HYPERSURFACES OF EUCLIDEAN SPACE AS GRADIENT RICCI SOLITONS

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Abstract. In this paper we obtain some necessary and sufficient conditions for a hypersurface of a Euclidean space to be a gradient Ricci soliton. We also study the geometry of a special type of compact Ricci solitons isometrically immersed into a Euclidean space.

Mathematics Subject Classification 2010: 53C20, 53C25, 53B21.
Key words: Ricci soliton, isometric immersion, support function, Euclidean space.

1. Introduction

A Ricci soliton is a Riemannian manifold \((M, g)\) which admits a smooth vector field \(\xi\) on \(M\) that satisfies

\[
Ric + \frac{1}{2} L_\xi g = \lambda g,
\]

where \(Ric\) is the Ricci tensor of the Riemannian manifold \(M\), \(L_\xi\) is the Lie derivative in the direction of \(\xi\) and \(\lambda\) is a constant. A compact Ricci soliton has constant curvature in dimension 2 (HAMILTON [9]), and also in dimension 3 (IVEY [10]). Clearly any Einstein metric gives a trivial Ricci soliton. Nontrivial compact Ricci solitons may exist only if \(\dim M \geq 4\) and they must have nonconstant positive scalar curvature (FERNANDEZ-LOPEZ and GARCIA-RIO [11]). Lately Ricci solitons are being studied quite rigorously (cf. [3],[4],[5],[6], [8], [13], [14]).

*This Work is supported by King Saud University, Deanship of Scientific Research, Research Group Project No. RGP-VPP-182.
A Ricci soliton is said to be a gradient Ricci soliton if the vector field $\xi$ is the gradient of some smooth function $f$ on $M$ (called the potential function), in this case the equation (1.1) takes the form

\begin{equation}
\text{Ric} + \text{Hess}_f = \lambda g,
\end{equation}

where $\text{Hess}_f$ is the Hessian of $f$. Hamilton has conjectured that a compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein, which was proved in [2] and since then obtaining conditions under which a Ricci soliton is an Einstein manifold has been a goal for many researchers, and we should also recall the significant result of Perelman [12] that a Ricci soliton on a compact manifold is a gradient Ricci soliton. In [1], we answered the question “under what conditions a gradient Ricci soliton admits a Riemannian metric which is isometric to a sphere?”.

In [7], we studied Ricci solitons of positive Ricci curvature. In this paper, we are interested in finding conditions for hypersurfaces of Euclidean space to be gradient Ricci solitons. A Ricci soliton $(M, g, \xi, \lambda)$ and a gradient Ricci soliton $(M, g, f, \lambda)$ is shrinking when $\lambda > 0$, steady when $\lambda = 0$ or expanding when $\lambda < 0$. For an $n$-dimensional Riemannian manifold $(M, g)$ immersed in $\mathbb{R}^{n+1}$ with the immersion $\psi : M \to \mathbb{R}^{n+1}$ with unit normal vector field $N$, we are interested in obtaining necessary and sufficient conditions for $M$ to be a Ricci soliton. In particular, we obtain three different characterizations for Ricci soliton as hypersurfaces one with potential field $t = \nabla f$, $f = \frac{1}{2} ||\psi||^2$, and another with potential function $\rho = \langle \psi, N \rangle$ where $\psi = t + \rho N$ and third for a hypersurface in the unit sphere. We also study compact Ricci soliton isometrically immersed into a Euclidean space and obtain conditions under which the hypersurface Ricci soliton has constant mean curvature, in particular we show that a certain compact steady Ricci soliton cannot be isometrically immersed into a Euclidean space and also obtain conditions for the Ricci soliton hypersurface be isometric a round sphere.

2. Preliminaries

Let $(M, g)$ be an $n$-dimensional orientable Riemannian manifold isometrically immersed into a Euclidean space $\mathbb{R}^{n+1}$ with immersion $\psi : M \to \mathbb{R}^{n+1}$. We denote by $N$ and $B$ the unit normal vector field and the shape operator of the hypersurface. Then we have the following fundamental equations of
the hypersurface

\[ \nabla_X Y = \nabla_X Y + g(BX, Y)N, \quad \nabla_X N = -BX, \quad X, Y \in \mathfrak{X}(M), \]

where \( \mathfrak{X}(M) \) is the Lie algebra of smooth vector fields on \( M \). The Ricci tensor \( \text{Ric} \) and the scalar curvature \( S \) of the hypersurface are given by

\[ \text{Ric}(X, Y) = n\alpha g(BX, Y) - g(BX, BY), \quad S = n^2 \alpha^2 - \|B\|^2, \quad X, Y \in \mathfrak{X}(M), \]

where \( n\alpha = \text{Tr}.B \) is the mean curvature function. The Codazzi equation for the hypersurface is

\[ (\nabla B)(X, Y) = (\nabla B)(Y, X), \quad X, Y \in \mathfrak{X}(M), \]

where the covariant derivative \( (\nabla B)(X, Y) = \nabla_X (BY) - B(\nabla_X Y) \).

