Abstract. The notion of $\Gamma$-symmetric space is a natural generalization of the classical notion of symmetric space based on $\mathbb{Z}_2$-grading on Lie algebras. We consider homogeneous spaces $G/H$ such that the Lie algebra $\mathfrak{g}$ of $G$ admits a $\Gamma$-grading where $\Gamma$ is a finite abelian group. In this work we study Riemannian metrics and Lorentzian metrics on the Heisenberg group $\mathbb{H}_3$ adapted to the symmetries of a $\Gamma$-symmetric structure on $\mathbb{H}_3$. We prove that the classification of Riemannian and Lorentzian $\mathbb{Z}_2^2$-symmetric metrics on $\mathbb{H}_3$ corresponds to the classification of its left-invariant Riemannian and Lorentzian metrics, up to isometry. We study also the $\mathbb{Z}_k^2$-symmetric structures on $G/H$ when $G$ is the $(2p + 1)$-dimensional Heisenberg group. This gives examples of non-Riemannian symmetric spaces. When $k \geq 1$, we show that there exists a family of flat and torsion free affine connections adapted to the $\mathbb{Z}_k^2$-symmetric structures.

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1. Introduction

A symmetric space can be considered as a reductive homogeneous space $G/H$ on which acts an abelian subgroup $\Gamma$ of the automorphisms group of $G$ with $\Gamma$ isomorphic to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $H$ the subgroup of $G$ composed of the fixed points of the automorphisms belonging to $\Gamma$. If we suppose that

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the Lie groups \( G \) and \( H \) are connected and that \( G \) is simply connected, it is equivalent to provide \( G/H \) with a symmetric structure or to provide the Lie algebra \( \mathfrak{g} \) of \( G \) with a \( \mathbb{Z}_2 \)-gradation \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) with \( [\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j \pmod{2}} \). Riemannian symmetric spaces form an interesting class of symmetric spaces. But there are symmetric spaces which are not Riemannian symmetric. We describe examples when \( G \) is the Heisenberg group. Nevertheless, a symmetric space is always provided with an affine connection \( \nabla \) which is torsion free and has a curvature tensor satisfying \( \nabla R = 0 \). When the symmetric space is Riemannian, this connection is the Levi-Civita connection of the metric. A natural generalization of the notion of symmetric space can be obtained by considering that the subgroup \( \Gamma \) is abelian, finite and not necessarily isomorphic to \( \mathbb{Z}_2 \). When \( \Gamma \) is cyclic isomorphic to \( \mathbb{Z}_k \) it corresponds to the generalized symmetric spaces of [2, 12, 14]. These structures are also characterized by \( \mathbb{Z}_k \)-graduations of the complexified Lie algebra \( \mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C} \) of \( \mathfrak{g} \). We get another interesting case when \( \Gamma = \mathbb{Z}_2^k \) because the characteristic graduation is defined on \( \mathfrak{g} \) and not on \( \mathfrak{g}_\mathbb{C} \). When \( \mathfrak{g} \) is simple the \( \mathbb{Z}_2^2 \)-graduations of \( \mathfrak{g} \) have been classified as well as the \( \mathbb{Z}_2^k \)-symmetric spaces \( G/H \) when \( G \) is simple connected (see [1, 11]). All these spaces are Riemannian (see [15]). But, in this paper, we provide some examples of non Riemannian symmetric spaces studying symmetric spaces \( G/H \) when \( G \) is the Heisenberg group \( \mathbb{H}_{2p+1} \). We study also, for \( k > 1 \), \( \mathbb{Z}_2^k \)-symmetric structures on these homogeneous spaces showing, in particular, that these spaces are Riemannian and affine. But contrary to the symmetric case, there exist on these spaces affine connections different from the canonical (or the Levi-Civita) connection and more adapted to the symmetries of \( G/H \) than the canonical one. We describe these connections and we prove that there exists connections adapted to the \( \mathbb{Z}_2^k \) symmetries which are flat and torsion free.

2. \( \mathbb{Z}_2^k \)-symmetric spaces

2.1. Recall on symmetric and Riemannian symmetric spaces

A symmetric space is a triple \((G, H, \sigma)\) where \( G \) is a connected Lie group, \( H \) a closed subgroup of \( G \) and \( \sigma \) an involutive automorphism of \( G \) such that \( G^\sigma_0 \subset H \subset G^\sigma \) where \( G^\sigma = \{ x \in G, \ \sigma(x) = x \} \), \( G^\sigma_0 \) the identity component of \( G^\sigma \). If \((G, H, \sigma)\) is a symmetric space, to each point \( \varpi \) of the homogeneous manifold \( M = G/H \) corresponds an involutive diffeomorphism \( \sigma_\varpi \) which has
as an isolated fixed point. Let $g$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$. The automorphism $\sigma \in \text{Aut}(G)$ induces an involutive automorphism of $g$, denoted by $\sigma$ again, such that $\mathfrak{h}$ consists of all elements of $g$ which are left fixed by $\sigma$. We deduce that the Lie algebra $g$ is $\mathbb{Z}_2$-graded: $g = \mathfrak{h} \oplus m$ with $m = \{ X \in g, \sigma(X) = -X \}$. $[\mathfrak{h}, m] \subset m$, $[m, m] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. If we assume that $G$ is simply connected and $H$ connected, then the $\mathbb{Z}_2$-grading $g = \mathfrak{h} \oplus m$ defines a symmetric space structure $(G, H, \sigma)$. Thus, under these hypothesis, it is equivalent to speak about $\mathbb{Z}_2$-grading of Lie algebras or symmetric spaces.

An important class of symmetric spaces consists of Riemannian symmetric spaces. A Riemannian symmetric space is a Riemannian manifold $M$ whose curvature tensor field associated with the Levi-Civita connection is parallel. In this case the geodesic symmetry at a point $u \in M$ attached to the Levi-Civita connection is an isometry and, if we fix $u$, it defines an involutive automorphism $\sigma$ of the largest group of isometries $G$ of $M$ which acts transitively on $M$. We deduce that $M$ is an homogeneous manifold $M = G/H$ and the triple $(G, H, \sigma)$ is a symmetric space. Let us note that, in this case, $H$ is compact. When $H \cap Z(G) = \{ e \}$, this last condition is equivalent to $ad_g(H)$ compact. Here $Z(G)$ denotes the center of $G$. Conversely, if $(G, H, \sigma)$ is a symmetric space such that the image $ad_g(H)$ of $H$ under the adjoint representation of $G$ is a compact subgroup of $GL(g)$, then $g$ admits an $ad_g(H)$-invariant inner product and $\mathfrak{h}$ and $m$ are orthogonal with respect to it. This inner product restricted to $m$ induces an $G$-invariant Riemannian metric on $G/H$ and $G/H$ is a Riemannian symmetric space. For example, if $H$ is compact, $ad_g(H)$ is also compact and $(G, H, \sigma)$ is a Riemannian symmetric space. Assume now that $H$ is connected, then $ad_g(H)$ is compact if and only if the connected Lie group associated with the linear algebra $ad_g(\mathfrak{h}) = \{ adX, X \in \mathfrak{h} \}$ is compact. In this case, $g$ admits an $ad_g(\mathfrak{h})$-invariant inner product $\varphi$, that is, $\varphi([X, Y], Z) + \varphi(Y, [X, Z]) = 0$ for all $X \in \mathfrak{h}$ and $Y, Z \in g$ such that $\varphi(\mathfrak{h}, m) = 0$. An interesting particular case is the following. Assume that $g$ is $\mathbb{Z}_2$-graded and that this grading is effective that is $\mathfrak{h}$ doesn’t contain non trivial ideal of $g$. If $ad_g(\mathfrak{h})$ is irreducible on $m$, then $g$ is simple, or a sum $g_1 + g_2$ with $g_1$ simple or $m$ abelian. In the first case, the Killing-Cartan form $K$ of $g$ induces a negative or positive defined bilinear form on $m$. It follows a classification of $\mathbb{Z}_2$-graded Lie algebras when $g$ is simple or semi-simple.

Many results on the problem of classifications concern more particularly the simple Lie algebras. For solvable or nilpotent Lie algebras, it is an open
problem. A first approach is to study induced grading on Borel or parabolic subalgebras of simple Lie algebras. In this work we describe $\Gamma$-grading of the Heisenberg algebras. Two reasons for this study

- Heisenberg algebras are nilradical of some Borel subalgebras.
- The Riemannian and Lorentzian geometries on the 3-dimensional Heisenberg group have been studied recently by many authors.

Thus it is interesting to study the Riemannian and Lorentzian symmetries with the natural symmetries associated with a $\Gamma$-symmetric structure on the Heisenberg group. In this paper we prove that these geometries are entirely determined by Riemannian and Lorentzian structures adapted to $\mathbb{Z}_2^2$-symmetric structures.

