \circ (x \to f) = [x \circ (x \to e)] \circ (x \to f) = (x \to e) \circ [x \circ (x \to f)] = (x \to e) \circ (x \land f) = f \circ (x \to e)\);
(c58). If \( e \in B(A) \) then \( e \lor e^* = 1 \). Since \( e^* \leq e \to x \) and \( e \leq x \to e \) we deduce that \( 1 = e^* \lor e \leq (e \to x) \lor (x \to e) \), so \( (e \to x) \lor (x \to e) = 1 \).

(c59). Denote \( t = [(e \to x) \to x] \land [(x \to e) \to e] \). We have \( e \leq (e \to x) \to x \) and \( x \leq (e \to x) \to x \); it follows that \( e \lor x \leq (e \to x) \to x \). Analogous, \( e \lor x \leq (x \to e) \to e \). Hence \( e \lor x \leq [(e \to x) \to x] \land [(x \to e) \to e] \), so \( e \lor x \leq t \).

We have \( t = t \circ 1 \) \((c59)\); but \( t \circ (e \to x) = [(e \to x) \to x] \land [(x \to e) \to e] \circ (e \to x) \leq [(e \to x) \to x] \circ (e \to x) \leq (e \to x) \land x \leq x \); similarly, \( t \circ (x \to e) \leq e \). Hence, \( t = [t \circ (e \to x)] \lor [t \circ (x \to e)] \leq x \lor e \). It follows that \( e \lor x = t \). \( \square \)

Proposition 8. If \( L \) is a divisible residuated lattice, then the assertions (i), (ii), (iii) and (iv) from Proposition 7 are equivalent.

Proof. (i) \( \Rightarrow \) (ii). See Proposition 7.

(ii) \( \Rightarrow \) (iii). We have that \( e \to e^* = e \to (e \to 0) = (e \circ e) \to 0 = e \to 0 = e^* \).

Hence, \( e \land e^* = e \circ (e \to e^*) = e \circ e^* = 0 \). Since \( e \land e^* = e^* \land e = e^* \circ (e^* \to e) = 0 \), we obtain that \( e^* \to e \leq e^{*\ast} = e^* \). But, \( e \circ e^* \leq e \) so \( e \leq e^* \to e \). We have that \( e^* \to e = e \).

(iii) \( \Rightarrow \) (i). Applying (c59), \( e \land e^* = 1 \iff (e \to e^*) \to e^* = 1 \) and \( (e^* \to e) \to 1 \). By (iii), \( e^* \to e = e \), hence \( (e^* \to e) \to 1 = 1 \). We also have that \( e \to e^* = e \to (e \to 0) = (e \circ e) \to 0 = e \to 0 = e^* \). So, \( (e \to e^*) \to e^* = 1 \). We deduce that \( e \land e^* = 1 \), that is, \( e \in B(A) \), from Proposition 6.

(i) \( \Rightarrow \) (iv). See Proposition 7.

(iv) \( \Rightarrow \) (i). If \( x \in L \), then from \( (e \to x) \) we deduce \( (e \to x) \circ [e \to x] = (e \to x) \circ e \), hence \( (e \to x) \land e = e \land x \). For \( x = 0 \) we obtain that \( e^* \to e = e \). Also, from hypothesis (for \( x = 0 \)) we obtain \( e^* \to e = e \). So, from (c59) we obtain \( e \lor e^* = [(e \to e^*) \to e^*] \land [(e^* \to e) \to e] = [(e \to e^*) \to e^*] \land (e \to e) \)

\((c59) \) \( \leq (e \lor e^*) \land 1 = (e \to e^*) \to e^* \) \( (c59) \) \( \leq [e \circ (e \to e^*)] = e \land e^* = 0 = 1 \), hence \( e \in B(L) \), from Proposition 6.

Proposition 9. Let \( L \) be a residuated lattice. For \( e \in L \) we consider the following assertions:

(i) \( e \in B(L) \);
\[(vi) \ (e \to e^*) \vee (e^* \to e) = 1.\]

Then \((i) \Rightarrow (vi)\) but \((vi) \not\Rightarrow (i)\).

**Proof.** \((i) \Rightarrow (vi)\) If \(e \in B(L)\) from \((c_{58})\) for \(x = e^*\) we deduce that 
\((e \to e^*) \vee (e^* \to e) = 1.\)

\( (vi) \not\Rightarrow (i)\). Consider the residuated lattice \(L = \{0, a, b, c, 1\}\) from the 
Example 4; it is easy to verify that \(B(L) = \{0, 1\}\). We have \((c \to c^*) \vee (c^* \to c) = (c \to 0) \vee (0 \to c) = 1\), but \(c \notin B(L)\). 

**Definition 3.** If \(L\) is a residuated lattice, we say that an element \(x \in L\) is regular if for every \(y \in L\) we have \((x \to y) \to x = x\). We denote by \(R(L)\) the set of all regular elements of \(L\). We say that an element \(x \in L\) is dense if for every \(r \in R(L)\) we have \(x \to r = r\). We denote by \(D(L)\) the set of all dense elements of \(L\).

We give a new characterization for regular elements:

**Theorem 8.** Let \(L\) be a residuated lattice. For \(x \in L\) the following assertions are equivalent:

\( (i) \) \(x \in R(L)\);

\( (ii) \) \(x^* \to x = x\).

\( (iii) \) \(x = x^{**}\) and \(x^* \odot (x^* \to x) = 0\).

**Proof.** \((i) \Rightarrow (ii)\). If \(x \in R(L)\), then \((x \to y) \to x = x\), so for \(y = 0\) we 
obtain \(x^* \to x = x\).

\( (ii) \Rightarrow (i)\). Suppose that \(x^* \to x = x\) and let \(y \in L\). Then \(0 \leq y \Rightarrow x \to 0 \leq x \to y \Rightarrow x \to y \Rightarrow (x \to y) \to x \leq x^* \to x = x\).

Since \(x \leq (x \to y) \to x\) we deduce that \((x \to y) \to x = x\), so \(x \in R(L)\).

\( (ii) \Rightarrow (iii)\). Let \(x \in L\) such that \(x^* \to x = x\). Then \(x^* \odot (x^* \to x) = x^* \odot x = 0\). To prove that \(x = x^{**}\) we use \((c_{20})\) and the relation \(x^{**} \leq x^* \to x = x\).