The immersion \( \psi \) is the position vector field of \( M \) in \( R^{n+1} \) and therefore can be expressed as \( \psi = t + \rho N \), where the tangential component \( t \in \mathfrak{X}(M) \) and \( \rho = g(\psi, N) \) is the support function. We immediately get the following

\[ \nabla_X t = X + \rho BX, \quad \nabla \rho = -Bt, \quad X \in \mathfrak{X}(M), \]

where \( \nabla \rho \) is the gradient of the support function.

3. Conditions for a hypersurface of \( R^{n+1} \) to be a Ricci Soliton

In this section we obtain some conditions for a hypersurface in a Euclidean space to be a gradient Ricci soliton.

**Theorem 3.1.** An \( M \) orientable hypersurface of the Euclidean space \( R^{n+1} \) with immersion \( \psi : M \to R^{n+1}, \psi = t + \rho N \) and shape operator \( B \), is a Ricci soliton \( (M, g, t, \lambda) \) if and only there exists a constant \( \lambda \) such that

\[ B^2 - (\rho + n\alpha) B + (\lambda - 1) I = 0, \]

where \( \alpha \) is the mean curvature of the hypersurface. Moreover this Ricci soliton is a gradient Ricci soliton.

**Proof.** Let \( \psi : M \to R^{n+1} \) be the immersion of the hypersurface \( M \). Then using the equation (2.4), we compute the Lie derivative \( (\mathcal{L}_t g) \) as
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\[(Lg) (X, Y) = 2g (X, Y) + 2 \rho g (B (X), Y), \quad X, Y \in \mathfrak{X} (M), \]
which together with the equation (2.2) gives

\[Ric (X, Y) + \frac{1}{2} (Lg)(X, Y) = g(X, Y) + g((\rho + \alpha) B - B^2)(X, Y). \tag{3.2}\]

Suppose the given condition holds, that is, there exists a constant \(\lambda\) such that the shape operator \(B\) satisfies \(B^2 - (\rho + \alpha) B + (\lambda - 1) I = 0\), then the equation (3.2) gives \(Ric (X, Y) + \frac{1}{2} (Lg)(X, Y) = \lambda g(X, Y),\) that is, the hypersurface \(M\) is a Ricci soliton with potential vector field \(t\).

Conversely, if \((M, g, t, \lambda)\) is the Ricci soliton, then it is easy to show that we get (3.1). In fact, this Ricci soliton is a gradient Ricci soliton. To see this, consider \(f = \frac{1}{2} \|\psi\|^2\), then \(X (f) = g(X, t), \quad X \in \mathfrak{X} (M),\) which gives \(t = \nabla f\), that is, the potential field is the gradient of the smooth function \(f\).

**Theorem 3.2.** An orientable hypersurface \(M\) of the Euclidean space \(\mathbb{R}^{n+1}\) with support function \(\rho\) and shape operator \(B\) is the gradient Ricci soliton \((M, g, \rho, \lambda)\) if and only if there exists a constant \(\lambda\) such that \((\nabla B)(X, t) + (1 - n \alpha) B(X) + (1 + \rho) B^2(X) = -\lambda X.\)

**Proof.** Suppose \(\psi : M \to \mathbb{R}^{n+1}\) is the immersion of the hypersurface \(M\). Then using the equation (2.4), we compute the Hessian \(Hess_\rho\) of the support function \(\rho, \quad Hess_\rho (X, Y) = g(\nabla_X \nabla \rho, Y) = -g((\nabla B)(X, t) + B(X) + \rho B^2(X), Y),\) which together with the equation (2.2) gives

\[Ric (X, Y) + Hess_\rho (X, Y) = -g((\nabla B)(X, t), Y) \tag{3.3}\]
\[+ g((1 - n \alpha) B(X) + (1 + \rho) B^2(X), Y).\]

If there exists a constant \(\lambda\) such that the condition in the hypothesis holds, then we have \(Ric (X, Y) + Hess_\rho (X, Y) = \lambda g(X, Y),\) that is, \(M\) is a gradient Ricci soliton with potential function \(\rho.\) Conversely, if the hypersurface \(M\) is the gradient Ricci soliton \((M, g, \rho, \lambda)\), then using equation (3.3) we get the condition in the hypothesis.