2.2. $\Gamma$-symmetric spaces

Let $\Gamma$ be a finite abelian group.

**Definition 1.** A $\Gamma$-symmetric space is a triple $(G, H, \tilde{\Gamma})$ where $G$ is a connected Lie group, $H$ a closed subgroup of $G$ and $\tilde{\Gamma}$ a finite abelian subgroup of the group $\text{Aut}(G)$ of automorphisms of $G$ isomorphic to $\Gamma$ such that $G_e^\tilde{\Gamma} \subset H \subset G^\Gamma$ where $G^\Gamma = \{x \in G, \sigma(x) = x \ \forall \sigma \in \tilde{\Gamma}\}$, $G_e^\Gamma$ the identity component of $G^\Gamma$.

If $\Gamma$ is isomorphic to $\mathbb{Z}_2$ then we find the notion of symmetric spaces again. If $\Gamma$ is isomorphic to $\mathbb{Z}_k$ with $k \geq 3$, then $\Gamma$ is a cyclic group generated by an automorphism of order $k$. The corresponding spaces are called generalized symmetric spaces and have been studied by Ledger, Obata [14], Gray and Wolf [18] and Kowalski [12]. The general notion of $\Gamma$-symmetric spaces was introduced by Lutz [13] and was algebraically reconsidered by Bahturin and Goze [1].

An equivalent and useful definition is the following:

**Definition 2.** Let $\Gamma$ be a finite abelian group. A $\Gamma$-symmetric space is an homogeneous space $G/H$ such that there exists an injective homomorphism $\rho : \Gamma \to \text{Aut}(G)$ where $\text{Aut}(G)$ is the group of automorphisms of the Lie group $G$, the subgroup $H$ satisfies $G_e^\rho \subset H \subset G^\Gamma$ where $G^\Gamma = \{x \in G/\rho(\gamma)(x) = x, \forall \gamma \in \Gamma\}$ and $G_e^\Gamma$ is the connected identity component of $G^\Gamma$ of $G$.

In [1], one proves that, if $G$ and $H$ are connected, then the triple $(G, H, \tilde{\Gamma})$ is a $\Gamma$-symmetric space if and only if the complexified Lie algebra $g_C = g \otimes_{\mathbb{R}} \mathbb{C}$ of $g$ is $\Gamma$-graded: $g_C = \bigoplus_{\gamma \in \Gamma} g_{\gamma}$ where $g_e = \mathfrak{h}$ is the
Lie algebra of $H$ with $\epsilon$ the unit of $\Gamma$. In this case, we have the relations $[\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}] \subset \mathfrak{g}_{\gamma+\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

In fact, the derivative of an automorphism $\sigma$ of $G$ belonging to $\tilde{\Gamma}$ is an automorphism of $\mathfrak{g}$, still denoted $\sigma$. So if $\gamma$ runs over $\tilde{\Gamma}$, we obtain a subgroup $\tilde{\Gamma}$ of the group of automorphisms of $\mathfrak{g}$ which is isomorphic to $\Gamma$. The elements of $\tilde{\Gamma}$ are automorphisms of $\mathfrak{g}$ of finite order, pairwise commuting and the $\Gamma$-grading corresponds to the spectral decomposition of $\mathfrak{g}_\mathbb{C}$ associated with the abelian finite group $\Gamma$. Conversely, if we have a $\Gamma$-grading of $\mathfrak{g}_\mathbb{C}$, and if we denote by $\tilde{\Gamma}$ the dual group of $\Gamma$, that is, the group of characters, thus $\tilde{\Gamma}$ is a finite abelian group isomorphic to $\Gamma$. Any element $\chi \in \tilde{\Gamma}$ can be considered as an automorphism of $\mathfrak{g}_\mathbb{C}$ by $\chi(X) = \chi(\gamma)X$ for any homogeneous vector $X \in \mathfrak{g}_\gamma$. Thus $\tilde{\Gamma}$ is an abelian subgroup of $Aut(\mathfrak{g}_\mathbb{C})$ isomorphic to $\Gamma$ and the $\Gamma$-grading of $\mathfrak{g}$ corresponds to the spectral decomposition associated with $\tilde{\Gamma}$ considered as an abelian finite subgroup of $Aut(\mathfrak{g}_\mathbb{C})$. Then, if we assume that $G$ is also simply connected, we have a one-to-one correspondence between the set of $\Gamma$-symmetric structures and the $\Gamma$-gradings of $\mathfrak{g}$.

In [13], it is shown that for any $\bar{x} \in M = G/H$, there exists a subgroup $\Gamma_{\bar{x}}$ of the group $\text{Diff}(M)$ of diffeomorphisms of $M$, isomorphic to $\Gamma$, such that $\bar{x}$ is the unique point of $M$ satisfying $\sigma(\bar{x}) = \bar{x}$ for any $\sigma \in \Gamma_{\bar{x}}$. By extension, the elements of $\Gamma_{\bar{x}}$ are also called symmetries of $M$.

2.3. $\mathbb{Z}_2^k$-symmetric spaces

Assume that $\Gamma = \mathbb{Z}_2^k$. In this case any element of $\tilde{\Gamma}$ is an involutive automorphism of $\mathfrak{g}$ and the eigenvalues are real. Since the elements of $\tilde{\Gamma}$ are pairwise commuting, we define a spectral decomposition of $\mathfrak{g}$ itself. This implies a $\mathbb{Z}_2^k$-grading defined on $\mathfrak{g}$: $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$. For example, if $k = 2$, then $\Gamma = \{a, b, c, e\}$ where $e$ is the identity, with $a^2 = b^2 = c^2 = e$, $ab = c$, $bc = a$, $ca = b$ and $\Gamma$ contains 4 elements, $\sigma_a, \sigma_b, \sigma_c$ and the identity $Id$. These maps are involutive and satisfy $\sigma_a \circ \sigma_b = \sigma_c$, $\sigma_b \circ \sigma_c = \sigma_a$, $\sigma_c \circ \sigma_a = \sigma_b$. Each one of these linear maps is diagonalizable, and because they are pairwise commuting, we can diagonalize all these maps simultaneously. Let $\mathfrak{g}_a = \{X \in \mathfrak{g}, \sigma_a(X) = X, \sigma_b(X) = -X\}$, $\mathfrak{g}_b = \{X \in \mathfrak{g}, \sigma_a(X) = -X, \sigma_b(X) = X\}$, $\mathfrak{g}_c = \{X \in \mathfrak{g}, \sigma_a(X) = -X, \sigma_b(X) = -X\}$ and $\mathfrak{g}_e = \{X \in \mathfrak{g}, \sigma_a(X) = X, \sigma_b(X) = X\}$ be the root spaces. We have $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c \oplus \mathfrak{g}_e$.

Let us return to the general case $\Gamma = \mathbb{Z}_2^k$. If $G$ is connected and simply connected and $H$ connected, then the $\Gamma$-grading of $\mathfrak{g}$ determine a structure
of $\Gamma$-symmetric space on the triple $(G, H, \tilde{\Gamma})$. We will say also that the homogeneous space $G/H$ is a $\mathbb{Z}_2^k$-symmetric space.

**Proposition 3.** Any $\mathbb{Z}_2^k$-symmetric homogeneous space $G/H$ is reductive.

**Proof.** In fact if $g = \bigoplus g_{\gamma}$ is the associated decomposition of $g$, thus putting $m = \bigoplus_{\gamma \in \mathbb{Z}_2^k} g_{\gamma}$, we have $g = g_e \oplus m$ with $[g_e, g_e] \subset g_e$ and $[g_e, m] \subset m$. The decomposition $g = g_e \oplus m$ is reductive.

In general $[m, m]$ is not a subset of $g_e$, except if $k = 1$.

Two $\mathbb{Z}_2^k$-gradings $g = \bigoplus g_{\gamma}$ and $g = \bigoplus g'_{\gamma'}$ of $g$ are called equivalent if there exist an automorphism $\pi$ of $g$ and an automorphism $\omega$ of $\mathbb{Z}_2^k$ such that $g'_{\gamma'} = \pi(g_{\omega(\gamma)})$ for any $\gamma' \in \mathbb{Z}_2^k$. If we consider only connected and simply connected groups $G$, and connected subgroups $H$, then the classification of $\mathbb{Z}_2^k$-symmetric spaces is equivalent to the classification, up to equivalence, to $\mathbb{Z}_2^k$-gradings on Lie algebras. For example, the $\mathbb{Z}_2^2$-grading of classical simple complex Lie algebras are classified in [1]. This classification is completed for exceptional simple algebras in [11].