\( (iii) \Rightarrow (ii)\). Since \(x^* \odot (x^* \to x) = 0\) we deduce that \(x^* \to x \leq x^{**} = x\). 
But, \(x \leq x^* \to x\), so \(x^* \to x = x\). \(\Box\)

From this theorem we obtain that:

**Corollary 4.** If \(L\) is a residuated lattice, then \(R(L) \subseteq MV(L)\).

**Corollary 5.** If \(x, y \in R(L)\), then \(x^{**} \in R(L)\) and \(x \land y \in R(L)\).
Proof. Let $x \in R(L)$; by Theorem 8, $x^{**} = x$ and $x^* \to x = x$, so $x^{**} \in R(L)$. From $x \land y \leq x \Rightarrow x^* \leq (x \land y)^* \Rightarrow (x \land y)^* \to x \leq x^* \to x = x \Rightarrow (x \land y)^* \to x = x$. Analogously we deduce that $(x \land y)^* \to y = y$. Then $(x \land y)^* \to (x \land y) \equiv ([x \land y]^* \to x] \land [(x \land y)^* \to y] = x \land y \Rightarrow x \land y \in R(L)$. 

A residuated lattice $L$ will be called $G$-algebra if $x^2 = x$, for every $x \in L$. $L$ is a $G$-algebra iff $x \circ (x \to y) = x \circ y = x \land y$ for every $x, y \in L$ (see [21]).

Remark 9. If $L$ is a $G$-algebra, then for every $x \in L$ we deduce that $x^* \in R(L)$. Indeed, $x^*$ verifies the conditions $(iii)$ from the Theorem 8 because $x^* = (x^*)^*$ and $(x^*)^* \circ [(x^*)^* \to x^*] = x^{**} \circ (x^{**} \to x^*) = x^{**} \circ x^* = 0$.

Lemma 4. Let $L$ be a residuated lattice. If $x \in L$ such that $x \circ x = x$ then $x \to x^* = x^*$.

Proof. $x \to x^* = x \to (x \to 0) = (x \circ x) \to 0 = x \to 0 = x^*$. 

We give a new characterization for boolean elements:

Theorem 9. Let $L$ be a residuated lattice. For $x \in L$ the following assertions are equivalent:

(i) $x \in B(L)$;

(ii) $x \in R(L)$, $x \circ x = x$ and $(x \to x^*) \lor (x^* \to x) = 1$.

Proof. (i) $\Rightarrow$ (ii). If $x \in B(L)$, then by Proposition 7, $x \circ x = x, x = x^{**}$ and $x^* \land x = 0$. Since $x^* \circ (x^* \to x) \leq x^* \land x = 0 \Rightarrow x^* \circ (x^* \to x) = 0$, so $x \in R(L)$ (by Theorem 8, (iii) $\Rightarrow$ (i)). By Lemma 4 and Proposition 7, (iii), since $x \in B(L)$ we deduce that $(x \to x^*) \lor (x^* \to x) = x^* \lor x = 1$.

(iii) $\Rightarrow$ (i). Since $x \circ x = x$, by Lemma 4, $x \to x^* = x^*$. Since $x \in R(L)$, then by Theorem 8, (ii), $x^* \to x = x$. Then $x \lor x^* = (x^* \to x) \lor (x \to x^*) = 1$. Using Proposition 6, we conclude that $x \in B(L)$. 

Corollary 6. Let $L$ be a MTL-algebra. For $x \in L$ the following assertions are equivalent:

(i) $x \in B(L)$;

(ii) $x \in R(L)$ and $x \circ x = x$. 

Corollary 7. In general, for a residuated lattice \( L \), \( B(L) \subseteq R(L) \).

Proof. By Theorem 9 we deduce that \( B(L) \subseteq R(L) \). To prove that \( B(L) \neq R(L) \) we consider the residuated lattice \( L = \{0, a, b, c, 1\} \) from Example 4; it is easy to verify that \( B(L) = \{0, 1\} \). We have \( a^* = b, b^* = a \), hence \( a^{**} = b^* = a \) and \( a^* \odot (a^* \to a) = b \odot (b \to a) = b \odot a = 0 \), hence \( a \in R(L) \) but \( a \notin B(L) \).

Corollary 8. If \( L \) is a semi-divisible residuated lattice, then \( B(L) \subseteq B(MV(L)) \subseteq R(L) \).

Proof. Since \( L \) is semi-divisible then \( MV(L) \) is an MV-algebra. Let \( e \in B(L) \). Then by Proposition 7, \( e = e^* = (e^*)^* \in MV(L) \) and \( e \land e^* = 0 \). Since \( e \land e^* = 0 \Rightarrow e \land_W e^* = 0 \Rightarrow e \in B(MV(L)) \), by Theorem 7. We deduce that \( B(L) \subseteq B(MV(L)) \).

Consider now \( e \in B(MV(L)) \). Then \( e = e^* = e \lor_W e^* = 1 \). Hence \( e \lor_W e^* = 1 \Rightarrow e^* \lor_W e^{**} = 1 \Rightarrow (e^* \to e^{**}) \to e^{**} = 1 \Rightarrow (e^* \to e) \to e = 1 \Rightarrow e^* \to e = e \), hence by Theorem 8, \( (i) \Rightarrow (i) \), we deduce that \( e \in R(L) \), so \( B(L) \subseteq B(MV(L)) \subseteq R(L) \).

Corollary 9. If \( L \) is a MTL-algebra and \( x \odot x = x \) for every \( x \in L \), then \( B(L) = R(L) \).

Corollary 10. If \( L \) is a semi-divisible MTL-algebra, then \( B(L) = B(MV(L)) \).

Proof. By Corollary 8, \( B(L) \subseteq B(MV(L)) \) for any semi-divisible residuated lattice \( L \). Consider now that \( L \) is a semi-divisible MTL-algebra and let \( e \in B(MV(L)) \). Then \( e \lor_W e^* = 1 \Rightarrow e^* \lor_W e^{**} = 1 \Rightarrow (e^{**} \to e^*) \to e^* = 1 \Rightarrow (e \to e^*) \to e^* = 1 \Rightarrow e \to e^* = e^* \). Also \( e \lor_W e^* = 1 \Rightarrow e^* \lor_W e^{**} = 1 \Rightarrow (e^* \to e^{**}) \to e^{**} = 1 \Rightarrow (e^* \to e) \to e = 1 \Rightarrow e^* \to e = e \).

