SEMI-SYMMETRY PROPERTIES OF S-MANIFOLDS
ENDOWED WITH A SEMI-SYMMETRIC NON-METRIC
CONNECTION

BY

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Abstract. In this study, S-manifolds endowed with a semi-symmetric non-metric connection naturally related with the S-structure are considered and some general results concerning the curvature of such connection are given. In particular, the conditions of semi-symmetry, Ricci semi-symmetry and projective semi-symmetry of this semi-symmetric non-metric connection are investigated.

Mathematics Subject Classification 2010: 53C05, 53C25.

Key words: S-manifold, semi-symmetric non-metric connection, semi-symmetry properties.

1. Introduction

In 1963, YANO [26] introduced the notion of f-structure on a $C^\infty$ manifold $M$ of dimension $2n + s$, as a non-vanishing tensor field $f$ of type $(1, 1)$ on $M$ which satisfies $f^3 + f = 0$ and has constant rank $r = 2n$. Also, the existence of an f-structure on $M$ is equivalent to a reduction of the structure group to $U(n) \times O(s)$ (see [3]). Almost complex $(s = 0)$ and almost contact $(s = 1)$ are well-known examples of f-structures. The case $s = 2$ appeared in the study of hypersurfaces in almost contact manifolds [5, 11] and it motivated that, in 1970, GOLDBERG and YANO [12] defined

*The third author is partially supported by the PAI group FQM-327 (Junta de Andalucía, Spain, 2011) and by the MEC project 2011-22621 (MEC, Spain, 2011).
globally framed $f$-manifolds (also called $f$-manifolds and $f$-$p_k$-manifolds). A wide class of globally framed $f$-manifolds was introduced in [3] by Blair by defining $K$-manifolds and the particular cases of $S$-manifolds and $C$-manifolds. The $S$-manifolds have been considered by several authors (see, for instance, [4, 6, 13, 16]) and they make a more general framework to study Sasakian manifolds (case $s = 1$).

On the other hand, given a Riemannian manifold $(M, g)$ of dimension $n \geq 3$ endowed with a linear connection $\nabla$ whose curvature tensor field is denoted by $R$, for any $(0, k)$-tensor field $T$ on $M$, $k \geq 1$, the $(0, k+2)$-tensor field $R.T$ is defined by

\begin{equation}
(R.T)(X_1, \ldots, X_k, X, Y) = - \sum_{i=1}^{k} T(X_1, \ldots, X_{i-1}, R(X, Y)X_i, X_{i+1}, \ldots, X_k),
\end{equation}

for any $X, Y, X_1, \ldots, X_k \in \mathcal{X}(M)$. In this context, $M$ is called semi-symmetric with respect to $\nabla$ if $R.R = 0$ and Ricci semi-symmetric if $R.S = 0$, where $S$ is denoting the Ricci tensor field of $\nabla$. Moreover, $M$ is said to be projectively semi-symmetric if $R.P = 0$, being $P$ the Weyl projective curvature tensor field of $\nabla$, defined by

\begin{equation}
P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1} \{S(Y, Z)X - S(X, Z)Y\},
\end{equation}

(alternatively, $P(X, Y, Z, W) = g(P(X, Y)Z, W)$), for $X, Y, Z, W \in \mathcal{X}(M)$. For the Riemannian connection it is known that the semi-symmetry implies the Ricci semi-symmetry (for more details, [8, 21] and references therein can be consulted; specifically, for the contact geometry case we recommend the papers [15, 19, 24]).

Further, in 1924 Friedmann and Schouten [10] introduced the notion of semi-symmetric linear connections. More precisely, if $\nabla$ is a linear connection in a differentiable manifold $M$, the torsion tensor $T$ of $\nabla$ is given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, for any vector fields $X$ and $Y$ on $M$. The connection $\nabla$ is said to be symmetric if the torsion tensor $T$ vanishes, otherwise it is said to be non-symmetric. In this case, $\nabla$ is said to be a semi-symmetric connection if its torsion tensor $T$ is of the form $T(X, Y) = \eta(Y)X - \eta(X)Y$, for any $X, Y$, where $\eta$ is a 1-form on $M$. Moreover, if $g$ is a (pseudo)-Riemannian metric on $M$, $\nabla$ is called a metric connection if $\nabla g = 0$, otherwise it is called non-metric. It is well
known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold. In 1932, Hayden [14] defined a metric connection with torsion on a Riemannian manifold and in 1970, Yano [27] made a systematic study of semi-symmetric metric connections on a Riemannian manifold.

In [1], Agashe and Chafle defined a semi-symmetric non-metric connection on a Riemannian manifold and studied some of its properties. This idea was further developed by De and Kamilya [7], Biswas and De [2], Sengupta, De and Binh [20], Dogru, Ozgür and Murathan [9] and others. Recently, different types of Riemannian manifolds endowed with a semi-symmetric non-metric connection and their submanifolds have been considered (see [17, 18, 22, 23]).