### 2.4. Riemannian and pseudo-Riemannian $\mathbb{Z}_2^k$-symmetric spaces

Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space with $G$ and $H$ connected. The homogeneous space $M = G/H$ is reductive. Then there exists a one-to-one correspondence between the $G$-invariant pseudo-Riemannian metrics $g$ on $M$ and the non-degenerated symmetric bilinear form $B$ on $m$ satisfying $B([X, Y], Z) + B(X, [Y, Z]) = 0$ for all $X, Y \in m$ and $Z \in g_e$.

**Definition 4** ([8]). A $\mathbb{Z}_2^k$-symmetric space $M = G/H$ with $Ad_G(H)$ compact, is called Riemannian $\mathbb{Z}_2^k$-symmetric if $M$ is provided with a $G$-invariant Riemannian metric $g$ whose associated bilinear form $B$ satisfies

1. $B(g_{\gamma}, g_{\gamma'}) = 0$ if $\gamma \neq \gamma' \neq e \neq \gamma$

2. The restriction of $B$ to $m = \bigoplus_{\gamma \neq e} g_{\gamma}$ is positive definite.

In this case the linear automorphisms which belong to $\tilde{\Gamma}$ are linear isometries. Some examples are described in [15].

**Proposition 5.** Let $(G, H, \mathbb{Z}_2^k)$ be a Riemannian $\mathbb{Z}_2^k$-symmetric space, $G$ and $H$ supposed to be connected. Then $H$ is compact.
**Proof.** In fact, $H$ coincides with the identity component of the isotropy group which is compact.

**Example:** $\mathbb{Z}_2^k$-symmetric nilpotent spaces. Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space with $G$ nilpotent. Such a space will be called a $\mathbb{Z}_2^k$-symmetric nilpotent space. If $k = 1$, we cannot have on $G/H$ a Riemannian symmetric metric except if $G$ is abelian. But, if $k \geq 2$, there exist Riemannian $\mathbb{Z}_2^k$-symmetric nilpotent spaces. For example, let $G$ be the 3-dimensional Heisenberg Lie group. Its Lie algebra $\mathfrak{h}_3$ admits a basis ${X_1, X_2, X_3}$ with $[X_1, X_2] = X_3$. We have a $\mathbb{Z}_2^2$-grading of $\mathfrak{h}_3$:

$$\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}$$

and the metric $g = \omega_1^2 + \omega_2^2 + \omega_3^2$ defines a structure of Riemannian $\mathbb{Z}_2^2$-symmetric nilpotent space on $H_3/\{e\} = H_3$ where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of ${X_1, X_2, X_3}$. We will develop this calculus in the next sections.

A Lorentzian metric on an $n$-dimensional differential manifold $M$ is a smooth field of non-degenerate quadratic forms of signature $(n - 1, 1)$. We say that a homogeneous space $(M = G/H, g)$ provided with a Lorentzian metric $g$ is Lorentzian if the canonical action of $G$ on $M$ preserves the metric.

If $M$ is reductive and if $g = g_e + m$, the Lorentzian metric is determinate by the $\text{ad}_{g_e}$-invariant non-degenerate bilinear form $B$ with signature $(n - 1, 1)$.

**Definition 6.** Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space. It is called Lorentzian if there exists on the homogeneous space $M = G/H$ a Lorentzian metric $g$ such that one of the two conditions is satisfied:

1. The homogeneous non trivial components $\mathfrak{h}_\gamma$ of the $\mathbb{Z}_2^k$-graded Lie algebra $\mathfrak{g}$ are orthogonal and non-degenerate with respect to the induced bilinear form $B$.

2. One non trivial component $\mathfrak{h}_{\lambda_0}$ is degenerate, the other components are orthogonal and non-degenerate.

Let us note that, in this case, $H$ is not necessarily compact. Some examples of Lorentzian $\mathbb{Z}_2^k$-symmetric nilpotent spaces are described in the next sections.

**3. Affine structures on $\mathbb{Z}_2^k$-symmetric spaces**

Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space. Since the homogeneous space
$G/H$ is reductive, from [10], Chapter X, we deduce that \( M = G/H \) admits two \( G \)-invariant canonical connections denoted by $\nabla$ and $\bar{\nabla}$. The first canonical connection, $\nabla$, satisfies

\[
\begin{aligned}
R(X,Y) &= -\text{ad}([X,Y]_h), \\
T(X,Y) &= -[X,Y]_m, \\
\nabla T &= 0 \\
\nabla R &= 0
\end{aligned}
\]

where $T$ and $R$ are the torsion and the curvature tensors of $\nabla$. The tensor $T$ is trivial if and only if $[X,Y]_m = 0$ for all $X, Y \in m$. This means that $[X,Y] \in \mathfrak{h}$ that is $[m,m] \subset \mathfrak{h}$. If the grading of $g$ is given by $\mathbb{Z}_k^2$ with $k > 1$, then $[m,m]$ is not a subset of $\mathfrak{h}$ and then the torsion $T$ need not to vanish. In this case the other connection $\bar{\nabla}$ is given by $\nabla_X Y = \nabla_X Y - T(X,Y)$. This is an affine invariant torsion free connection on $G/H$ which has the same geodesics as $\nabla$. This connection is called the second canonical connection or the torsion-free canonical connection.

Remark. Actually, there is another way of writing the canonical affine connection of a $\Gamma$-symmetric space, without any reference to Lie algebras. This is done by an intrinsic construction of $\Gamma$-symmetric spaces proposed by Lutz in [13].

3.1. Associated affine connection

Any symmetric space $G/H$ is an affine symmetric space, that is, it is provided with an affine connection $\nabla$ whose torsion tensor $T$ and curvature tensor $R$ satisfy

\[
T = 0, \quad \nabla R = 0,
\]

where

\[
\nabla R(X_1, X_2, X_3, Y) = \nabla(Y, R(X_1, X_2, X_3)) - R(\nabla(Y, X_1), X_2, X_3) - R(X_1, \nabla(Y, X_2), X_3) - R(X_1, X_2, \nabla(Y, X_3))
\]

for any vector fields $X_1, X_2, X_3, Y$ on $G/H$. It is the only affine connection which is invariant by the symmetries of $G/H$. This means that the two canonical connections, which are defined on an homogeneous reductive space, coincide if the reductive space is symmetric. For example, if $G/H$ is a Riemannian symmetric space, this connection $\nabla$ coincides with the Levi-Civita connection associated with the Riemannian metric.
Let us return to the general case. Let us assume that $G/H$ is a reductive homogeneous space, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the reductive decomposition of $\mathfrak{g}$. Any connection on $G/H$ is given by a linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ satisfying $\Lambda[X,Y] = [\Lambda(X), \lambda(Y)]$ for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{h}$, where $\lambda$ is the linear isotropy representation of $\mathfrak{h}$. The corresponding torsion and curvature tensors are given by:

$$T(X,Y) = \Lambda(X)Y - \Lambda(Y)X - [X,Y]_m$$

and $R(X,Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]_h)$ for any $X, Y \in \mathfrak{m}$.

Let $(G, H, \mathbb{Z}_2)$ be a $\mathbb{Z}_2$-symmetric space. We have recalled that, when $k = 1$, the homogeneous space $G/H$ is an affine symmetric space. But, as soon as $k > 1$, in general the two canonical connections do not coincide and the torsion tensor of the first one is not trivial. We can consider connections adapted to the $\mathbb{Z}_2$-symmetric structures.

**Definition 7.** Let $\nabla$ be an affine connection on the $\mathbb{Z}_2$-symmetric space $G/H$ defined by the linear map $\Lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$. Then this connection is called adapted to the $\mathbb{Z}_2$-symmetric structure, if $\Lambda(X)(g_\gamma) \subset g_{\gamma'\gamma}$ for any $\gamma, \gamma' \in \mathbb{Z}_2$, $\gamma, \gamma' \neq \epsilon$. The connection is called homogeneous if any homogeneous component $g_\gamma$ of $\mathfrak{m}$ is invariant by $\Lambda$.