Then \( e^* \lor e = (e \to e^*) \lor (e^* \to e) = 1 \), since \( L \) is a MTL-algebra, so \( e \in B(L) \) and \( B(MV(L)) \subseteq B(L) \),

We deduce that \( B(L) = B(MV(L)) \).

Corollary 11. If \( L \) is a \( G \)-semi-divisible MTL-algebra, then \( B(L) = B(MV(L)) = R(L) \).

Remark 10. 1. We consider the residuated lattice from Remark 3, (2) which is not semi-divisible. Let \( c = \frac{1}{2} \). We have \( 0^* = 1, x^* = \frac{1}{2} - x \), if \( x \in (0, \frac{1}{2}) \), \( x^* = 0 \), if \( x \in [\frac{1}{2}, 1] \). Then, \( 0^{**} = 0, x^{**} = x \), if \( x \in (0, \frac{1}{2}) \), \( x^{**} = 1 \),
if $x \in [\frac{1}{2}, 1]$. Also, $0^* \rightarrow 0 = 0, x^* \rightarrow x = x$, if $x \in (0, \frac{1}{2}), x^* \rightarrow x = 1$, if $x \in [\frac{1}{3}, 1]$. We conclude that $MV(L) = \{x \in L : x^{**} = x\} = [0, \frac{1}{2}) \cup \{1\}$ (but $MV(L)$ is not an MV-algebra), $R(L) = \{x \in L : x^* \rightarrow x = x\} = [0, \frac{1}{4}) \cup \{1\}$ and $B(L) = \{x \in L : x^* \lor x = 1\} = \{0, 1\}$. So, $B(L) \subseteq R(L) \subseteq MV(L)$.

2. If consider the residuated lattice from Remark 3, (3) which is semi-divisible we have $0^* = 1, x^* = \frac{1}{2}$, if $x \in (0, \frac{1}{2}], x^* = 0$, if $x \in (\frac{1}{2}, 1]$. Then, $0^{**} = 0, x^{**} = \frac{1}{2}$, if $x \in (0, \frac{1}{2}], x^{**} = 1$, if $x \in (\frac{1}{2}, 1]$. Also, $0^* \rightarrow 0 = 0, x^* \rightarrow x = \frac{1}{2}$, if $x \in (0, \frac{1}{2}), x^* \rightarrow x = 1$, if $x \in [\frac{1}{2}, 1]$. We conclude that $MV(L) = \{x \in L : x^{**} = x\} = \{0, \frac{1}{2}, 1\}$ (in this case $MV(L)$ is an MV-algebra), $R(L) = \{x \in L : x^* \rightarrow x = x\} = \{0, 1\}$ and $B(L) = \{x \in L : x^* \lor x = 1\} = \{0, 1\}$. Since $B(MV(L)) = \{x \in MV(L) : x \land x^* = 0\} = \{0, 1\}$ we have $B(L) = R(L) \subseteq MV(L)$ and $B(L) = B(MV(L))$ (since $L$ is semi-divisible, see Corollary 8).

We characterize the residuated lattices which are Boolean algebras:

**Theorem 10.** For a residuated lattice $L$ the following assertions are equivalent:

(i) $L$ is a Boolean algebra relative to the natural ordering;

(ii) $L$ is a $G$-algebra and $x^{**} = x$, for every $x \in L$.

**Proof.** (i) $\Rightarrow$ (ii). If $L$ is a Boolean algebra relative to the natural ordering, then $L$ becomes a residuated lattice (see Example 3) and $x \circ x = x \land x = x, x^{**} = x$, for every $x \in L$.

(ii) $\Rightarrow$ (i). Let $L$ be a $G$-algebra such that $x^{**} = x$, for every $x \in L$. Then $x \circ y = x \land y$ for every $x, y \in L$. First we shall prove that $x \lor y = x^* \rightarrow y$ for every $x, y \in L$.

Indeed, $x, y \leq x^* \rightarrow y$. Let $t \in L$ such that $x, y \leq t$. From $x \leq t$ we deduce that $t^* \leq x^*$, hence $x^* \rightarrow y \leq t^* \rightarrow y \leq t \circ t^* = (t^* \circ t^*) \rightarrow 0 = t^* \rightarrow 0 = t^{**} = t$. Following Proposition 6, to prove that $L$ is a Boolean algebra it will suffice to prove that $x \lor x^* = 1$, for every $x \in L$.

Obviously, $x \lor x^* = x^* \rightarrow x^* = 1$.

**Theorem 11.** Let $L$ be a residuated lattice. For $x \in L$ the following assertions are equivalent:

(i) $x \in D(L)$;

(ii) $x^* = 0$. 

Proof. (i) ⇒ (ii). Since \((0 \to y) \to 0 = 1 \to 0 = 0\) for every \(y \in L\), we deduce that \(0 \in R(L)\). Let \(x \in D(L)\); since \(0 \in R(L)\), we obtain \(x \to 0 = 0\), hence \(x^* = 0\).

(ii) ⇒ (i). Let now \(x \in L\) such that \(x^* = 0\) and \(r \in R(L)\) (hence \(r^{**} = r\), by Theorem 8). Then \(x \to r = x \to r^{**} = x \to (r^{*} \to 0) \overset{\text{(c1)}}{=} r^{*} \to (x \to 0) = r^{*} \to x^* = r^{*} \to 0 = r^{**} = r\), hence \(x \in D(L)\).

Proposition 10. If \(L\) is a \(G\)-algebra, then:

(i) For every \(x \in L\), \(x^{**} \to x \in D(L)\);

(ii) \(x \in D(L)\) iff \(x = y^{**} \to y\) for some \(y \in L\);

(iii) For every \(x \in L\), \((x^{**} \to x) = x\) and \((x^{*} \to x) \to x = x^{**}\).