The purpose of this paper is to link the three notions commented above by investigating semi-symmetry properties of $S$-manifolds endowed with certain semi-symmetric non-metric connection naturally related with the $S$-structure. To this end, in Section 2 we give a brief introduction about $S$-manifolds. In Section 3 we define a semi-symmetric non-metric connection on an $S$-manifold, obtaining some general results and, in Section 4, we investigate the curvature and the Ricci tensor fields of such connection. Specially and since we show that the sectional curvature has no sense in this case, we consider the $L$-sectional curvature, giving an example of the constant case. Furthermore, we prove that an $S$-manifold has constant $f$-sectional curvature with respect to this semi-symmetric non-metric connection if and only if it has the same constant $f$-sectional curvature with respect to the Riemannian connection. Consequently, the curvature of the semi-symmetric non-metric connection is completely determined by its $f$-sectional curvature. In Section 5 we study the semi-symmetry properties of the Riemannian connection and we prove that, in this case, a semi-symmetric $S$-manifold has constant $f$-sectional curvature equal to $s$, generalizing the well-known result of Takahashi [24] concerning Sasakian manifolds (case $s = 1$). Moreover, we study the Ricci semi-symmetry and the projective semi-symmetry of the Riemannian connection. Finally, in last section, we present the results concerning the semi-symmetry properties of the semi-symmetric non-metric connection. In particular, we prove that if the $S$-manifold is projectively semi-symmetric with respect to such connection, then it is of constant $f$-sectional curvature equal to $s/2$. 
2. Preliminaries on $S$-manifolds

A $(2n + s)$-dimensional differentiable manifold $M$ is called a metric $f$-manifold if there exist an $(1, 1)$ type tensor field $f$, $s$ vector fields $\xi_1, \ldots, \xi_s$, $s$ 1-forms $\eta^1, \ldots, \eta^s$ and a Riemannian metric $g$ on $M$ such that

\begin{equation}
\tag{2.1}
f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta_{ij}, \quad f \xi_i = 0, \quad \eta^i \circ f = 0,
\end{equation}

\begin{equation}
\tag{2.2}
g(fX, fY) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X) \eta^i(Y),
\end{equation}

for any $X, Y \in \mathcal{X}(M)$, $i, j \in \{1, \ldots, s\}$. In addition we have:

\begin{equation}
\tag{2.3}
\eta^i(X) = g(X, \xi_i), \quad g(X, fY) = -g(fX, Y).
\end{equation}

Then, a 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(M)$, called the fundamental 2-form. In what follows, we denote by $\mathcal{M}$ the distribution spanned by the structure vector fields $\xi_1, \ldots, \xi_s$ and by $\mathcal{L}$ its orthogonal complementary distribution. Then, $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$. If $X \in \mathcal{M}$ we have $fX = 0$ and if $X \in \mathcal{L}$ we have $\eta^i(X) = 0$, for any $i \in \{1, \ldots, s\}$, that is, $f^2 X = -X$.

In a metric $f$-manifold, special local orthonormal basis of vector fields can be considered: let $U$ be a coordinate neighborhood and $E_1$ a unit vector field on $U$ orthogonal to the structure vector fields. Then, from (2.1)-(2.3), $fE_1$ is also a unit vector field on $U$ orthogonal to $E_1$ and the structure vector fields. Next, if it is possible, let $E_2$ be a unit vector field on $U$ orthogonal to $E_1, fE_1$ and the structure vector fields and so on. The local orthonormal basis $\{E_1, \ldots, E_n, fE_1, \ldots, fE_n, \xi_1, \ldots, \xi_s\}$, so obtained is called an $f$-basis.

Moreover, a metric $f$-manifold is normal if $[f, f] + 2 \sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0$, where $[f, f]$ is denoting the Nijenhuis tensor field associated to $f$. A metric $f$-manifold is said to be an $S$-manifold if it is normal, $\eta^1 \wedge \cdots \wedge \eta^s \wedge (d\eta^i)^n \neq 0$ and $\Phi = d\eta^i$, $1 \leq i \leq s$.

Examples of $S$-manifolds can be found in [3, 4, 13].

Now, if $\nabla^g$ denotes the Riemannian connection associated with $g$, then ([3]),

\begin{equation}
\tag{2.4}
(\nabla^g_X f)Y = \sum_{i=1}^{s} \{g(fX, fY) \xi_i + \eta^i(Y) f^2 X\},
\end{equation}
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for all $X, Y \in \mathcal{X}(M)$. From (2.4), it is deduced that

$$\nabla^*_X \xi_i = -fX,$$

for any $X \in \mathcal{X}(M), i \in \{1, \ldots, s\}$. With respect to the curvature tensor field $R^*$ of $\nabla^*$, the following formulas are proved in [6], for all $X, Y \in \mathcal{X}(M), i \in \{1, \ldots, s\}$:

$$R^*(X, Y)\xi_i = \sum_{j=1}^{s} (\eta^j(X)f^2Y - \eta^j(Y)f^2X),$$

(2.6)

$$R^*(X, \xi_i)Y = -\sum_{j=1}^{s} \{g(fX, fY)\xi_j + \eta^j(Y)f^2X\}.$$

(2.7)

Moreover, by using the above formulas, in [6] it is obtained that

$$R^*(\xi_i, X, \xi_j, Y) = -g(fX, fY),$$

(2.8)

$$K^*(\xi_i, X) = g(fX, fX),$$

(2.9)

$$S^*(X, \xi_i) = 2n \sum_{i=1}^{s} \eta^i(X),$$

(2.10)

for all $X, Y \in \mathcal{X}(M), i, j \in \{1, \ldots, s\}$, where $K^*$ and $S^*$ denote, respectively, the sectional curvature and the Ricci tensor field of the Riemannian connection.