**Examples.**

1. If $k = 1$, the affine canonical connection is adapted and homogeneous.

2. Let us consider the 5-dimensional nilpotent Lie algebra, $\mathfrak{l}_5$ whose Lie brackets are given in a basis $\{X_1, \cdots, X_5\}$ by $[X_1, X_i] = X_{i+1}, i = 2, 3, 4$. This algebra admits a $\mathbb{Z}_2$-grading $\mathfrak{l}_5 = \mathbb{R}\{X_3, X_5\} \oplus \mathbb{R}\{X_1, X_2, X_4\}$. Thus $\Lambda(X_1), \Lambda(X_2), \Lambda(X_4)$ are matrices of order 3. If we assume that the torsion $T$ is zero, we obtain

$$\Lambda(X_1) = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & \epsilon \end{pmatrix}, \quad \Lambda(X_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ d & f & \epsilon \end{pmatrix}, \quad \Lambda(X_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{e}{2} & 0 & 0 \end{pmatrix}.$$

The linear isotropy representation of $H$ whose Lie algebra is $\mathfrak{h}$ is given by taking the differential of the map $\mathfrak{l}_5/H \to \mathfrak{l}_5/H$ corresponding to the left multiplication $\pi \to h\pi$ with $\pi = xH_4$. We obtain

$$\lambda(X_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda(X_5) = (0).$$

We deduce that the curvature is always non zero.
4. The \( \mathbb{Z}_2^k \)-symmetric spaces \((\mathbb{H}_3, H, \mathbb{Z}_2^k)\)

We denote by \(\mathbb{H}_3\) the 3-dimensional Heisenberg group, that is the linear group of dimension 3 consisting of matrices

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}, \quad a, b, c \in \mathbb{R}.
\]

Its Lie algebra, \(\mathfrak{h}_3\) is the real Lie algebra whose elements are matrices

\[
\begin{pmatrix}
0 & x & z \\
0 & 0 & y \\
0 & 0 & 0
\end{pmatrix} \quad \text{with} \quad x, y, z \in \mathbb{R}.
\]

The elements of \(\mathfrak{h}_3\), \(X_1, X_2, X_3\), corresponding to \((x, y, z) = (1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\) form a basis of \(\mathfrak{h}_3\) and the Lie brackets are given in this basis by \([X_1, X_2] = X_3, [X_1, X_3] = [X_2, X_3] = 0\).

4.1. Description of \(\text{Aut}(\mathfrak{h}_3)\)

Denote by \(\text{Aut}(\mathfrak{h}_3)\) the group of automorphisms \(\mathfrak{h}_3\). Every \(\tau \in \text{Aut}(\mathfrak{h}_3)\) admits in the basis \(\{X_1, X_2, X_3\}\) the following matricial representation:

\[
(1) \quad \begin{pmatrix}
\alpha_1 & \alpha_2 & 0 \\
\alpha_3 & \alpha_4 & 0 \\
\alpha_5 & \alpha_6 & \Delta
\end{pmatrix} \quad \text{with} \quad \Delta = \alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0.
\]

We will denote by \(\tau(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)\) any element of \(\text{Aut}(\mathfrak{h}_3)\) in this representation. Let \(\Gamma\) be a finite abelian subgroup of \(\text{Aut}(\mathfrak{h}_3)\). It admits a cyclic decomposition. If \(\Gamma\) contains a component of the cyclic decomposition which is isomorphic to \(\mathbb{Z}_k\), then there exists an automorphism \(\tau\) satisfying \(\tau^k = \text{Id}\). The aim of this section is to determine the cyclic decomposition of any finite abelian subgroup \(\Gamma\).

**Subgroups of \(\text{Aut}(\mathfrak{h}_3)\) isomorphic to \(\mathbb{Z}_2\)**

Let \(\tau \in \text{Aut}(\mathfrak{h}_3)\) satisfying \(\tau^2 = \text{Id}\). If we consider the matricial representation (1) of \(\tau\), we obtain:

\[
\begin{pmatrix}
\alpha_1^2 + \alpha_2\alpha_3 & \alpha_1\alpha_2 + \alpha_2\alpha_4 & 0 \\
\alpha_1\alpha_3 + \alpha_3\alpha_4 & \alpha_2\alpha_3 + \alpha_4^2 & 0 \\
\alpha_1\alpha_5 + \alpha_3\alpha_6 + \Delta\alpha_5 & \alpha_2\alpha_5 + \alpha_4\alpha_6 + \Delta\alpha_6 & \Delta^2
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
Proposition 8. Any involutive automorphism \( \tau \) of \( Aut(\mathfrak{h}_3) \) is equal to one of the following automorphisms

\[
Id, \quad \tau_1(\alpha_3, \alpha_6) = \begin{pmatrix}
-1 & 0 & 0 \\
\alpha_3 & 1 & 0 \\
\alpha_6 & 0 & -1
\end{pmatrix}, \quad \tau_2(\alpha_3, \alpha_5) = \begin{pmatrix}
1 & 0 & 0 \\
\alpha_3 & -1 & 0 \\
\alpha_5 & 0 & -1
\end{pmatrix},
\]

\[
\tau_3(\alpha_1, \alpha_2 \neq 0, \alpha_6) = \begin{pmatrix}
\frac{1 - \alpha_1^2}{\alpha_2} & \frac{\alpha_2}{\alpha_1} & 0 \\
-\alpha_1 & 0 & 0 \\
\frac{1 + \alpha_1}{\alpha_2} & \alpha_6 & -1
\end{pmatrix}, \quad \tau_4(\alpha_5, \alpha_6) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
\alpha_5 & \alpha_6 & 1
\end{pmatrix}.
\]

Corollary 9. Any subgroup of \( Aut(\mathfrak{h}_3) \) isomorphic to \( \mathbb{Z}_2 \) is one of the following:

\( \Gamma_1(\alpha_3, \alpha_6) = \{Id, \tau_1(\alpha_3, \alpha_6)\} \), \( \Gamma_2(\alpha_3, \alpha_5) = \{Id, \tau_2(\alpha_3, \alpha_5)\} \), \( \Gamma_3(\alpha_1, \alpha_2, \alpha_6) = \{Id, \tau_3(\alpha_1, \alpha_2, \alpha_6), \alpha_2 \neq 0\} \), \( \Gamma_4(\alpha_5, \alpha_6) = \{Id, \tau_4(\alpha_5, \alpha_6)\} \).

- Subgroups of \( Aut(\mathfrak{h}_3) \) isomorphic to \( \mathbb{Z}_k \), \( k \geq 3 \). If \( \tau = \tau(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in Aut(\mathfrak{h}_3) \) satisfies \( \tau^k = Id \), then \( \Delta = \alpha_1 \alpha_4 - \alpha_2 \alpha_3 = 1 \) and its minimal polynomial has 3 simple roots and it is of degree 3. More precisely, it is written

\[
m_\tau(x) = (x - 1)(x - \mu_k)(x - \overline{\mu_k})
\]

where \( \mu_k \) is a root of order \( k \) of 1. Since we can assume that \( \tau \) is a generator of a cyclic subgroup of \( Aut(\mathfrak{h}_3) \) isomorphic to \( \mathbb{Z}_k \), the root \( \mu_k \) is a primitive root of 1. There exists \( m \) relatively prime with \( k \) such that \( \mu_k = \exp(\frac{2m\pi i}{k}) \). We have \( \alpha_1 + \alpha_4 = \mu_k + \overline{\mu_k} \) and \( \alpha_1 + \alpha_4 = 2 \cos \frac{2m\pi}{k} \). Thus

\[
\alpha_1 = \cos \frac{2m\pi}{k} - \sqrt{\cos^2 \frac{2m\pi}{k} - 1 + \alpha_2 \alpha_3},
\]

\[
\alpha_4 = \cos \frac{2m\pi}{k} + \sqrt{\cos^2 \frac{2m\pi}{k} - 1 + \alpha_2 \alpha_3}
\]

or

\[
\alpha_1 = \cos \frac{2m\pi}{k} + \sqrt{\cos^2 \frac{2m\pi}{k} - 1 - \alpha_2 \alpha_3},
\]

\[
\alpha_4 = \cos \frac{2m\pi}{k} - \sqrt{\cos^2 \frac{2m\pi}{k} - 1 - \alpha_2 \alpha_3}.
\]
If \( \tau' \) and \( \tau'' \) denote the automorphisms corresponding to these solutions, we have, for a good choice of the parameters \( \alpha_1, \tau' \circ \tau'' = \text{Id} \) and \( \tau'' = (\tau')^{k-1} \). Thus these automorphisms generate the same subgroup of \( \text{Aut}(h_3) \). Moreover, with same considerations, we can choose \( m = 1 \). Thus we have determined the automorphism \( \tau_5(\alpha_2, \alpha_3, \alpha_5, \alpha_6) \) whose matrix is

\[
\begin{pmatrix}
\cos \frac{2\pi}{k} + \sqrt{\cos^2 \frac{2\pi}{k} - 1 - \alpha_2\alpha_3} & \alpha_2 & 0 \\
\alpha_3 & \cos \frac{2\pi}{k} - \sqrt{\cos^2 \frac{2\pi}{k} - 1 - \alpha_2\alpha_3} & 0 \\
\alpha_5 & \alpha_6 & 1
\end{pmatrix}
\]

**Proposition 10.** Any abelian subgroup of \( \text{Aut}(h_3) \) isomorphic to \( \mathbb{Z}_{k, k} \geq 3 \), is equal to

\[
\Gamma_{6,k}(\alpha_2, \alpha_3, \alpha_5, \alpha_6) = \left\{ \text{Id}, \tau_6(\alpha_2, \alpha_3, \alpha_5, \alpha_6), \cdots, \tau_6^{k-1}, \alpha_2\alpha_3 \leq -1 + \cos^2 \frac{2\pi}{k} \right\}.
\]

**General case.** Suppose now that the cyclic decomposition of a finite abelian subgroup \( \Gamma \) of \( \text{Aut}(h_3) \) is isomorphic to \( \mathbb{Z}_{k_2}^2 \times \mathbb{Z}_{k_3}^3 \times \cdots \times \mathbb{Z}_{k_p}^p \) with \( k_i \geq 0 \).