Proof. (i). Since \(x, x^* \leq x^{**} \to x\), we deduce that \((x^{**} \to x)^* \leq x^* \leq x^{**} \to x\), hence \((x^{**} \to x)^* \leq x^{**} \to x\) so \((x^{**} \to x)^* \circ (x^{**} \to x)^* \leq (x^{**} \to x)^* \circ (x^{**} \to x) \to x \circ (x^{**} \to x)^* \Rightarrow (x^{**} \to x)^* \leq 0 \Rightarrow (x^{**} \to x)^* = 0 \Rightarrow x^{**} \to x \in D(L)\).

(ii). By (i), if \(x = y^{**} \to y\) for some \(y \in L\), we deduce that \(x \in D(L)\).

Conversely, let \(x \in D(L)\). Then for \(y = x\) we obtain \(x^{**} \to x = 0^* \to x = 1 \to x = x\).

(iii). First we prove that \((x^{**} \to x) \to x = x\). Clearly, \(x \leq x^{**}, x^{**} \to x\).

Let \(t \in A\) such that \(t \leq x^{**}, x^{**} \to x\). We deduce that \(x^{**} \leq t \to x\), hence \(t \leq x^{**} \leq t \to x\), so \(t \circ t = t \leq x\), that is, \(x^{**} \to (x^{**} \to x) = x\).

We prove now that \((x^{**} \to x) \to x = x^{**}\).

Since \((x^{**} \to x) \to x = x\) and \(L\) is a \(G\)-algebra we deduce that \((x^{**} \to x)^* \circ (x^{**} \to x) = x\), so \((x^{**} \to x)^* \circ (x^{**} \to x) \leq x\). Hence \(x^{**} \leq (x^{**} \to x) \to x\).

Since \(x \leq x^{**} \Rightarrow (x^{**} \to x) \to x \leq (x^{**} \to x) \to x^{**} \overset{\text{(c2)}}{=} x^{*} \to (x^{**} \to x)^*\). From (i), \(x^{**} \to x \in D(L)\) for every \(x \in L\), so \((x^{**} \to x) \to x \leq x^{*} \to 0 = x^{**}\). We deduce that \((x^{**} \to x) \to x = x^{**}\).

Corollary 12. If \(L\) is a \(G\)-algebra, then for every \(x \in L\) there are \(y \in R(L)\) and \(z \in D(L)\) such that \(x = y \wedge z\).

Proof. We have \(y = x^{**} \in R(L)\) and \(z = x^{**} \to x \in D(L)\).

5. Semi-\(G\)-algebras

Definition 4 ([12], [23]). A non empty subset \(D \subseteq L\) is called a deductive system of \(L\), ds for short, if the following conditions are satisfied:
A ds $D$ of $L$ is called proper if $D \neq L$.

Remark 11 ([12], [23]). A nonempty subset $D \subseteq L$ is a ds of $L$ iff:

(Ds$_1'$) If $x, y \in D$, then $x \circ y \in D$;

(Ds$_2'$) If $x \in D, y \in L, x \leq y$, then $y \in D$.

We denote by $Ds(L)$ the set of all deductive systems of $L$. For a nonempty subset $S \subseteq L$, the smallest ds of $L$ which contains $S$, i.e. $\cap \{ D \in Ds(L) : S \subseteq D \}$, is said to be the ds of $L$ generated by $S$ and will be denoted by $< S >$. For $D_1, D_2 \in Ds(L)$ we put $D_1 \wedge D_2 = D_1 \cap D_2$ and $D_1 \vee D_2 = < D_1 \cup D_2 >$. The lattice $(Ds(L), \subseteq)$ is a complete Brouwerian lattice (hence distributive).

We say ([21]) that $D \in Ds(L)$ is prime if $D \neq L$ and $D$ verify one of the equivalent assertions:

(i) If $D = D_1 \cap D_2$ with $D_1, D_2 \in Ds(L)$, then $D = D_1$ or $D = D_2$;

(ii) For $a, b \in L$, if $a \vee b \in D$, then $a \in D$ or $b \in D$.

We denote by $Spec(L)$ the set of all prime deductive systems of $L$. A ds of $L$ is maximal if it is proper and it is not contained in any other proper ds. In a nontrivial residuated lattice $L$, every proper ds can be extended to a maximal ds.

We shall denote by $Max(L)$ the set of all maximal ds of $L$. Obviously, $Max(L) \subseteq Spec(L)$.

In what follow, we present new results relative to lattice of maximal deductive systems of a residuated lattice. We prove that if $L$ is semi-divisible, then there is a bijective correspondence between maximal deductive systems of $L$ and maximal deductive systems of $MV(L)$. Also, we characterize $Rad(MV(L))$, for a semi-divisible residuated lattice $L$.

Theorem 12 ([5]). Let $L$ be a residuated lattice and $M$ a proper ds of $L$. Then the following are equivalent:

(i) $M \in Max(L)$,
(ii) For any \( x \in L, x \notin M \) iff \( (x^n)^* \in M \), for some \( n \geq 1 \).

**Corollary 13.** Let \( L \) be a residuated lattice. If \( M \in \text{Max}(L) \) and \( x, y \in L \) such that \( (x^n)^* \to y \in M \), for any \( n \geq 1 \), then \( x \in M \) or \( y \in M \).

**Proof.** Let \( x, y \in L \) such that \( (x^n)^* \to y \in M \) for any \( n \geq 1 \) and suppose that \( x \notin M \) and \( y \notin M \). From Theorem 12 we deduce that there is \( n_0 \geq 1 \) such that \( (x^{n_0})^* \in M \). Since \( (x^{n_0})^*, (x^{n_0})^* \to y \in M \Rightarrow y \in M \), a contradiction. \( \square \)

**Theorem 13.** Let \( L \) be a residuated lattice. If \( M \) is a proper ds of \( L \), then the following are equivalent:

(i) \( M \in \text{Max}(L) \),

(ii) If \( x \notin M \), then there is \( n \geq 1 \) such that \( x^n \to y \in M \), for every \( y \in L \).

**Proof.** (i) \( \Rightarrow \) (ii). If we suppose that \( x \notin M \) then by Theorem 12 there is \( n \geq 1 \) such that \( (x^n)^* \in M \). Because \( (x^n)^* \leq x^n \to y \) for every \( y \in L \) we deduce that \( x^n \to y \in M \), for every \( y \in L \).