In view of (2.9), an $S$-manifold cannot have constant sectional curvature. For this reason, it is necessary to introduce a more restrictive curvature. In general, a plane section $\pi$ on a metric $f$-manifold is said to be an $f$-section if it is determined by a unit vector $X$, normal to the structure vector fields and $fX$. The sectional curvature of $\pi$ is called an $f$-sectional curvature. An $S$-manifold is said to be an $S$-space-form if it has constant $f$-sectional curvature $c$ and then, it is denoted by $M(c)$. In such case, the curvature tensor field $R^*$ of $M(c)$ satisfies [16]:

$$R^*(X, Y, Z, W) = \sum_{i,j=1}^{s} \{g(fX, fW)\eta^i(Y)\eta^j(Z) - g(fX, fZ)\eta^i(Y)\eta^j(W) + g(fY, fZ)\eta^i(X)\eta^j(W) - g(fY, fW)\eta^i(X)\eta^j(Z)\}$$

(2.11)

$$+ \frac{c + 3s}{4} \{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\}$$

$$+ \frac{s}{2} g(fX, fY)(fZ, fW).$$

In the sequel, the sectional curvatures of $\pi$ are calculated with the help of (2.7). In fact, if $\pi$ is an $f$-sectional curvature, then

$$S^*(\pi, X) = g(fX, fX).$$

(2.12)

Moreover, in the case of $S$-space-forms, one has

$$K^*(\pi, X) = g(fX, fX).$$

(2.13)

for all $X \in \mathcal{X}(M)$.
+ \frac{c - s}{4} \{ \Phi(X, W)\Phi(Y, Z) - \Phi(X, Z)\Phi(Y, W) - 2\Phi(X, Y)\Phi(Z, W) \},

for any \( X, Y, Z, W \in \mathcal{X}(M) \).

3. A semi-symmetric non-metric connection on \( S \)-manifolds

From now on, let \( M \) denote a \((2n + s)\)-dimensional manifold \((M, f, \xi_i, \eta^i, g)\). Let us consider the 1-form and the vector field given by:

\[
\begin{align*}
    u &= \sum_{i=1}^{s} \eta^i \quad \text{and} \\
    U &= \sum_{i=1}^{s} \xi_i.
\end{align*}
\]

It is clear that \( u(X) = g(U, X) \), for any \( X \in \mathcal{X}(M) \). Now, we define

\[
(3.1) \quad \nabla_X Y = \nabla_X^* Y + \sum_{j=1}^{s} \eta^j(Y^i)X_i,
\]

for any \( X, Y \in \mathcal{X}(M) \). Then, applying the results presented in [1], we deduce:

**Theorem 3.1.** Let \( M \) be an \( S \)-manifold. The map \( \nabla \) defined in \((3.1)\) is a semi-symmetric non-metric linear connection on \( M \).

**Example 3.2.** Let us consider \( \mathbb{R}^{2n + s} \) with its standard \( S \)-structure given by ([13]):

\[
\begin{align*}
    \eta^\alpha &= \frac{1}{2} \left( dz^\alpha - \sum_{i=1}^{n} y^i dx^i \right), \quad \xi_\alpha = 2 \frac{\partial}{\partial z^\alpha}, \\
    g &= \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4} \left( \sum_{i=1}^{n} \left( dx^i \otimes dx^i + dy^i \otimes dy^i \right) \right), \\
    f &= \sum_{i=1}^{n} \left( X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial y^i} \right) + \sum_{\alpha=1}^{s} \frac{Z^\alpha}{\partial z^\alpha} \\
    &= \sum_{i=1}^{n} \left( Y^i \frac{\partial}{\partial x^i} - X^i \frac{\partial}{\partial y^i} \right) + \sum_{\alpha=1}^{s} \frac{Y_i y^i}{\partial z^\alpha},
\end{align*}
\]

where \((x^i, y^i, z^\alpha), i = 1, \ldots, n \) and \( \alpha = 1, \ldots, s \), are the cartesian coordinates. It is known that, with this structure, \( \mathbb{R}^{2n + s} \) is an \( S \)-space-form.
of constant $f$-sectional curvature $c = -3s$. If, following [13], we denote $(x^1, \ldots, x^n, y^1, \ldots, y^n, z^1, \ldots, z^s) = (x^1, \ldots, x^{2n+s})$, the Christoffel symbols of the semi-symmetric non-metric connection defined in (3.1) are given by

$$\Gamma_{ai}^b = \Gamma_{ai}^{sb} - \frac{1}{2} s y_i \delta_{ab}, \quad \Gamma_{aa}^b = \Gamma_{aa}^{sb} + \frac{1}{2} \delta_{ab},$$

for any $a, b \in \{1, \ldots, 2n+s\}$, $i \in \{1, \ldots, n\}$ and $a \in \{2n+1, \ldots, 2n+s\}$, where $\Gamma^{sb}$ and $\Gamma_{aa}^{sb}$ are denoting the Christoffel symbols of the Riemannian connection of $\mathbb{R}^{2n+s}$ and the not-written symbols are the same as the Riemannian connection ones (see [13] for the details concerning them).

Through this paper, we always use the letter $\nabla$ to denote the semi-symmetric non-metric connection defined in (3.1). Now, we can prove:

**Proposition 3.3.** Let $M$ be an $S$-manifold. Then,

$$\nabla_X \xi_i = -f X + X,$$

(3.2)

$$(\nabla_X \eta^i)Y = g(X, fY) - \sum_{j=1}^{s} \eta^j(Y) \eta^i(X),$$

(3.3)

for any $X \in \mathcal{X}(M)$, $i \in \{1, \ldots, s\}$.