**Lemma 11.** Let \( \Gamma \) be an abelian finite subgroup of \( \text{Aut}(h_3) \) with a cyclic decomposition isomorphic to \( \mathbb{Z}_{k_2}^2 \times \mathbb{Z}_{k_3}^3 \times \cdots \times \mathbb{Z}_{k_p}^p \). Then

1. If there is \( i \geq 3 \) such that \( k_i \neq 0 \), then \( k_2 \leq 1 \).
2. If \( k_2 \geq 2 \), then \( \Gamma \) is isomorphic to \( \mathbb{Z}_{k_2}^2 \).

**Proof.** Assume that there is \( i \geq 3 \) such that \( k_i \geq 1 \). If \( k_2 \geq 1 \), there exist two automorphisms \( \tau \) and \( \tau' \) satisfying \( \tau'^2 = \text{Id} \) and \( \tau' \circ \tau = \tau \circ \tau' \). Thus \( \tau' \) and \( \tau \) can be reduced simultaneously in the diagonal form and admit a common basis of eigenvectors. Since for any \( \sigma \in \text{Aut}(h_3) \) we have \( \sigma(X_3) = \Delta X_3 \), \( X_3 \) is an eigenvector for \( \tau' \) and \( \tau \) associated to the eigenvalue 1 for \( \tau' \) and \( \pm 1 \) for \( \tau \). As the two other eigenvalues of \( \tau' \) are complex conjugate numbers, the corresponding eigenvectors are complex conjugate. This implies that the eigenvalues of \( \tau \) distinguished of \( \Delta = \pm 1 \) are equal and from Proposition 8, \( \tau = \tau_4(\alpha_5, \alpha_6) \). If we assume that \( k_2 \geq 2 \), there exist \( \tau \) and \( \tau'' \) not equal and belonging to \( \mathbb{Z}_{k_2}^2 \). Thus we have \( \tau = \tau_4(\alpha_5, \alpha_6) \) and \( \tau'' = \tau_4(\alpha'_5, \alpha'_6) \). But \( \tau_4(\alpha_5, \alpha_6) \circ \tau_4(\alpha'_5, \alpha'_6) = \tau_4(\alpha'_5, \alpha'_6) \circ \tau_4(\alpha_5, \alpha_6) \), so if and only if \( \alpha_5 = \alpha'_5, \alpha_6 = \alpha'_6 \) and \( \tau = \tau'' \), this contradicts the hypothesis. □
From this lemma, we have to determine, in a first step, the subgroups \( \Gamma \) of \( \text{Aut}(h_3) \) isomorphic to \((\mathbb{Z}_2)^k\) with \( k \geq 2 \).

- Any involutive automorphism \( \tau \) commuting with \( \tau_1(a_3, a_6) \) with \( \tau \neq \tau_1(a_3, a_6) \) is equal to \( \tau_2(-a_3, a_5) \) or \( \tau_3(-a_3, a_5) = \tau_4(-\frac{\alpha_3\alpha_6}{2} - a_5, -a_6) \) and \( [\tau_2(-a_3, a_5), \tau_4(-\frac{\alpha_3\alpha_6}{2} - a_5, -a_6)] = 0 \). Thus

\[
\Gamma_7(a_3, a_5, a_6) = \left\{ \text{Id}, \tau_1(a_3, a_6), \tau_2(-a_3, a_5), \tau_4\left(-\frac{\alpha_3\alpha_6}{2} - a_5, -a_6\right) \right\}
\]

is a subgroup of \( \text{Aut}(h_3) \) isomorphic to \( \mathbb{Z}_2^2 \). Moreover it is the only subgroup of \( \text{Aut}(h_3) \) of type \((\mathbb{Z}_2)^k\), \( k \geq 2 \), containing an automorphism of type \( \tau_1(a_3, a_6) \).

- A direct computation shows that any abelian subgroup \( \Gamma \) containing \( \tau_2(a_3, a_5) \) is either isomorphic to \( \mathbb{Z}_2 \) or equal to \( \Gamma_7 \).

Assume that \( \tau_3(a_1, a_3, a_6) \in \Gamma \). The automorphisms \( \tau_3(-a_1, -a_2, a_6') \) and \( \tau_4(a_5, a_6') \) commute with \( \tau_3(a_1, a_3, a_6) \). Since

\[
\tau_3(a_1, a_2, a_6) \circ \tau_3(-a_1, -a_2, a_6') = \tau_4\left(\frac{a_6'(1 - a_1) - a_6(1 + a_1)}{a_2}, -a_6 - a_6'\right)
\]

we obtain the following subgroup, denoted \( \Gamma_8(a_1, a_2, a_6, a_6') \):

\[
\left\{ \text{Id}, \tau_3(a_1, a_2, a_6), \tau_3(-a_1, -a_2, a_6'), \tau_4\left(\frac{a_6'(1 - a_1) - a_6(1 + a_1)}{a_2}, -a_6 - a_6'\right) \right\}
\]

which is isomorphic to \( \mathbb{Z}_2^2 \).

- We suppose that \( \tau_4(a_5, a_6) \in \Gamma \). If \( \Gamma \) is not isomorphic to \( \mathbb{Z}_2 \), then \( \Gamma \) is one of the groups \( \Gamma_7, \Gamma_8 \).

**Theorem 12.** Any finite abelian subgroup \( \Gamma \) of \( \text{Aut}(h_3) \) isomorphic to \((\mathbb{Z}_2)^k\) is one of the following

1. \( k = 1 \), \( \Gamma = \Gamma_1(a_3, a_6), \Gamma_2(a_3, a_5), \Gamma_3(a_1, a_2, a_6), \alpha_2 \neq 0, \Gamma_4(a_5, a_6), \Gamma_5(a_1, a_3, a_6), \Gamma_6(a_1, a_2, a_6, a_6'). \)

2. \( k = 2 \), \( \Gamma = \Gamma_7(a_3, a_5, a_6), \Gamma_8(a_1, a_2, a_6, a_6'). \)

Assume now that \( \Gamma \) is isomorphic to \( \mathbb{Z}_2^{k_3} \) with \( k_3 \geq 2 \). If \( \tau \in \Gamma_5 \), its matricial representation is

\[
\begin{pmatrix}
-1 - \sqrt{-3 - 4\alpha_2\alpha_3} & \alpha_2 & 0 \\
2 & -1 + \sqrt{-3 - 4\alpha_2\alpha_3} & 0 \\
\alpha_3 & \frac{\alpha_2}{2} & 1 \\
\alpha_5 & 0 & 1
\end{pmatrix}
\]
To simplify, we put $\lambda = \frac{1 - \sqrt{-3 - 4\alpha_2 \alpha_3}}{2}$. The eigenvalues of $\tau$ are $1, j, j^2$ and the corresponding eigenvectors $X_3, V, \overline{V}$ with

$$V = \left(1, -\frac{\lambda - j}{\alpha_2}, -\frac{\alpha_5}{1 - j} + \frac{\alpha_6(\lambda - j)}{\alpha_2(1 - j)}\right)$$

if $\alpha_2 \neq 0$. If $\tau'$ is an automorphism of order 3 commuting with $\tau$, then $\tau'V = jV$ or $j^2V$. But the two first components of $\tau'(V)$ are $\lambda' - \frac{\beta_2}{\alpha_2}(\lambda - j)$, $\beta_3 - \frac{\lambda'(\lambda - j)}{\alpha_2}$ where $\beta_i$ and $\lambda'$ are the corresponding coefficients of the matrix of $\tau'$. This implies $\alpha_2\lambda' - \beta_2(\lambda - j) = \alpha_2j$ or $\alpha_2j^2$. Considering the real and complex parts of this equation, we obtain

$$\begin{cases}
\alpha_2\lambda' - \beta_2\lambda = 0, \\
\beta_2j = \alpha_2j \quad \text{or} \quad \alpha_2j^2.
\end{cases}$$

As $\alpha_2 \neq 0$, we obtain $\alpha_2 = \beta_2$ and $\lambda = \lambda'$. Let us compare the second component of $\tau'(V)$. We obtain $\beta_3\alpha_2 - \lambda'(\lambda - j) = -(\lambda - j)j$ or $-(\lambda - j)j^2$. As $\lambda = \lambda'$, we have in the first case $2\lambda j = j^2$ and in the second case $2\lambda j = j^3 = 1$. In any case, this is impossible. Thus $\alpha_2 = 0$ and, from Section 2.2, $\tau = Id$. This implies that $k_3 = 1$ or 0.