(ii) \( \Rightarrow \) (i). Let \( D \in Ds(L) \) proper such that \( M \nsubseteq D \), hence there is \( x_0 \in D \) such that \( x_0 \notin M \). Then there is \( n \geq 1 \) such that \( x_0^n \to y \in M \), for every \( y \in L \). Hence \( x_0^n \to y \in D \). Since \( D \in Ds(L) \) and \( x_0 \in D \) we deduce that \( x_0^n \in D \). From \( x_0^n \to y \in D \Rightarrow y \in D \), for every \( y \in L \). Hence \( D = L \), a contradiction. \( \square \)

**Proposition 11.** Let \( M \in \text{Max}(L) \) and \( x \in L \). Then \( x \in M \) iff \( x^{**} \in M \).

**Proof.** If \( x \in M \) since \( x \leq x^{**} \), then \( x^{**} \in M \). Conversely, let \( x \in L \) such that \( x^{**} \in M \) and suppose by contrary that \( x \notin M \). Since \( M \in \text{Max}(L) \), then there is \( n \geq 1 \) such that \( (x^n)^* \in M \). Following \((c_{35})\), \( (x^{**})^n \leq (x^n)^{**} \), so \( (x^n)^{**} \in M \). From \( (x^n)^*, (x^n)^{**} \in M \Rightarrow 0 \in M \Rightarrow M = L \), a contradiction. \( \square \)

**Remark 12.** In a residuated lattice \( L \), the following are equivalent:

(i) \( x \to x^* = x^* \) for every \( x \in L \);

(ii) \( (x^2)^* = x^* \) for every \( x \in L \);

(iii) \( (x^n)^* = x^* \) for every \( x \in L \) and \( n \geq 2 \);
(iv) \(x \odot (x \to x^*) = 0\) for every \(x \in L\).

**Definition 5.** \(L\) is called **semi G-algebra** if it verifies one of equivalent conditions of Remark 12.

We recall that a residuated lattice \(L\) is called a **G-algebra** if \(x^2 = x\) for every \(x \in L\).

We will show that not every residuated lattice is semi G-algebra and, moreover, there are residuated lattices that are semi G-algebras but not G-algebras. Indeed, consider \(L = [0,1]\) such that \((L, \max, \min, \odot, \to, 0, 1)\) is a Product structure or Gaines structure, see Example 2. For \(x = 0\), \((0^2)^* = 0^* = 1 = 0^*\) and for \(x > 0 \Rightarrow x^* = x \to 0 = 0/x = 0\). Since \(x^2 > 0 \Rightarrow (x^2)^* = x^2 \to 0 = 0/x^2 = 0\), so \((x^2)^* = x^* = 0\), hence \(L\) is a semi G-algebra. Since \((1^2)^2 = 1^2 \neq 0\), \(L\) is not a G-algebra.

Consider \(L\) from Example 1 for \(p = 1\). Then \((1^2)^* = 1/2 \to 0 = \min\{1, 1 - 1/2 + 0\} = \min\{1, 1/2\} = 1/2\) and \((1^2)^2 = 1/2 \odot 1/2 = \max\{0, 1/2 + 1/2 - 1\} = 0\), so \([(1^2)^*]^2 = 1 \neq 1/2 = (1^2)^*,\) hence \(L\) is not a semi G-algebra.

**Lemma 5.** If \(L\) is a semi G-algebra and \(x, y \in R(L)\), then \(x \to y \in R(L)\). In particular \(x \in R(L) \Rightarrow x^* \in R(L)\).

**Proof.** If \(x, y \in R(L)\), following Theorem 8, \(x^{**} = x\) and \(y^{**} = y\). By \((c_{34})\), \((x \to y)^{**} \leq x^{**} \to y^{**} = x \to y\), so \((x \to y)^{**} = x \to y\). By \((c_{15})\), \((x \to y)^* \to (x \to y) \leq [(x \to y)^* \odot (x \to y)^*] \to [(x \to y) \odot (x \to y)^*] = [(x \to y)^*]^2 \to 0 = [(x \to y)^*]^2 \to 0 = ([(x \to y)^*]^2)^* = [(x \to y)^*]^* = (x \to y)^{**} \to 0 = x \to y\), hence \((x \to y)^* \to (x \to y) \leq x \to y \Rightarrow (x \to y)^* \to (x \to y) = x \to y \Rightarrow x \to y \in R(L)\).

For \(D \in Ds(L)\), we consider \(\varphi_D : L \to \{0,1\}\) defined for \(x \in L\) by \(\varphi_D(x) = 1\) if \(x \in D\) and \(\varphi_D(x) = 0\) if \(x \notin D\).

**Lemma 6.** Let \(D \in Ds(L)\) proper. Then:

(i) \(\varphi_D(0) = 0, \varphi_D(1) = 1, \varphi_D(x \odot y) = \varphi_D(x) \odot \varphi_D(y)\) and \(\varphi_D(x \to y) = \varphi_D(x) \to \varphi_D(y)\), for every \(x, y \in L\);

(ii) If \(D \in Spec(L)\), then \(\varphi_D(x \lor y) = \varphi_D(x) \lor \varphi_D(y)\), for every \(x, y \in L\);