In the rest of this section we consider an orientable hypersurface \(M\) in the unit sphere \(S^{n+1}\) and obtain a condition for this hypersurface to be a gradient Ricci soliton. If \(\psi : M \to S^{n+1}\) is an immersion we denote by \(g\) the induced metric on the hypersurface \(M\) as well as that on the unit sphere \(S^{n+1}\) and by \(N\) and \(B\) the unit normal vector field and the shape
operator of the hypersurface $M$ in the unit sphere $S^{n+1}$. Let $\langle \cdot, \cdot \rangle$ be the Euclidean metric on the Euclidean space $\mathbb{R}^{n+2}$ and $Z$ be a constant vector field on the Euclidean space $\mathbb{R}^{n+2}$. If we denote by $\vec{N}$ the unit normal vector field of the unit sphere $S^{n+1}$ in the Euclidean space $\mathbb{R}^{n+2}$, we have smooth functions $f, \sigma$ defined on the hypersurface $M$ by $f = \langle Z, N \rangle |_M$ and $\sigma = \langle Z, \vec{N} \rangle |_M$. We can express the restriction of the constant vector field $Z$ to the hypersurface $M$ as $Z |_M = u + fN + \sigma \vec{N}$, where $u \in \mathcal{X}(M)$.

We use these smooth function to obtain a necessary and sufficient condition for a hypersurface $M$ of the unit sphere $S^{n+1}$ to be a gradient Ricci soliton. We prove the following:

**Theorem 3.3.** Let $M$ be an orientable hypersurface of $S^{n+1}$, with immersion $\psi : M \to S^{n+1}$ and $Z$ be a constant vector field on the Euclidean space $\mathbb{R}^{n+2}$. The hypersurface $M$ is the Ricci soliton $(M, g, u, \lambda)$ if and only if $B^2 - (n\alpha + f) B + (\rho + 1 + \lambda - n) I = 0$. Moreover, the Ricci soliton is a gradient Ricci soliton with potential function $\sigma$.

**Proof.** If we denote the Riemannian connections on $M$, $S^{n+1}$ and the Euclidean space $\mathbb{R}^{n+2}$, by $\nabla$, $\nabla$ and $D$ respectively and the tangential component of the constant vector field $Z$ to the unit sphere $S^{n+1}$ by $\xi$, then

$$
Z |_{S^{n+1}} = \xi + \sigma \vec{N}
$$

Taking covariant derivative in the equation (3.4), we get $D_X Z = 0 = D_X (\xi + \sigma \vec{N})$, $X \in \mathcal{X}(S^{n+1})$, which gives $\nabla_X \xi - g(X, \xi) \vec{N} + X(\sigma) \vec{N} + \sigma X = 0$. Thus, on equating tangential and normal components on the unit sphere $S^{n+1}$ as hypersurface of the Euclidean space $\mathbb{R}^{n+2}$, $\nabla_X \xi = -\sigma X$, $X(\sigma) = g(X, \xi)$, $X \in \mathcal{X}(S^{n+1})$. Hence $\nabla \sigma = \xi$, where $\nabla \sigma$ is gradient of $\sigma$ on $S^{n+1}$. Now, as $\nabla_X \xi = -\sigma X$ using it in $\xi = u + fN$ we have $\nabla_X \xi = \nabla_X u + X(f) N - fB(X)$, $X \in \mathcal{X}(M)$ that is, $-\sigma X = \nabla_X u + g(Bu, X) N + X(f) N - fB(X)$, which gives on equating tangential and normal components on the hypersurface $M$, $\nabla_X u = -\sigma X + fB(X)$, $\nabla f = -Bu$, where $B$ is the shape operator of the hypersurface $M$. Now

$$
(L_u g)(X, Y) = -2\sigma g(X, Y) + 2f g(B(X), Y), \quad X, Y \in \mathcal{X}(M).
$$

Using the Gauss equation for the hypersurface $M$ in $S^{n+1}$, we have

$$
\text{Ric}(X, Y) = (n - 1) g(X, Y) + n a g(BX, Y) - g(BX, BY).
$$
Combining the above two equations, we get
\[
Ric(X,Y) + \frac{1}{2}(\mathcal{L}_ug)(X,Y) = (n-1-\sigma)g(X,Y)
\]
\[
+ (n\alpha + f)g(B(X),Y) - g(B^2(X),Y).
\]
(3.7)
Suppose there exists a constant \(\lambda\) such that the condition in the hypothesis holds, then we get \(Ric(X,Y) + \frac{1}{2}(\mathcal{L}_ug)(X,Y) = \lambda g(X,Y)\), that is, the hypersurface \(M\) is a Ricci soliton \((M, g, u, \lambda)\). The converse is obvious.