**Theorem 13.** Let $\Gamma$ be a finite abelian subgroup of $\text{Aut}(h_3)$. Thus $\Gamma$ is isomorphic to one of the following group

1. $\mathbb{Z}_2 \times \mathbb{Z}_2$,
2. $\mathbb{Z}_2^{k_2} \times \mathbb{Z}_3^{k_3} \times \cdots \times \mathbb{Z}_p^{k_p}$ with $k_i = 0$ or 1 for $i = 2, \ldots, p$.

To prove the second part, we show as in the case $i = 3$ that $k_i = 1$ as soon as $k_i \neq 0$.

**Remark.** We have determined the finite abelian subgroups of $\text{Aut}(h_3)$. There are non-abelian finite subgroups with elements of order at most 3. Take for example the subgroup generated by

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} -\frac{1}{2} & \alpha & 0 \\ -\frac{\alpha}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha \neq 0.$$ 

The relations on the generators are $\sigma_1^2 = Id$, $\sigma_2^2 = Id$, $\sigma_1\sigma_2\sigma_1 = \sigma_2^2$. Thus the group generated by $\sigma_1$ and $\sigma_2$ is isomorphic to the symmetric group $\Sigma_3$ of degree 3.
4.2. Description of the $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$-gradings of $\mathfrak{h}_3$

Let $\Gamma$ be a finite abelian subgroup of $\text{Aut}(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_2^k$ ($k = 1$ or 2).

- If $\Gamma = \mathbb{Z}_2$, we have obtained $\Gamma = \Gamma_i$, $i = 1, 2, 3, 4$. Up to equivalence of gradings, the $\mathbb{Z}_2$-grading of $\mathfrak{h}_3$ are $\mathfrak{h}_3 = \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_1, X_3\}$ and $\mathfrak{h}_3 = \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2, X_3\}$.
- If $\Gamma = \mathbb{Z}_2^2$ then $\Gamma = \Gamma_7$ or $\Gamma = \Gamma_8$.

**Lemma 14.** There is an automorphism $\sigma \in \text{Aut}(\mathfrak{h}_3)$ such that $\sigma^{-1}\Gamma_7\sigma = \Gamma_8$.

The proof is a simple computation. There also exists $\sigma \in \text{Aut}(\mathfrak{h}_3)$ such that

\[
\begin{align*}
\sigma^{-1}\tau_1(\alpha_3, \alpha_6)\sigma &= \tau_1(0, 0), \\
\sigma^{-1}\tau_2(-\alpha_3, \alpha_5)\sigma &= \tau_2(0, 0).
\end{align*}
\]

We deduce:

**Proposition 15.** Every $\mathbb{Z}_2^2$-grading on $\mathfrak{h}_3$ is equivalent to the grading defined by $\Gamma_7(0, 0, 0) = \{\text{Id}, \tau_1(0, 0), \tau_2(0, 0), \tau_4(0, 0)\}$.

This grading corresponds to $\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}$.

4.3. Non existence of Riemannian symmetric structures on $\mathbb{H}_3/H$

Consider the symmetric space $\mathbb{H}_3/H_1$ associated with the grading

\[
\mathfrak{h}_3 = \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_1, X_3\}.
\]

Let $\{\omega_1, \omega_2, \omega_3\}$ be the dual basis of $\{X_1, X_2, X_3\}$. Any pseudo-Riemannian metric on the symmetric space $\mathbb{H}_3/H_1$ where $H_1$ is a one-dimensional connected Lie group whose Lie algebra $\mathfrak{g}_0 = \mathbb{R}\{X_2\}$ is given by a non-degenerate $\text{ad}_{\mathfrak{g}_0}$-invariant bilinear form $B = a\omega_1^2 + b\omega_1 \wedge \omega_3 + c\omega_3^2$ on $\mathfrak{g}_1 = \mathbb{R}\{X_1, X_3\}$. This implies $B([X_2, X_1], X_3) = -B(X_3, X_3) = -c = 0$. But we have also $B([X_2, X_1], X_1) + B(X_1, [X_2, X_1]) = -2B(X_3, X_1) = -2b = 0$. We deduce

**Proposition 16.** The nilpotent symmetric space $\mathbb{H}_3/H$ associated to the grading $\mathfrak{h}_3 = \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_1, X_3\}$ doesn't admit any pseudo-Riemannian symmetric metric.
Consider now the symmetric space $\mathbb{H}_3/H_2$ associated with the grading $\mathfrak{h}_3 = \mathbb{R}\{X_3\} \oplus \mathbb{R}\{X_1, X_2\}$. Then $H_2$ is the Lie subgroup whose Lie algebra is $\mathbb{R}\{X_3\}$ and the bilinear form $B = a\omega_1^2 + b\omega_1 \land \omega_2 + c\omega_2^2$ on $\mathfrak{g}_1 = \mathbb{R}\{X_1, X_2\}$ is $adX_3$-invariant because $adX_3 = 0$. But $Ad_G$ is an homomorphism of $G$ onto the group of inner automorphisms of $\mathfrak{g}$ with kernel the center of $G$, we deduce that $Ad_G(H)$ is compact in this case and any non-degenerate bilinear form $B$ on $\mathfrak{g}_1$ defines a Riemannian or a Lorentzian structure on the symmetric space $\mathbb{H}_3/H_2$.

**Proposition 17.** The nilpotent symmetric space $\mathbb{H}_3/H_2$ associated to the grading $\mathfrak{h}_3 = \mathbb{R}\{X_3\} \oplus \mathbb{R}\{X_1, X_2\}$ admits a structure of Riemannian symmetric space. It admits also a structure of Lorentzian symmetric space.

### 4.4. Riemannian $\mathbb{Z}_2^2$-symmetric structures on $\mathbb{H}_3$

Consider on $\mathbb{H}_3$ a $\mathbb{Z}_2^2$-symmetric structure. It is determined, up to equivalence, by the $\mathbb{Z}_2^2$-grading of $\mathfrak{h}_3$

$$\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}.$$ 

Since every automorphism of $\mathfrak{h}_3$ is an isometry of any invariant Riemannian metric on $\mathbb{H}_3$, we deduce

**Theorem 18.** Any Riemannian $\mathbb{Z}_2^2$-symmetric structure on $\mathbb{H}_3$ is isometric to the Riemannian structure associated with the grading $\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}$ and the Riemannian $\mathbb{Z}_2^2$-symmetric metric is written $g = \omega_1^2 + \omega_2^2 + \lambda^2\omega_3^2$ with $\lambda \neq 0$, where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of $\{X_1, X_2, X_3\}$.

**Proof.** Indeed, since the components of the grading are orthogonal, the Riemannian metric $g$, which coincides with the form $B$ satisfies $g = \alpha_1\omega_1^2 + \alpha_2\omega_2^2 + \alpha_3\omega_3^2$ with $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$.

According to [6], we reduce the coefficients to $\alpha_1 = \alpha_2 = 1$. $\Box$

**Remark.** According to [7] and [9], this metric is naturally reductive for any $\lambda$.

**Corollary 19.** A Riemannian tensor $g$ on $\mathbb{H}_3$ determines a Riemannian $\mathbb{Z}_2^2$-symmetric structure over $\mathbb{H}_3$ if and only if it is a left-invariant metric on $\mathbb{H}_3$.

This is a consequence of the previous theorem and of the classification of left-invariant metrics on Heisenberg groups (see [6]).
4.5. Lorentzian $\mathbb{Z}_2^2$-symmetric structures on $H_3$

We say that an homogeneous space $(M = G/H, g)$ is Lorentzian if the canonical action of $G$ on $M$ preserves a Lorentzian metric (i.e. a smooth field of non-degenerate quadratic forms of signature $(n - 1, 1)$) (see [3]).

**Proposition 20 ([5]).** Modulo an automorphism and a multiplicative constant, there exists on $H_3$ one left-invariant metric assigning a strictly positive length on the center of $h_3$.

The Lie algebra $h_3$ is generated by the central vector $X_3$ and $X_1$ and $X_2$ such that $[X_1, X_2] = X_3$. The automorphisms of the Lie algebra preserve the center and then send the element $X_3$ on $\lambda X_3$, with $\lambda \in \mathbb{R}^*$. Such an automorphism acts on the plane generated by $X_1$ and $X_2$ as an automorphism of determinant $\lambda$.

It is shown in [16] and [17] that, modulo an automorphism of $h_3$, there are three classes of invariant Lorentzian metrics on $H_3$, corresponding to the cases where $||X_3||$ is negative, positive or zero.