**Proof.** First, taking $Y = \xi_i$ in (3.1), from (2.5) we deduce:

$$\nabla_X \xi_i = \nabla_X^* \xi_i + \sum_{j=1}^{s} \eta^j(\xi_i) X = -f X + X.$$  

Now, by using (2.3), (2.5) and (3.1) again:

$$(\nabla_X \eta^i)(Y) = X \eta^i(Y) - \eta^i(\nabla_X Y) = X g(Y, \xi_i) - \eta^i(\nabla_X Y)$$

$$= g(\nabla_X Y, \xi_i) + g(Y, \nabla_X^* \xi_i) - \eta^i(\nabla_X Y)$$

$$= -g(Y, f X) - \sum_{j=1}^{s} \eta^j(Y) \eta^i(X).$$

\[\square\]

**Theorem 3.4.** Let $M$ be an $S$-manifold. Then,

$$\nabla_X f Y = \sum_{i=1}^{s} \{g(f X, fY) \xi_i + \eta^i(Y)(f^2 X - f X)\},$$

(3.4)

for all $X, Y \in \mathcal{X}(M)$. 

Proof. From (3.1), we get $(\nabla_X f)Y = (\nabla_X^* f)Y - \sum_{i=1}^{s} \eta^i(Y) fX$. Therefore, from (2.4) the proof is complete. □

By using (2.1) and (3.4), we easily prove:

**Corollary 3.5.** Let $M$ be an $S$-manifold. Then we have,

(3.5) $$(\nabla_X f)\xi_i = -f \nabla_X \xi_i = f^2 X - fX,$$

(3.6) $$\nabla_{\xi_i} (fX) = f \nabla_{\xi_i} X,$$

for all $X \in \mathcal{X}(M)$, $i \in \{1, \ldots, s\}$.

4. The curvature of $\nabla$

Let $M$ be an $S$-manifold endowed with the semi-symmetric non-metric connection $\nabla$ defined in (3.1). From Formula (3.2) in [1], denoting by $R$ and $R^*$ the curvature tensor fields of $\nabla$ and $\nabla^*$, respectively, we obtain that

$$R(X, Y)Z = R^*(X, Y)Z + s (g(X, fZ)Y - g(Y, fZ)X)$$

$$+ \sum_{i,j=1}^{s} \{\eta^i(Y)\eta^j(Z)X - \eta^j(X)\eta^i(Z)Y\},$$

for all $X, Y, Z \in \mathcal{X}(M)$. Thus, from (2.6), (2.7) and (4.1), we get:

**Proposition 4.1.** Let $M$ be an $S$-manifold. Then,

$$R(X, \xi_i)Y = sg(X, fY)\xi_i + 2 \sum_{j=1}^{s} \eta^j(Y)X$$

$$- \sum_{j=1}^{s} g(fX, fY)\xi_j - \sum_{j,k=1}^{s} \eta^j(X)\eta^k(Y)(\xi_i + \xi_j);$$

$$R(X, \xi_j)\xi_i = 2X - \sum_{k=1}^{s} \eta^k(X)(\xi_k + \xi_j);$$

$$R(\xi_i, \xi_j)X = \sum_{k=1}^{s} \eta^k(X)(\xi_i - \xi_j);$$

$$R(\xi_k, \xi_j)\xi_i = \xi_k - \xi_j,$$

for all $X, Y \in \mathcal{X}(M)$, $i, j, k \in \{1, \ldots, s\}$. 
Consequently, to consider the sectional curvature of the semi-symmetric non-metric connection $\nabla$ has no sense because, from (4.2) we have that

$$R(\xi_i, X, X, \xi_i) = g(R(\xi_i, X)X, \xi_i) = 1,$$

while from (4.3), $R(X, \xi_i, \xi_i, X) = g(R(X, \xi_i)\xi_i, X) = 2$, for any unit vector field $X \in \mathcal{L}$ and any $i \in \{1, \ldots, s\}$. However, if we consider only pairs of orthonormal vector fields $X, Y \in \mathcal{L}$, from (4.1) we deduce that:

(4.6) $$R(X, Y, Y, X) = R(Y, X, X, Y) = K^*(X, Y).$$

Therefore, we can consider the so-called $\mathcal{L}$-sectional curvature with respect to $\nabla$, denoted by $K_\mathcal{L}$ and defined by $K_\mathcal{L}(X, Y) = R(X, Y, Y, X)$, for any orthonormal vector fields $X, Y \in \mathcal{L}$. From (4.6), it satisfies that $K_\mathcal{L} = K^*_\mathcal{L}$, where $K^*_\mathcal{L}$ is the $\mathcal{L}$-sectional curvature with respect to the Riemannian connection $\nabla^*$, obtained in a similar way. Then, we can prove the following theorem.

**Theorem 4.2.** Let $M$ be a $(2n+s)$-dimensional $S$-manifold with $n \geq 2$. If the $\mathcal{L}$-sectional curvature $K_\mathcal{L}$ with respect to the semi-symmetric non-metric connection $\nabla$ is constant equal to $c$, then $c = s$.