(iii) If \(L\) is a semi G-algebra and \(D \in Max(L)\) then \(\varphi_D(x \to y) = \varphi_D(x) \to \varphi_D(y)\), for every \(x, y \in L\).
Proof. (i). Since $0 \notin D$ and $1 \in D$, then $\varphi_D(0) = 0$ and $\varphi_D(1) = 1$.
Consider $x, y \in L$. If $x \circ y \in D$, then $x, y \in D$, so $\varphi_D(x \circ y) = 1 = 1 \circ 1 = \varphi_D(x) \circ \varphi_D(y)$. If $x \circ y \notin D$, then we deduce that $x \notin D$ or $y \notin D$, so $\varphi_D(x \circ y) = \varphi_D(x) \circ \varphi_D(y) = 0$.
If $x \land y \in D$ then $x, y \in D$, so $\varphi_D(x \land y) = \varphi_D(x) \land \varphi_D(y) = 1$.
If $x \land y \notin D$, then $x \notin D$ or $y \notin D$ (since if by contrary $x \in D$ and $y \in D \Rightarrow x \circ y \notin D$, then $\varphi_D(x \land y) = \varphi_D(x) \land \varphi_D(y) = 0$).
(ii). If $x \lor y \in D$, since $D$ is prime, then $x \in D$ or $y \in D$, so $\varphi_D(x \lor y) = \varphi_D(x) \lor \varphi_D(y) = 1$.
If $x \lor y \notin D$, then $x \notin D$ and $y \notin D$, so $\varphi_D(x \lor y) = \varphi_D(x) \lor \varphi_D(y) = 0$.
(iii). Consider $x, y \in L$ such that $x \rightarrow y \in D$.
If $x \in D$, then $y \in D$, so $\varphi_D(x \rightarrow y) = \varphi_D(x) \rightarrow \varphi_D(y) = 1$.
If $x \notin D$ and $y \in D$, then $\varphi_D(x) \rightarrow \varphi_D(y) = 0 \rightarrow 1 = 1 \Rightarrow \varphi_D(x \rightarrow y) = \varphi_D(x) \rightarrow \varphi_D(y) = 1$.
If $x \notin D$ and $y \notin D$, then $\varphi_D(x) \rightarrow \varphi_D(y) = 0 \rightarrow 0 = 1 \Rightarrow \varphi_D(x \rightarrow y) = \varphi_D(x) \rightarrow \varphi_D(y) = 1$.
To prove the equality $\varphi_D(x \rightarrow y) = \varphi_D(x) \rightarrow \varphi_D(y)$ it is necessary to prove that $\varphi_D(x) = 1$, that is $x \in D$.
If by contrary $x \notin D$, then there is $n \geq 1$ such that $(x^n)^* \in D$. Since $L$ is supposed to be a semi G-algebra, then $(x^n)^* = x^*$, so $x^* \in D$. Since $x^* \leq x \rightarrow y$, then $x \rightarrow y \in D$, a contradiction. \hfill \qed

Corollary 14. If $L$ is a semi G-algebra and $M \in \text{Max}(L)$, then $\varphi_M : L \rightarrow \{0, 1\}$ is a morphism of residuated lattices and $\text{Ker}(f_M) = M$.

Proof. Using Lemma 6, $\varphi_M$ is a morphism of residuated lattices.
Obviously, $\text{Ker}(\varphi_M) = M$, because $\text{Ker}(\varphi_M) = \{x \in L : \varphi_M(x) = 1\} = \{x \in L : x \in M\} = M$. \hfill \qed

Theorem 14. If $L$ is a semi G-algebra, then there is a bijection between $\text{Max}(L)$ and $\mathcal{RL}(L, \{0, 1\}) = \{ f : L \rightarrow \{0, 1\} \mid f \text{ is a morphism of residuated lattices} \}$.

Proof. We define $\alpha : \text{Max}(L) \rightarrow \mathcal{RL}(L, \{0, 1\})$ by $\alpha(M) = f_M$, for every $M \in \text{Max}(L)$ and $\beta : \mathcal{RL}(L, \{0, 1\}) \rightarrow \text{Max}(L)$ by $\beta(f) = \text{Ker}(f)$, for every $f \in \mathcal{RL}(L, \{0, 1\})$.
$\text{Ker}(f) \in \text{Max}(L)$ because if $x \notin \text{Ker}(f)$ then $f(x) = 0$. Since $f(x^*) = (f(x))^* = 0^* = 1$, we deduce that $x^* \in \text{Ker}(f)$, so $\text{Ker}(f) \in \text{Max}(L)$. For $M \in \text{Max}(L), (\beta \circ \alpha)(M) = \beta(\alpha(M)) = \text{Ker}(f_M) = M$, from Corollary 14,
If \( \alpha \circ \beta = 1_{\text{Max}(L)} \). If \( f \in \mathcal{RL}(L, \{0, 1\}) \), then \((\alpha \circ \beta)(f) = \alpha(\beta(f)) = f_{\text{Ker}(f)}\).

We prove that \( f_{\text{Ker}(f)} = f \). If \( x \in \text{Ker}(f) \), then \( f(x) = 1 \), so \( f_{\text{Ker}(f)}(x) = 1 \) and if \( x \notin \text{Ker}(f) \), then \( f(x) = 0 \), so \( f_{\text{Ker}(f)}(x) = 0 \). We deduce that \( \alpha \circ \beta = 1_{\mathcal{RL}(L, \{0,1\})} \), so \( \alpha, \beta \) are bijections.

We recall that an MV-algebra is an algebra \((M, \oplus, \ast, 0)\) of type \((2, 1, 0)\) such that \((M, \oplus, 0)\) is a commutative monoid, \(x^{**} = x\) for every \(x \in M\) and \((x \ast y)^* \ast y = (y \ast x)^* \ast x\) for every \(x, y \in M\). We denote \(0^* = 1\). For \(x, y \in M\) we denote \(x \circ y = (x \ast y)^*\) and \(x \rightarrow y = x \ast y\). Then ([23]), \((M, \lor, \land, \rightarrow, 0, 1)\) is a BL-algebra, where for \(x, y \in M\), \(x \lor y = (x \rightarrow y) \rightarrow y\) and \(x \land y = (x \land y)^*\). Conversely, a BL-algebra \((L, \lor, \land, \rightarrow, 0, 1)\) is an MV-algebra if \(x^{**} = x\) for every \(x \in L\), where \(x^* = x \rightarrow 0\).

**Lemma 7.** Let \( L \) be a residuated lattice. If \( D \subseteq Ds(L) \), then \( D \cap MV(L) = \{y \in MV(L) : y = x^{**} \text{ for some } x \in D\} \).

**Proof.** If \( y \in MV(L) \) such that \( y = x^{**} \) with \( x \in D \), since \( x \leq x^{**} \), then \( y \in D \), hence \( y \in D \cap MV(L) \). Conversely, if \( y \in D \cap MV(L) \), then \( y \in D \) and \( y = x^* \) (with \( x \in L \)). We have \( y = x^* = x^{**} = y^{*\ast} \).

Suppose \( L \) is a semi-divisible residuated lattice. Then \( MV(L) \) is an MV-algebra (by Corollary 2), hence a BL-algebra and by \( Ds(MV(L)) \) we denote the set of all deductive systems of \( MV(L) \). We recall that for \(x, y \in MV(L), x \oplus y = x^* \rightarrow y\) (so \(x \rightarrow y = x \oplus y\)).