We propose to look at the Lorentzian metrics that are associated with the $\mathbb{Z}_2^2$-symmetric structures over $H_3$. If $g$ is the Heisenberg algebra equipped with a $\mathbb{Z}_2^2$-grading, then by automorphism, we can reduce to the case where $\Gamma = \Gamma_7$. In this case, the grading of $h_3$ is given by:

$$h_3 = g_0 + g_{+-} + g_{--} + g_{--},$$

with $g_0 = \{0\}$, and

$$g_{+-} = \mathbb{R} \left\{ X_2 - \frac{\alpha_6}{2} X_3 \right\}, g_{--} = \mathbb{R} \left\{ X_1 - \frac{\alpha_3}{2} X_2 + \frac{\alpha_5}{2} X_3 \right\}, g_{---} = \mathbb{R} \left\{ X_3 \right\}.$$

Assume $Y_1 = X_1 - \frac{\alpha_3}{2} X_2 + \frac{\alpha_5}{2} X_3$, $Y_2 = X_2 - \frac{\alpha_6}{2} X_3$, $Y_3 = X_3$. The dual basis is

$$\vartheta_1 = \omega_1, \quad \vartheta_2 = \omega_2 + \frac{\alpha_3}{2} \omega_1, \quad \vartheta_3 = \omega_3 - \frac{\alpha_6}{2} \omega_2 - \left( \frac{\alpha_3 \alpha_6}{4} + \frac{\alpha_5}{2} \right) \omega_1,$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of the base $\{X_1, X_2, X_3\}$.

**Case I.** The components $g_{+-}$, $g_{--}$, $g_{---}$ are non-degenerate. The quadratic form induced on $h_3$ therefore writes

$$g = \lambda_1 \omega_1^2 + \lambda_2 \left( \omega_2 + \frac{\alpha_3}{2} \omega_1 \right)^2 + \lambda_3 \left( \omega_3 - \frac{\alpha_6}{2} \omega_2 - \left( \frac{\alpha_3 \alpha_6}{4} + \frac{\alpha_5}{2} \right) \omega_1 \right)^2,$$
with \( \lambda_1, \lambda_2, \lambda_3 \neq 0 \). The change of basis associated with the matrix

\[
\begin{pmatrix}
1 & 0 \\
\frac{\alpha_3}{2} & 1
\end{pmatrix}
\begin{pmatrix}
0 & \frac{-\alpha_5}{2} - \frac{\alpha_3 \alpha_6}{4} \\
1 & 0
\end{pmatrix}
\]

is an automorphism. Thus \( g \) is isometric to \( g = \lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2 \). Since the signature is \((2, 1)\) one of the \( \lambda_i \) is negative and the two others positive.

**Proposition 21.** Every Lorentzian \( \mathbb{Z}_2^2 \)-symmetric metric \( g \) on \( \mathbb{H}_3 \) such that the components of the grading of \( \mathfrak{h}_3 \) are non-degenerate, is reduced to one of these two forms: \( g = -\omega_1^2 + \omega_2^2 + \lambda^2 \omega_3^2 \) or \( g = \omega_1^2 + \omega_2^2 - \lambda^2 \omega_3^2 \).

**Case II.** Suppose that a component is degenerate. When this component is \( \mathbb{R}\{X_2 + \frac{\alpha_3}{2}X_3\} \) or \( \mathbb{R}\{X_1 - \frac{\alpha_3}{2}X_2 + \frac{\alpha_5}{2}X_3\} \) then, by automorphism, it reduces to the above case.

Suppose then that the component containing the center is degenerate.

Thus the quadratic form induced on \( \mathfrak{h}_3 \) is written

\[
g = \omega_1^2 + \left[ \omega_3 - \frac{\alpha_6}{2} \omega_2 - \frac{2\alpha_5 + \alpha_3 \alpha_6}{4} \omega_1 \right]^2 \\
- \left[ \omega_2 - \omega_3 + \frac{\alpha_6}{2} \omega_2 + \frac{2\alpha_5 + \alpha_3 \alpha_6}{4} \omega_1 \right]^2.
\]

The change of basis associated with the matrix

\[
\begin{pmatrix}
1 & 0 \\
\frac{\alpha_3}{2} & 1
\end{pmatrix}
\begin{pmatrix}
0 & \frac{-\alpha_5}{2} - \frac{\alpha_3 \alpha_6}{4} \\
1 & 0
\end{pmatrix}
\]

is given by an automorphism. Thus \( g \) is isomorphic to \( g = \omega_1^2 + \omega_3^2 - (\omega_2 - \omega_3)^2 \).

**Proposition 22.** Every Lorentzian \( \mathbb{Z}_2^2 \)-symmetric metric \( g \) on \( \mathbb{H}_3 \) such that the component of the grading of \( \mathfrak{h}_3 \) containing the center is degenerate, is reduced to the form \( g = \omega_1^2 + \omega_3^2 - (\omega_2 - \omega_3)^2 \).

**Corollary 23.** A Lorentzian tensor \( g \) on \( \mathbb{H}_3 \) determines a Lorentzian \( \mathbb{Z}_2^2 \)-symmetric structure over \( \mathbb{H}_3 \) if and only if it is a left-invariant Lorentzian metric on \( \mathbb{H}_3 \).

The classification, up to isometry, of left-invariant Lorentzian metrics on \( \mathbb{H}_3 \) is described in [4] and in [17]. It corresponds to the previous classification of Lorentzian \( \mathbb{Z}_2^2 \)-symmetric metrics.
5. \( \mathbb{Z}_2^k \)-symmetric spaces based on \( \mathbb{H}_{2p+1} \)

5.1. \( \mathbb{Z}_2^k \)-gradings of \( \mathfrak{h}_{2p+1} \)

Let \( \sigma \) be an involutive automorphism of the \((2p+1)\)-dimensional Heisenberg algebra \( \mathfrak{h}_{2p+1} \). Let \( \{ X_1, \ldots, X_{2p+1} \} \) be a basis of \( \mathfrak{h}_{2p+1} \) whose structure constants are given by

\[
[X_1, X_2] = \cdots = [X_{2p-1}, X_{2p}] = X_{2p+1}.
\]

Since the center \( \mathbb{R}\{X_{2p+1}\} \) is invariant by \( \sigma \), it is contained in an homogeneous component of the grading \( \mathfrak{h}_{2p+1} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) associated with \( \sigma \). But for any \( X \in \mathfrak{h}_{2p+1}, X \neq 0 \), there exists \( Y \neq 0 \) such that \( [X, Y] = aX_{2p+1} \) with \( a \neq 0 \). We deduce that any \( \mathbb{Z}_2 \)-grading is equivalent to one of the following:

1. If \( X_{2p+1} \in \mathfrak{g}_0 \), then
   \[ \mathfrak{h}_{2p+1} = \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_2, \ldots, X_{2p}\} \]

2. If \( X_{2p+1} \in \mathfrak{g}_1 \), then
   \[ \mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_2, X_3, \ldots, X_{2k}, X_{2p+1}\} \oplus \mathbb{R}\{X_{2k+1}, X_{2k+2}, \ldots, X_{2p}\} \]

Let \( \mathfrak{h}_{2p+1} = \bigoplus_{\gamma \in \mathbb{Z}_2^k} \mathfrak{g}_\gamma \) be a \( \mathbb{Z}_2^k \)-grading of the Heisenberg algebra. The support of this grading is the subset \( \{ \gamma \in \mathbb{Z}_2^k, \mathfrak{g}_\gamma \neq 0 \} \). We will say that this grading is irreducible if the subgroup of \( \mathbb{Z}_2^k \) generated by its support is the full group \( \mathbb{Z}_2^k \).

**Lemma 24.** If \( \mathfrak{h}_{2p+1} \) admits an irreducible \( \mathbb{Z}_2^k \)-grading, then \( k = 1 \) or \( k = 2 \).

In fact, this is a consequence of the previous classification of the \( \mathbb{Z}_2 \)-gradings of \( \mathfrak{h}_{2p+1} \). We deduce also that any \( \mathbb{Z}_2^2 \)-grading is equivalent to \( \mathfrak{h}_{2p+1} = \{0\} \oplus \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \ldots, X_{2p}\} \).
5.2. Pseudo-Riemannian symmetric spaces $\mathbb{H}_{2p+1}/H$

We consider the symmetric spaces $\mathbb{H}_{2p+1}/H$ corresponding to the previous symmetric decomposition of $\mathfrak{h}_{2p+1}$, where $H$ is a connected Lie subgroup of $\mathbb{H}_{2p+1}$ whose Lie algebra is $\mathfrak{g}_0$.

- With the $\mathbb{Z}_2$-grading $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_2, \ldots, X_{2p}\}$. Since $ad(X_{2p+1})$ is zero any non-degenerate bilinear form on $\mathfrak{g}_1$ defines a symmetric pseudo-Riemannian metric on $\mathbb{H}_{2p+1}/H$ where $H$ is a connected one-dimensional Lie Group.