**Proof.** It is clear that if $K_\mathcal{L}$ is constant equal to $c$, since $K_\mathcal{L} = K^*_\mathcal{L}$, then $M$ is an $S$-space-form $M(c)$. Consequently, from (2.11), we have

(4.7) $$K_\mathcal{L}(X, Y) = K^*_\mathcal{L}(X, Y) = \frac{c + 3s}{4} + \frac{3(c - s)}{4} g(X, fY)^2,$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$. Now, since $n \geq 2$, we can choose $X$ and $Y$ such that $g(X, fY) = 0$. Thus, from (4.7) we deduce

$$\frac{c + 3s}{4} = c,$$

that is, $c = s$. $\square$

**Example 4.3.** Let us consider $\mathbb{R}^{2n+2+(s-1)}$ with coordinates

$$(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}, z_1, \ldots, z_{s-1})$$

and with its standard $S$-structure of constant $f$-sectional curvature $-3(s-1)$ (see either [13] or Example 3.2 for the definitions of the structure elements).
Let $S^{2n+1}(2)$ be a $(2n + 1)$-dimensional ordinary sphere of radius 2 and $M = S^{2n+1}(2) \times \mathbb{R}^{s-1}$ a hypersurface of $\mathbb{R}^{2n+2+s-1}$. Put

$$\xi_s = \sum_{i=1}^{n+1} \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right) - \sum_{i=1}^{n+1} s^{-1} \sum_{i=1}^{s} y_i \frac{\partial}{\partial z_{\alpha i}}$$

and $\eta^s(X) = g(X, \xi_s)$, for any vector field $X$ tangent to $M$. Then, if we take

$$\tilde{\xi}_\alpha = s \xi_\alpha; \quad \tilde{\eta}^\alpha = \frac{1}{s} \eta^\alpha; \quad \alpha = 1, \ldots, s;$$

$$\tilde{f} = f; \quad \tilde{g} = \frac{1}{s} g + \frac{1-s}{s^2} \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha,$$

it is known ([13]) that $(M, \tilde{f}, \tilde{\xi}_1, \ldots, \tilde{\xi}_s, \tilde{\eta}^1, \ldots, \tilde{\eta}^s, \tilde{g})$ is an $S$-space-form of constant $f$-sectional curvature $c = s$. Moreover, from (2.11), it is easy to show that the $L$-sectional curvature $K^L_s$ of $M$ is also constant and equal to $s$. Consequently, since $K^L_s = K^L_s$, we deduce that $M$ is of constant $L$-sectional curvature $c = s$ with respect to the semi-symmetric non-metric connection $\nabla$.

Next, what about the $f$-sectional curvature of the semi-symmetric non-metric connection $\nabla$ defined in (3.1)? First, we have:

**Proposition 4.4.** Let $M$ be an $S$-manifold. Then,

$$R(fX, fY, fZ, fW) = R(X, Y, Z, W),$$

for any $X, Y, Z, W \in \mathcal{L}$.

**Proof.** It is a direct computation from (4.1) taking into account that

$$R^s(fX, fY, fZ, fW) = R^s(X, Y, Z, W), \quad \text{for any } X, Y, Z, W \in \mathcal{L} \text{ (see [3])}. \quad \square$$

Therefore, the $f$-sectional curvature of $\nabla$ is well defined, since, by using (4.1), we obtain that, for any unit vector field $X \in \mathcal{L}$:

$$R(X, fX, fX, X) = R^s(X, fX, fX, X).$$

Observe that, if $\dim M = 2 + s$, that is, if $n = 1$, $K^L_s$ is actually the $f$-sectional curvature.

Next, taking into account (2.11), from (4.1) and (4.9) we prove the following theorem.
Theorem 4.5. Let $M$ be an $S$-manifold. Then, the $f$-sectional curvature associated with the semi-symmetric non-metric connection $\nabla$ is constant if and only if the $f$-sectional curvature associated with the Riemannian connection is constant too. In this case, both constants are the same and the curvature of $\nabla$ is given by

$$R(X, Y, Z, W) = \sum_{i,j=1}^{s} \{2g(X, W)\eta^i(Y)\eta^j(Z) - 2g(Y, W)\eta^i(X)\eta^j(Z)$$
$$+ g(fY, fZ)\eta^i(X)\eta^j(W) - g(fX, fZ)\eta^i(Y)\eta^j(W)\}$$
$$+ \sum_{i,j,k=1}^{s} \{\eta^i(X)\eta^k(Y)\eta^j(Z)\eta^k(W) - \eta^k(X)\eta^i(Y)\eta^j(Z)\eta^k(W)\}$$
$$+ \frac{c+3s}{4} \{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\}$$
$$+ \frac{c-s}{4} \{\Phi(X, W)\Phi(Y, Z) - \Phi(X, Z)\Phi(Y, W)$$
$$- 2\Phi(X, Y)\Phi(Z, W)\}$$
$$+ s\{g(X, fZ)g(Y, W) - g(X, W)g(Y, fZ)\},$$

for any $X, Y, Z, W \in \mathcal{X}(M)$, where $c$ denotes the constant $f$-sectional curvature.

Hence, we observe that the constant $f$-sectional curvature determines completely the curvature tensor field of $\nabla$.