In fact this Ricci soliton is a gradient Ricci soliton. To see this, notice that \(\nabla \sigma = \xi\), and consequently, for \(X \in \mathfrak{X}(M)\), \(X(\sigma) = g(X,\xi) = g(X,u+fN) = g(X,u), X \in \mathfrak{X}(M)\), that is \(u = \nabla \rho\).

4. Compact Ricci solitons

In this section, we study the impact of compactness on the isometric immersion of a Ricci soliton into a Euclidean space. First we have the following nonexistence result for the compact Ricci solitons of Theorem 3.1.

**Theorem 4.1.** Let \((M, g, t, \lambda)\) be a compact steady Ricci soliton. Then there does not exist an isometric immersion \(\psi : M \to \mathbb{R}^{n+1}\) with \(\psi = t + \rho N\).

**Proof.** Suppose there exists an isometric immersion \(\psi : M \to \mathbb{R}^{n+1}\). Then as \(f = \frac{1}{2} \|\psi\|^2\) gives \(t = \nabla f\), that is \((M, g, f, \lambda)\) is the gradient Ricci soliton. Then using defining equation (1.4) for steady Ricci soliton and the equation (2.4), on taking trace, we get
\[
\Delta f = -S,
\]
(4.1)
where \(S\) is the scalar curvature and \(\Delta f\) is the Laplacian of the smooth function \(f\). Using the expression for the scalar curvature in the equation (2.2) and the Schwarz inequality \(\|B\|^2 \geq n^2\alpha\), we conclude that \(S \leq n\alpha^2 (n-1)\), which together with the equation (4.1) gives \(\Delta f \geq n\alpha^2 (n-1)\).

Integrating this equation over the compact manifold \(M\), gives \(0 \geq \int_{M} n\alpha^2 (n-1) dv\), that is, \(\alpha = 0\) and consequently the hypersurface is minimal. This gives a contradiction as there does not exist a compact minimal hypersurface in the Euclidean space \(\mathbb{R}^{n+1}\). This proves the Theorem.

**Theorem 4.2.** If a compact Ricci soliton \((M, g, t, \lambda)\) is isometrically immersed into the Euclidean space \(\mathbb{R}^{n+1}\) with the immersion \(\psi : M \to \mathbb{R}^{n+1}\)
$R^{n+1}, \psi = t + \rho N$ as a hypersurface of constant mean curvature $\alpha$, then $\lambda \leq (n - 1) \alpha^2$.

**Proof.** As this Ricci soliton is gradient Ricci soliton (cf. Theorem 3.1), we have $\text{Ric}(X,Y) = \lambda g(X,Y) - g(A_fX,Y)$, where $\text{Hess}_f(X,Y) = g(A_fX,Y)$. Using the expression for $\text{Ric}$ in the equation (3.2) and the above equation, we arrive at $\lambda g(X,Y) - g(A_fX,Y) = n \alpha g(B(X),Y) - g(B(X),B(Y))$. Taking trace of the above equation, we get $\lambda n - \Delta f = n^2 \alpha^2 - \|B\|^2$ which together with Schwarz inequality implies

$$\lambda n \leq n \alpha^2 (n - 1) + \Delta f. \quad (4.2)$$

Integrating the above equation and noticing that $\alpha$ is a constant we get the desired result.

**Theorem 4.3.** Let $(M,g,t,\lambda)$ be a compact shrinking Ricci soliton isometrically immersed into $R^{n+1}$, with the immersion $\psi : M \to R^{n+1}$, $\psi = t + \rho N$. If the mean curvature $\alpha$ satisfies $\lambda \geq (n - 1) \alpha^2$, then $\alpha$ is a constant and $M$ is isometric to $S^n(c), c = \alpha^2$.

**Proof.** As $M$ is isometrically immersed into $R^{n+1}$, the equation (4.2) gives $0 \leq \int ((n - 1) \alpha^2 - \lambda) \, dg$. If the given condition holds, the above inequality gives $(n - 1) \alpha^2 = \lambda$, that is, $\alpha$ is a constant. As the potential vector field $t = \nabla f, f = \frac{1}{2} \|\psi\|^2$, using above value of $\lambda$ in the equation of gradient Ricci soliton with potential function $f$, we get

$$\Delta f = \|B\|^2 - n \alpha^2, \quad (4.3)$$

where we used the expression for $\text{Ric}$ in the equation (3.2). Note that the Schwarz inequality implies $\|B\|^2 \geq n \alpha^2$ and the equality holds if and only if $B = \alpha I$. Integrating equation (4.3) and using Schwarz inequality we get $B = \alpha I$, that is, $M$ is totally umbilical hypersurface and hence it is isometric to the round sphere $S^n(\alpha^2)$.

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Received: 10.V.2012
Accepted: 3.IX.2012

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