- Consider the $\mathbb{Z}_2$-grading

$$\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_2, X_3, \ldots, X_{2k}, X_{2p+1}\} \oplus \mathbb{R}\{X_{2k+1}, X_{2k+2}, \ldots, X_{2p}\}. $$

In this case, $H$ is a Lie subgroup isomorphic to $\mathbb{H}_{2k+1}$. Since we have $[\mathfrak{g}_0, \mathfrak{g}_1] = 0$, any non-degenerate bilinear form on $\mathfrak{g}_1$ defines a symmetric pseudo-Riemannian metric on $\mathbb{H}_{2p+1}/\mathbb{H}_{2k+1}$.

- We consider the $\mathbb{Z}_2$-gradings

$$\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_2, X_3, \ldots, X_{2p-1}\} \oplus \mathbb{R}\{X_{2n}, X_{2n}, X_{2p}\} \oplus \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}, X_{2p+1}\}. $$

In this case, any bilinear form on $\mathfrak{g}_1$ which is $ad(\mathfrak{g}_0)$-invariant is degenerate. In fact, if $B$ is such a form, we have $B([X_{2k+1}, X_{2k+2}], X_1) = B(X_{2p+1}, X_{2p+1}) = 0$ and for any $k = 0, \ldots, p-1$ and $s \neq k+1$, $B([X_{2k+1}, X_{2k+2}], X_{2s}) = B(X_{2p+1}, X_{2s}) = 0$, and $X_{2p+1}$ is in the kernel of $B$. We have the same proof for the second grading.

**Proposition 25.** The symmetric spaces $\mathbb{H}_{2p+1}/H$ corresponding to the $\mathbb{Z}_2$-grading of $\mathfrak{h}_{2p+1}$:

- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \ldots, X_{2p}, X_{2p+1}\}$

- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_2, X_4, \ldots, X_{2p}\} \oplus \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}, X_{2p+1}\}$

are not pseudo-Riemannian symmetric spaces.

5.3. Riemannian $\mathbb{Z}_2^2$-symmetric spaces $\mathbb{H}_{2p+1}/H$

Let us consider the $\mathbb{Z}_2^2$-grading of the Heisenberg algebra $\mathfrak{h}_{2p+1} = \{0\} \oplus \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \ldots, X_{2p}\}$. Since $\mathfrak{g}_0 = \{0\}$, then $H$ is reduced to the identity and the $\mathbb{Z}_2^2$-symmetric space $\mathbb{H}_{2p+1}/H$ is isomorphic to $\mathbb{H}_{2p+1}$. The reductive decomposition $\mathfrak{h}_{2p+1} = \mathfrak{g}_0 \oplus \mathfrak{m}$ is
reduced to \( m \). Since \( g_0 = \{0\} \), any bilinear definite positive form on \( m \) for which the homogeneous components \( \mathbb{R}\{X_{2p+1}\}, \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}\} \) and \( \mathbb{R}\{X_1, X_4, \ldots, X_{2p}\} \) are pairwise orthogonal defines a Riemannian \( \mathbb{Z}_2^2 \)-symmetric structure on \( \mathbb{H}_{2p+1} \).

The Levi-Civita connection associated with this Riemannian metric is an affine connection. In case of Riemannian symmetric space, the Levi-Civita connection associated with the Riemannian symmetric metric is torsion-free and the curvature tensor \( \nabla R = 0 \), where \( \nabla \) is the covariant derivative of this connection, and corresponds to the canonical connection defined in [10] which defines the natural affine structure on a symmetric space. This is not the case for Riemannian \( \mathbb{Z}_2^2 \)-symmetric spaces. In the next section, we define a class of affine connections adapted to the \( \mathbb{Z}_2^2 \)-symmetric structures, and we prove, in case of the Riemannian \( \mathbb{Z}_2^2 \)-symmetric space \( \mathbb{H}_{2p+1}/H \), that there exist adapted connections with torsion and curvature-free.

5.4. Adapted affine connections on the \( \mathbb{Z}_2^2 \)-symmetric spaces \( \mathbb{H}_{2p+1}/H \)

Let \( G/H \) be a \( \mathbb{Z}_2^k \)-symmetric space. Since \( G/H \) is a reductive homogeneous space, that is \( g \) admits a decomposition \( g = g_0 + m \) with \([g_0, g_0] \subset g_0 \) and \([g_0, m] \subset m \), any connection is given by a linear map \( \Lambda : m \rightarrow gl(m) \) satisfying \( \Lambda [X, Y] = [\Lambda (X), \lambda(Y)] \), for all \( X \in m \) and \( Y \in g_0 \), where \( \lambda \) is the linear isotropy representation of \( g_0 \). The corresponding torsion and curvature tensors are given by:

\[
T(X, Y) = \Lambda (X)(Y) - \Lambda (Y)(X) - [X, Y]_m
\]

and

\[
R(X, Y) = \left[ \Lambda (X), \Lambda (Y) \right] - \Lambda [X, Y] - \lambda([X, Y]_{g_0}),
\]

for any \( X, Y \in m \).

**Definition 26.** Consider the affine connection on the \( \mathbb{Z}_2^2 \)-symmetric space \( G/H \) defined by the linear map \( \Lambda : m \rightarrow gl(m) \). Then this connection is called adapted to the \( \mathbb{Z}_2^k \)-symmetric structure, if any \( \Lambda (X)(g_{\gamma}) \subset g_{\gamma \gamma'} \), for any \( \gamma, \gamma' \in \mathbb{Z}_2^k \), \( \gamma, \gamma' \neq 0 \). The connection is called homogeneous if any homogeneous component \( g_\gamma \) of \( m \) is invariant by \( \Lambda \).

Now we consider the case where \( G/H = \mathbb{H}_{2p+1}/H \) is the \( \mathbb{Z}_2^2 \)-symmetric space defined by the grading

\[
h_{2p+1} = \{0\} \oplus \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_3, \ldots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \ldots, X_{2p}\}.
\]
We have seen that $H$ is reduced to the identity and $\mathbb{H}_{2p+1}/H$ is isomorphic to $\mathbb{H}_{2p+1}$. Consider an adapted connection and let $\wedge$ be the associated linear map. Since the connection is adapted to the $\mathbb{Z}_2^2$-symmetric structure, $\wedge$ satisfies:

\[
\begin{align*}
\wedge(X_{2k+1})(X_{2l+1}) &= \wedge(X_{2s})(X_{2t}) = 0, \quad k, l = 0, \ldots, p - 1, s, t = 1, \ldots, p, \\
\wedge(X_{2k+1})(X_{2s}) &= C^{2k+1}_{s} X_{2p+1}, s = 1, \ldots, p, k = 0, \ldots, p - 1, \\
\wedge(X_{2s})(X_{2k+1}) &= C^{2k+1}_{k} X_{2p+1}, s = 1, \ldots, p, k = 0, \ldots, p - 1, \\
\wedge(X_{2k+1})(X_{2p+1}) &= \sum_{s=1}^{p} a^{2k+1}_{s} X_{2s}, k = 0, \ldots, p - 1, \\
\wedge(X_{2s})(X_{2p+1}) &= \sum_{k=0}^{p} a^{2k+1}_{s} X_{2k+1}, s = 1, \ldots, p.
\end{align*}
\]

**Theorem 27.** Any adapted connection $\nabla$ on the $\mathbb{Z}^2_2$-symmetric space $\mathbb{H}_{2p+1}/H = \mathbb{H}_{2p+1}$ satisfies $T = 0$ and $R = 0$ where $T$ and $R$ are respectively the torsion and the curvature of $\nabla$ if and only if the corresponding linear map $\wedge$ satisfies

\[
\begin{align*}
\wedge(X_{2k+1})(X_{2s}) &= C^{2k+1}_{s} X_{2p+1}, \quad s = 1, \ldots, p, k = 0, \ldots, p - 1, \\
\wedge(X_{2s})(X_{2i}) &= 0, k = 0, \ldots, p - 1, i \notin \{2, \ldots, 2p\}, \\
\wedge(X_{2s})(X_{2k+1}) &= C^{2k+1}_{k} X_{2p+1}, s = 1, \ldots, p, k = 0, \ldots, p - 1, k \neq s - 1, \\
\wedge(X_{2s})(X_{2s-1}) &= (C^{2k+1}_{s} - 1) X_{2p+1}, s = 1, \ldots, p, \\
\wedge(X_{2s})(X_{i}) &= 0, s = 1, \ldots, p, i \notin \{1, \ldots, 2p - 1\}.
\end{align*}
\]

In fact, we determine in a first step, all the connection adapted to the $\mathbb{Z}^2_2$-symmetric structure and which are torsion-free. In this case, $\wedge$ satisfies

\[
\wedge(X)(Y) - \wedge(Y)(X) - [X, Y] = 0, \quad \text{for any } X, Y \in \mathfrak{h}_{2p+1}.
\]

**REFERENCES**