With respect to the Ricci tensor field $S$ of the connection $\nabla$ we know that it is not a symmetric tensor field. In fact, since $d\eta^i = \Phi$, for any $i \in \{1, \ldots, s\}$, from Formulas (3.4) and (3.14) in [1] we deduce that

$$S(X, Y) - S(Y, X) = -2(2n + s - 1)sg(X, fY),$$

for any $X, Y \in \mathcal{X}(M)$, where $\dim(M) = 2n + s$. Moreover,

$$S(X, Y) = S^*(X, Y) - (2n + s - 1)\{sg(X, fY) - \sum_{i,j=1}^{s} \eta^i(X)\eta^j(Y)\},$$

for any $X, Y \in \mathcal{X}(M)$. Therefore, by using (2.10):

**Proposition 4.6.** Let $M$ be an $S$-manifold. Then, we have

$$S(X, \xi_i) = S(\xi_i, X) = (4n + s - 1)\sum_{j=1}^{s} \eta^j(X),$$
for any $X \in \mathcal{X}(M)$, $i \in \{1, \ldots, s\}$.

**Corollary 4.7.** Let $M$ be an $S$-manifold. Then, we have

\[
S(\xi_j, \xi_i) = 4n + s - 1,
\]

for any $i, j \in \{1, \ldots, s\}$.

Moreover, we can prove:

**Proposition 4.8.** Let $M$ be an $S$-manifold. Then,

\[
S(fX, fY) = S(X, Y)
\]

for any $X, Y \in \mathcal{L}$

**Proof.** Let $\{E_1, \ldots, E_n, fE_1, \ldots, fE_n, \xi_1, \ldots, \xi_s\}$ be an $f$-basis. Then, since from (4.2),

\[
R(\xi_j, fX, fY, \xi_j) = g(X, Y) - sg(X, fY),
\]

for any $j \in \{1, \ldots, s\}$, by using (4.8) and taking into account that $X, Y \in \mathcal{L}$, we deduce:

\[
S(fX, fY) = \sum_{i=1}^{n} \{ R(E_i, fX, fY, E_i) + R(fE_i, fX, fY, fE_i) \}
\]

\[
+ \sum_{j=1}^{s} R(\xi_j, fX fY, \xi_j)
\]

\[
= \sum_{i=1}^{n} \{ R(fE_i, f^2X, f^2Y, fE_i) + R(E_i, X, Y, E_i) \}
\]

\[
+ sg(X, Y) - s^2 g(X, fY)
\]

\[
=S(X, Y) - \sum_{j=1}^{s} R(\xi_j, X, Y, \xi_j) + sg(X, Y) - s^2 g(X, fY).
\]

But, from (4.2) again, $R(\xi_j, X, Y, \xi_j) = g(X, Y) - sg(X, fY)$, for any $j \in \{1, \ldots, s\}$ and this completes the proof.

**Corollary 4.9.** Let $M$ be an $S$-manifold. Then we have,

\[
S(X, Y) = S(fX, fY) + (4n + s - 1) \sum_{i=1}^{s} \eta^i(X)\eta^i(Y),
\]

for all $X, Y \in \mathcal{X}(M)$.
**Proof.** We can put

\[ X = X_0 + \sum_{i=1}^{s} \eta^i(X)\xi_i \quad \text{and} \quad Y = Y_0 + \sum_{j=1}^{s} \eta^j(Y)\xi_j, \]

where \(X_0, Y_0 \in \mathcal{L}\). Then, from (4.12) and (4.13):

\[ S(X, Y) = S(X_0, Y_0) + (4n + s - 1) \sum_{i,j=1}^{s} \eta^i(X)\eta^j(Y). \]

Now, by using (4.14), \(S(X_0, Y_0) = S(fX_0, fY_0) = S(fX, fY)\) and the proof is complete. \(\square\)

5. Semi-symmetry properties of \(S\)-manifolds with respect to the Riemannian connection

Let \((M, f, \xi_i, \eta^i, g)\) be an \(S\)-manifold. With respect to the Riemannian connection \(\nabla^*\) of this \(S\)-manifold we can prove:

**Theorem 5.1.** Any semi-symmetric \(S\)-manifold \((M, f, \xi_i, \eta^i, g)\) is an \(S\)-space-form of constant \(f\)-sectional curvature equal to \(s\).

**Proof.** Let \(X \in \mathcal{L}\) be a unit vector field. Since the \(S\)-manifold \((M, f, \xi_i, \eta^i, g)\) is semi-symmetric, then

\[ (R^* R^*) (X, \xi_i, X, fX, fX, \xi_j) = 0, \]

for any \(i, j \in \{1, \ldots, s\}\). Expanding this formula from (1.1) and taking into account (2.7), we get \(R^* (X, fX, fX, X) = s\), which completes the proof. \(\square\)

Observe that, in the case \(s = 1\), by using (2.8) we obtain that a semi-symmetric Sasakian manifold is of constant curvature equal to 1. This result was firstly proved by Takahashi (see [24]).

**Theorem 5.2.** Let \((M, f, \xi_i, \eta^i, g)\) be a Ricci semi-symmetric \(S\)-manifold. The Ricci tensor field \(S^*\) with respect to the Riemannian connection satisfies

\[ S^*(X, Y) = 2n \{sg(fX, fY) + \sum_{i,j=1}^{s} \eta^i(X)\eta^j(Y)\}, \]

for any \(X, Y \in \mathcal{X}(M)\).
Proof. Since \((M, f, \xi, \eta, g)\) is Ricci semi-symmetric, then, by using (1.1),
\[ S^s(R^s(X, \xi)X, Y) + S^s(\xi, g^s(X, \xi)Y) = 0, \]
for any \(X, Y \in \mathcal{X}(M)\) and \(i, j \in \{1, \ldots, s\}\). Now, from (2.7) and (2.10) we get the desired result. \(\square\)

Corollary 5.3. Any Ricci semi-symmetric Sasakian manifold is an Einstein manifold.