**Lemma 8.** If \( L \) is a semi-divisible residuated lattice and \( D \subseteq Ds(L) \), then \( D \cap MV(L) \subseteq Ds(MV(L)) \).

**Proof.** Clearly \( 1 \in D \cap MV(L) \). Let \( x, y \in MV(L) \) such that \( x, x \rightarrow y \in D \cap MV(L) \). Since \( D \subseteq Ds(L) \) and \( x, x \rightarrow y \in D \), hence \( y \in D \cap MV(L) \), so \( D \cap MV(L) \subseteq Ds(MV(L)) \).

**Lemma 9.** If \( L \) is a semi-divisible residuated lattice and \( M \in \text{Max}(L) \), then \( M \cap MV(L) \subseteq \text{Max}(MV(L)) \).

**Proof.** We have \( M \neq L \). If \( M \cap MV(L) = MV(L) \), then \( MV(L) \subseteq M \Rightarrow 0 \in M \Rightarrow M = L \), a contradiction. So, \( M \cap MV(L) \) is proper in \( MV(L) \). To prove \( M \cap MV(L) \) is maximal in \( MV(L) \), suppose by contrary that there is \( N \in Ds(MV(L)) \) such that \( M \cap MV(L) \subseteq N \subseteq MV(L) \). Then there is \( x_0 \in N \) such that \( x_0 \notin M \cap MV(L) \). Then \( x_0 \notin M \) hence there is
n ≥ 1 such that \((x_0^n)^* \in M \Rightarrow (x_0^n)^* \in M \cap MV(L) \subseteq N \Rightarrow (x_0^n)^* \in N\).

Since \(x_0^n \in N \Rightarrow 0 \in N \Rightarrow N = MV(L), \text{so } M \cap MV(L) \in Max(MV(L)). \]

Let \(L\) be a semi-divisible residuated lattice. For \(D \in Ds(MV(L))\) we denote by \(\overline{D} = \{x \in L : x^{**} \in D\}\) (since \(x \leq x^{**}\) we deduce that \(D \subseteq \overline{D}\).

**Lemma 10.** Let \(L\) be a semi-divisible residuated lattice.

(i) If \(D \in Ds(MV(L))\), then \(\overline{D} \in Ds(L)\);

(ii) If \(M \in Max(MV(L))\), then \(\overline{M} \in Max(L)\).

**Proof.** (i). Since \(1 \in D\) and \(1^{**} = 1 \Rightarrow 1 \in \overline{D}\).

If \(x,y \in L, x \leq y\) and \(x \in \overline{D}\) then \(x^{**} \in D\) and from \(x^{**} \leq y^{**} \Rightarrow y^{**} \in D \Rightarrow y \in \overline{D}\).

If \(x,y \in \overline{D}\), then \(x^{**}, y^{**} \in D \Rightarrow x^{**} \circ y^{**} \in D \subseteq MV(L) \Rightarrow x^{**} \circ y^{**} = (x^{**} \circ y^{**})^{**} = [x^{**} \rightarrow (y^{**} \rightarrow 0)]^{**} = (x^{**} \rightarrow y^{**})^{**} = (x^{**} \rightarrow y)^{**} = (y \rightarrow x^{**})^{**} = (y \rightarrow x)^{**} = (x \rightarrow y)^{**} = [(x \circ y)^{**}] = (x \circ y)^{**} \Rightarrow x \circ y \in \overline{D}\).

(ii). Suppose \(M \in Max(MV(L))\). Then \(M \neq MV(L)\). If \(\overline{M} = L\), then \(0 \in \overline{M} \Rightarrow 0^{**} = 0 \in M \Rightarrow M = MV(L)\), a contradiction. Hence \(\overline{M}\) is proper in \(L\).

To prove \(\overline{M} \in Max(L)\), consider \(N \in Ds(L)\) such that \(\overline{M} \subseteq N \subseteq L\).

Then there is \(x_0 \in N\) such that \(x_0 \notin \overline{M}\); hence \((x_0^n)^* \notin M\). Since \(M \in Max(MV(L))\), there is \(n \geq 1\) such that \([x_0^n]^{**} \in M \subseteq \overline{M} \Rightarrow [x_0^n]^{**} \in \overline{M} \subseteq N \Rightarrow \overline{[x_0^n]^{**}} \in N\). But \(x_0 \notin N \Rightarrow x_0^{**} \notin N \Rightarrow (x_0^{**})^n \notin N \Rightarrow 0 \in N \Rightarrow N = L\).

**Lemma 11.** Let \(L\) be a semi-divisible residuated lattice. Then:

(i) If \(M \in Max(L)\), then \(M = \overline{M} \cap MV(L)\);

(ii) If \(N \in Max(MV(L))\), then \(N = \overline{N} \cap MV(L)\).

**Proof.** (i). If \(M \in Max(L)\), by Lemma 9, \(M \cap MV(L) \in Max(MV(L))\) and from Lemma 10, (ii), we deduce that \(\overline{M} \cap MV(L) \in Max(L)\).

Since for \(x \in M \Rightarrow x^{**} \in M \cap MV(L)\), then \(x \in \overline{M} \cap MV(L) \Rightarrow M \subseteq \overline{M} \cap MV(L) \Rightarrow \overline{M} = M \cap MV(L)\).

(ii). Since \(N \in Max(MV(L))\), by Lemma 10, (ii), \(\overline{N} \in Max(MV(L))\), so by Lemma 9, \(\overline{N} \cap MV(L) \in Max(MV(L))\).

If \(y \in \overline{N} \cap MV(L)\), then \(y = x^{**}\) for some \(x \in \overline{N} \Leftrightarrow x^{**} \in N \Rightarrow \overline{N} \cap MV(L) \subseteq N \Rightarrow \overline{N} \cap MV(L) = N\).
**Theorem 15.** If \( L \) is a semi-divisible residuated lattice, then there is a bijection between \( \text{Max}(L) \) and \( \text{Max}(\text{MV}(L)) \).