Proof. Considering \(s = 1\) in (5.1), we deduce
\[ S^s(X, Y) = 2n(g(fX, fY) + \eta(X)\eta(Y)) = 2ng(X, Y), \]
for any \(X, Y \in \mathcal{X}(M)\). \(\square\)

Observe that this result was firstly proved by Tanno in [25] for the more general \(K\)-contact manifolds.

For the Weyl projective curvature tensor field, we have the following theorem:

Theorem 5.4. Any projectively semi-symmetric \(S\)-manifold \((M, f, \xi, \eta, g)\) is an \(S\)-space-form of constant \(f\)-sectional curvature equal to \(s\).

Proof. Let \(X \in \mathcal{L}\) a unit vector field. Then, from (1.2) and taking into account (2.7) and (2.8), we have
\[ (R^s \cdot P^s)(X, \xi, X, fX, fX, \xi_j) = (R^s \cdot R^s)(X, \xi, X, fX, fX, \xi_j) = s - R^s(X, fX, fX, X), \]
for any \(i, j = 1, \ldots, s\) and this completes the proof. \(\square\)

6. Semi-symmetry properties of an \(S\)-manifold with respect to the semi-symmetric non-metric connection \(\nabla\)

Let \(M\) be an \(S\)-manifold and \(\nabla\) the semi-symmetric non-metric connection defined in (3.1). Then, if \(R \cdot R = 0\), from (1.1) we deduce that
\[ R(R(X, \xi)X, fX, fX, \xi_j) + R(X, R(X, \xi)X, fX, fX, \xi_j) \]
\[ + R(X, fX, R(X, \xi)X, fX, \xi_j) + R(X, fX, fX, R(X, \xi)X, \xi_j) = 0, \]
for any unit vector field \(X \in \mathcal{L}\) and any \(i, j = 1, \ldots, s\). By using (4.2) and (4.3), a direct expansion of (6.1) gives:
\[ 2R(X, fX, fX, X) = s - s^2 \delta_{ij}, \quad i, j = 1, \ldots, s. \]
Consequently, if \( s \geq 2 \), \( M \) can not be a semi-symmetric manifold with respect to the semi-symmetric non-metric connection \( \nabla \). However, for \( s = 1 \), that is, for Sasakian manifolds, we have:

**Theorem 6.1.** Let \( M \) be a \((2n+1)\)-dimensional Sasakian manifold. If \( M \) is semi-symmetric with respect to the semi-symmetric non-metric connection \( \nabla \), then the \( f \)-sectional curvature of \( \nabla \) is constant and equal to 0.

Consequently, from Theorem 4.5, \( M \) is a Sasakian space-form of constant \( f \)-sectional curvature \( c = 0 \). This result should be compared with that one of [24] concerning the Riemannian connection.

Concerning the Ricci semi-symmetry, we can prove:

**Theorem 6.2.** Let \( M \) be a \((2n+1)\)-dimensional Ricci semi-symmetric Sasakian manifold with respect to the semi-symmetric non-metric connection \( \nabla \). Then, the Ricci tensor field of \( \nabla \) satisfies

\[
S(X, Y) = 2n\{g(X, Y) - g(X, fY) + \eta(X)\eta(Y)\},
\]

for any \( X, Y \in \mathcal{X}(M) \).

**Proof.** By virtue of (2.1) and (4.16), it is sufficient to prove that \( S(X, Y) = 2n\{g(X, Y) - g(X, fY)\} \), for any \( X, Y \in \mathcal{L} \).

Then, let \( X, Y \in \mathcal{L} \). So, since \( R.S = 0 \), from (1.1) we have that:

(6.2) \[ S(R(X, \xi)\xi, Y) + S(\xi, R(X, \xi)Y) = 0. \]

But, by using (4.3) we get \( S(R(X, \xi)\xi, Y) = 2S(X, Y) \) and by using (4.2) and (4.13) we obtain \( S(\xi, R(X, \xi)Y) = 4n\{g(X, fY) - g(X, Y)\} \). Replacing these results into (6.2), we complete the proof. \( \square \)

Next, suppose that \( s \geq 2 \) and \( R.S = 0 \). Then, from (1.1) we have that

(6.3) \[ S(R(\xi, X)\xi_j, fX) + S(\xi_j, R(\xi, X)fX) = 0, \]

for any unit vector field \( X \in \mathcal{L} \) and any \( i, j \in \{1, \ldots, s\} \). Now, by using (4.3) and (4.12):

(6.4) \[ S(R(\xi, X)\xi_j, fX) = -2S(X, fX). \]

Next, from (4.2) and (4.12):

(6.5) \[ S(\xi_j, R(\xi, X)fX) = (4n + s - 1)s. \]
Consequently, replacing (6.4) and (6.5) into (6.3), we get

\[
2S(X,fX) = (4n + s - 1)s,
\]
that is, \(S(X,fX) = S(fX,X) = (4n + s - 1)s\). But, from (4.10) we obtain that \(S(X,fX) - S(fX,X) = 2(2n + s - 1)s\), which is a contradiction since \(s \geq 2\). Therefore, \(M\) can not be Ricci semi-symmetric with respect to \(\nabla\) if \(s \geq 2\).

Due to the above results, it is natural to consider the Weyl projective curvature tensor field of \(\nabla\) (see (1.2)). For this tensor field we obtain the following theorem.

**Theorem 6.3.** Let \(M\) be a \((2n+s)\)-dimensional \(S\)-manifold \(M\). If \(M\) is projectively semi-symmetric with respect to the semi-symmetric non-metric connection \(\nabla\), then the Ricci tensor field \(S\) of \(M\) with respect to \(\nabla\) satisfies

\[
S(X,Y) = -(2n + s - 1)sg(X,fY) + (4n + s - 1) \sum_{i,j=1}^{s} \eta^i(X)\eta^j(Y),
\]
for any \(X,Y \in \mathcal{X}(M)\). Moreover, \(M\) is an \(S\)-space-form of constant \(f\)-sectional curvature equal to \(s/2\).

**Proof.** From (1.1) we have that

\[
(R.P)(X,Y,X_1,X_2,X_3,X_4) = (R.R)(X,Y,X_1,X_2,X_3,X_4)
\]
\[
+ \frac{1}{2n + s - 1} \{ S(X_2,X_3)g(R(X,Y)X_1,X_4) 
\]
\[
- S(R(X,Y)X_1,X_3)g(X_2,X_4) 
\]
\[
+ S(R(X,Y)X_2,X_3)g(X_1,X_4) - S(X_1,X_3)g(R(X,Y)X_2,X_4) 
\]
\[
+ S(X_2,R(X,Y)X_3)g(X_1,X_4) - S(X_1,R(X,Y)X_3)g(X_2,X_4) 
\]
\[
+ S(X_2,X_3)g(X_1,R(X,Y)X_4) - S(X_1,X_3)g(X_2,R(X,Y)X_4) \}.
\]

Considering \(X = \xi_i, Y = \xi_j, X_1 = X_4 = \xi_i\) and \(X_2, X_3 \in \mathcal{L}\), if \(R.P = 0\), from (6.8) and taking into account (4.4) and (4.5), a straightforward computation gives

\[
\{ sg(X_2,fX_3) + \frac{1}{2n + s - 1} S(X_2,X_3) \} (1 - \delta_{ij}) = 0,
\]
that is,

\[
S(X_2,X_3) = -(2n + s - 1)sg(X_2,fX_3),
\]
for any $X_2, X_3 \in \mathcal{L}$. Now, by using (4.16) and (6.9) we get (6.7).

Next, given a unit vector field $X \in \mathcal{L}$, since

$$(R.P)(X, \xi_{i1}, X, fX, fX, \xi_{ij}) = 0,$$

by expanding this formula we obtain (6.10)

$$s - s^2 \delta_{ij} - 2R(X, fX, fX, X) + \frac{1}{2n + s - 1} \{S(fX, fX) + S(X, fX)s\delta_{ij}\} = 0,$$

where we have used (4.2) and (4.3). From (6.9), we have that $S(fX, fX) = 0$ and $S(X, fX) = (2n + s - 1)s$. Then, we deduce that (6.10) is

$$2R(X, fX, fX, X) = s$$

in both cases $i = j$ and $i \neq j$ and the proof is complete. \hfill \Box

Finally, we say that a Riemannian manifold $(M, g)$ is Ricci projectively semi-symmetric with respect to a linear connection $\nabla$ if $P:S = 0$, where


(6.11)

$$= (R.S)(X, Y, Z, W) + \frac{1}{2n + s - 1} \{S(X, W)(S(Y, Z) - S(Z, Y)) + S(Y, W)(S(Z, X) - S(X, Z))\},$$

for any $X, Y, Z, W \in \mathcal{X}(M)$. Note that, since the Ricci tensor field is symmetric with respect to the Riemannian connection of $g$, for this connection $M$ is Ricci-projectively semi-symmetric if and only if it is Ricci semi-symmetric.

Now, let $M$ be a $(2n + s)$-dimensional $S$-manifold and $\nabla$ the semi-symmetric non-metric connection given by (3.1). Then, from (4.10) we deduce that (6.11) can be written as


(6.12)

$$+ 2s \{S(Y, W)g(X, fZ) - S(X, W)g(Y, fZ)\},$$

for any $X, Y, Z, W \in \mathcal{X}(M)$. Thus, we can prove:

**Theorem 6.4.** Let $M$ be a $(2n + s)$-dimensional $S$-manifold. Then,

(i) If $s \geq 2$, $M$ can not be Ricci-projectively semi-symmetric with respect to $\nabla$. 
(ii) If $M$ is a Ricci-projectively semi-symmetric Sasakian manifold (that is, if $s = 1$) with respect to $\nabla$, the Ricci tensor field of $\nabla$ satisfies

$$S(X, Y) = 2n\{g(X, Y) - g(X, fY) + \eta(X)\eta(Y)\},$$

for any $X, Y \in \mathcal{X}(M)$.

**Proof.** By using (2.1) and (6.12), we get

$$(P.S)(\xi_i, X, \xi_j, Y) = (R.S)(\xi_i, X, \xi_j, Y),$$

for any $X, Y \in \mathcal{L}$ and $i, j \in \{1, \ldots, s\}$. Consequently, we complete the proof by using the same line of reasoning as above. $\square$

**Acknowledgements.** The authors wish to express their gratitude to the referee of this paper for his/her valuable comments in order to improve it.

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