**Proof.** Following Lemma 9 and Lemma 10 we deduce that we can define \( f : \text{Max}(L) \to \text{Max}(\text{MV}(L)) \), by \( f(M) = M \cap \text{MV}(L) \), for every \( M \in \text{Max}(L) \) and \( g : \text{Max}(\text{MV}(L)) \to \text{Max}(L) \), by \( g(N) = \overline{N} \), for every \( N \in \text{Max}(\text{MV}(L)) \). By Lemma 11 we deduce that \( f \circ g = 1_{\text{Max}(\text{MV}(L))} \) and \( g \circ f = 1_{\text{Max}(L)} \); hence we deduce that \( f \) is bijective and \( f^{-1} = g \). \( \square \)

**Definition 6 ([23]).** A residuated lattice \( L \) is called semilocal (local) if it contains only a finite number (only one) of maximal deductive systems.

Clearly, if \( L \) is a chain, then \( L \) is local.

Following Theorem 15 we have:

**Theorem 16.** If \( L \) is a semi-divisible residuated lattice, then \( L \) is a semilocal residuated lattice iff \( \text{MV}(L) \) is a semilocal MV-algebra.

6. The radical of a semi-divisible residuated lattice

**Definition 7 ([12]).** The intersection of the maximal deductive systems of a residuated lattice \( L \) is called the radical of \( L \) and will be denoted by \( \text{Rad}(L) \).

For \( n \geq 1 \) and \( x \in L \) we denote \( \tilde{n}x = [(x^*)^n]^* \). Following [11], [12], \( \text{Rad}(L) = \{ x \in L : \text{for every } n \geq 1 \text{ there is } k_n \geq 1 \text{ such that } k_n(x^n) = 1 \} = \{ x \in L : \text{for every } n \geq 1 \text{ there is } k_n \geq 1 \text{ such that } [(x^n)^{k_n}] = 0 \} \).

If \( L \) is a BL-algebra, then \( \text{Rad}(L) = \{ x \in L : (x^n)^* \leq x, \text{ for every } n \geq 1 \} \) (see [23]).

**Proposition 12.** If \( L \) is a chain, then \( \text{Rad}(L) = \{ x \in L : x^n > x^*, \text{ for every } n \geq 1 \} \).

**Proof.** If \( L \) is a chain, then there is an unique maximal deductive system \( M \) of \( L \), so \( \text{Rad}(L) = M \).

If \( x \in M \) then \( x^n \in M \), for every \( n \geq 1 \), hence \( x^n > (x^n)^* \) (because if by contrary \( x^n \leq (x^n)^* \Rightarrow (x^n)^* \in M \Rightarrow 0 \in M \), a contradiction). Since \( x^n < x \Rightarrow (x^n)^* > x^* \), so \( x^n > x^* \).

Conversely, let \( x \in L \) such that \( x^n > x^* \), for every \( n \geq 1 \). Then \( x^n \neq 0 \) for every \( n \geq 1 \) \( \Rightarrow \) \( x \) is proper \( \Rightarrow x \in < x \supseteq M = \text{Rad}(L) \). \( \square \)

Now, we characterize \( \text{Rad}(\text{MV}(L)) \), for a semi-divisible residuated lattice \( L \).
Theorem 17. If $L$ is a semi-divisible residuated lattice, then $\text{Rad}(\text{MV}(L)) \subseteq \text{Rad}(L) \cap \text{MV}(L)$.

Proof. Clearly, $\text{Rad}(\text{MV}(L)) \subseteq \text{MV}(L)$. We prove now that $\text{Rad}(\text{MV}(L)) \subseteq \text{Rad}(L)$. Suppose that there is an element $x^* \in \text{Rad}(\text{MV}(L))$ such that $x^* \notin \text{Rad}(L)$. Then there is $M \in Max(L)$ such that $x^* \notin M$. Thus, there is $n \geq 1$ such that $[(x^*)^n]^* \in M$. Because $[(x^*)^n]^* = (((x^*)^n)^*)^*$ we deduce that $[(x^*)^n]^* \in M \cap \text{MV}(L)$, which is maximal in $\text{MV}(L)$, see Lemma 9. Since $x^* \in \text{Rad}(\text{MV}(L)) \subseteq M \cap \text{MV}(L)$ we obtain that $(x^*)^n \in M \cap \text{MV}(L)$, so $0 = (x^*)^n \circ [(x^*)^n]^* \in M \cap \text{MV}(L)$, a contradiction because $M \cap \text{MV}(L)$ is proper.

So, $\text{Rad}(\text{MV}(L)) \subseteq \text{Rad}(L) \cap \text{MV}(L)$.

Theorem 18. If $L$ is a semi-divisible residuated lattice and $x \in \text{Rad}(L)$, then $x^{**} \in \text{Rad}(\text{MV}(L))$.

Proof. Let $x \in \text{Rad}(L)$ and suppose that $x^{**} \notin \text{Rad}(\text{MV}(L))$. Then there is $M \in Max(\text{MV}(L))$ such that $x^{**} \notin M$. Thus, there is $n \geq 1$ such that $[(x^{**})^n]^* \in M$. Because $[(x^{**})^n]^* = (((x^{**})^n)^*)^*$ we deduce that $[(x^{**})^n]^* \in M$, which is maximal in $L$.

Since $x \in \text{Rad}(L) \subseteq \overline{M}$ we obtain that $(x^{**})^n \in \overline{M}$, so $0 = (x^{**})^n \circ [(x^{**})^n]^* \in \overline{M}$, a contradiction because $\overline{M}$ is proper.

So, if $x \in \text{Rad}(L)$, then $x^{**} \in \text{Rad}(\text{MV}(L))$.

Theorem 19. Let $L$ be a semi-divisible residuated lattice and $x \in L$. Then, $x^* \in \text{Rad}(L) \iff x^* \in \text{Rad}(\text{MV}(L))$.

Proof. Let $x^* \in \text{Rad}(L)$. From Theorem 18, $(x^*)^{**} = x^* \in \text{Rad}(\text{MV}(L))$.

Conversely, let $x^* \in \text{Rad}(\text{MV}(L))$. By Theorem 17, we deduce that $x^* \in \text{Rad}(L)$.

Corollary 15. $\text{Rad}(L) \cap \text{MV}(L) \subseteq \text{Rad}(\text{MV}(L))$.

From Theorem 17 and Corollary 15 we deduce that:

Corollary 16. For a semi-divisible residuated lattice $L$, $\text{Rad}(\text{MV}(L)) = \text{Rad}(L) \cap \text{MV}(L)$. 
REFERENCES


