DIFFERENTIAL GEOMETRY OF HOLOMORPHIC HYPERSURFACES

BY

MÜZEYYEN GÜLSAH KARTAL and AHMET YÜCESAN

Abstract. We study differential geometry of holomorphic hypersurfaces on an anti-Kaehler manifold with constant totally real sectional curvature. We give the conditions for being an almost Einstein manifold and having parallel Ricci tensor of holomorphic hypersurfaces. Then, we find considerable a result that if a holomorphic hypersurface is an almost Einstein manifold, then it has parallel Ricci tensor.

Mathematics Subject Classification 2010: 53B30, 53C15, 53C50, 53C56.

Key words: Anti-Kaehler manifold, holomorphic hypersurface, constant totally real sectional curvature, almost Einstein manifold, parallel Ricci tensor.

1. Introduction

An anti-Kaehler manifold (a Kaehler manifold with Norden metric or $B$-manifold) is defined by Norden in 1972 and studied by mathematicians like Borowiec, Francaviglia, Volovich, Sluka, Djelepov in the following years (see [1], [2], [3], [6]).

Hypersurface of an anti-Kaehler manifold is said to be a holomorphic hypersurface if the restriction of the metric on hypersurface of an anti-Kaehler manifold has a maximal rank and vectors on tangent space of hypersurface are $J$-invariant. Ganchev, Gribachev and Mihova published fundamental study related with holomorphic hypersurfaces in 1985 (see [4]). In this study, they proved classification theorem for the holomorphic hypersurfaces of $(\mathbb{R}^{2n+2}, g, J)$ with constant totally real sectional curvature. In 2007, Pranović examined the intrinsic properties of minimal holomorphic
hypersurface of an anti-Kaehler manifold with constant totally real sectional curvature and denoted that second fundamental form of its can be expressed in terms of the curvatures and the Ricci tensor (see [5]).

The purpose of this paper examine differential geometry of holomorphic hypersurfaces of an anti-Kaehler manifold with constant totally real sectional curvature. For this purpose, in chapter 2, an anti-Kaehler manifold and holomorphic hypersurfaces of its are introduced and gave the equations of Gauss and Codazzi for holomorphic hypersurface of an anti-Kaehler manifold. Finally, in chapter 3, the require conditions for being an almost Einstein manifold and parallel Ricci tensor of a holomorphic hypersurface of an anti-Kaehler manifold with constant totally real sectional curvature are obtained and as a result of these achievements is found that if a holomorphic hypersurface is an almost Einstein manifold, then the hypersurface has parallel Ricci tensor.

2. Preliminaries

Let \( \tilde{M} \) be a real connected \( 2(n + 1) \)-dimensional differentiable manifold endowed with an almost complex structure \( J \) (\( J^2 = -I \), \( I \) being the identity transformation) and a semi-Riemannian metric \( g \) of Norden type (that is, of signature \( (n + 1, n + 1) \)) and such that \( g(JX, JY) = -g(X, Y) \) and \((\tilde{\nabla}_X J)Y = 0\), for any \( X, Y \in \chi(\tilde{M}) \), where \( \tilde{\nabla} \) is the Levi-Civita connection of \( g \) and \( \chi(\tilde{M}) \) is the Lie algebra of smooth vector fields on \( \tilde{M} \). Then \( \tilde{M} \) are called an anti-Kaehler manifold (a Kaehler manifold with Norden metric, a Kähler-Norden manifold or \( B \)-manifold) (see [1], [6]).

A submanifold \( M \) (\( \dim M = 2n \)) is said to be a holomorphic hypersurface of \( \tilde{M} \) if the restriction of \( g \) on \( M \) has a maximal rank and 

\[
JT_p M = T_p M, \quad p \in M.
\]

We denote the restrictions of \( g \) and \( J \) on \( M \) by the same letters. Then \( M \) is a called as an anti-Kaehler manifold (see [4]).

Example 1 ([4]). Let \( \tilde{M} = \mathbb{R}^{2n+2} \) be equipped with the canonical complex structure \( J \) and the metric \( g \). Then \( (\tilde{M}, g, J) \) is an anti-Kaehler manifold. The position vector of \( z_0 \) and \( z \) respectively is given by \( Z_0 \) and \( Z \), we define the submanifold \( S^{2n} \) with a center \( z_0 \) by the equalities

\[
\begin{align*}
g(Z - Z_0, Z - Z_0) &= 1, \\
g(Z - Z_0, J(Z - Z_0)) &= 0.
\end{align*}
\]
$S^{2n}$ is a $2n$-dimensional submanifold of $\tilde{M}$ and the vectors $Z - Z_0, J(Z - Z_0)$ are perpendicular to $T_z S^{2n}$. The rank of $g$ on $T_z S^{2n}$ is equal to $2n$ and $T_z S^{2n}$ is $J$-invariant, i.e. $S^{2n}$ is a holomorphic hypersurface of $\tilde{M}$.

Let $\nabla$ and $\tilde{\nabla}$ be the Levi-Civita connection of $M$ and $\tilde{M}$, respectively. Then the Gauss formula is given by

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on $M$ and $\tilde{\nabla}_X Y$ is tangent component of $\tilde{\nabla}_X Y$ and $B(X, Y)$, called second fundamental form tensor field, is normal component of $\tilde{\nabla}_X Y$. Choose symmetric covariant tensor fields $h$ and $k$ of degree 2 on $M$ such that

$$(2.2) \quad h(X, Y) = k(X, JY) \quad \text{and} \quad k(X, Y) = -h(X, JY),$$

then second fundamental form tensor becomes

$$B(X, Y) = h(X, Y) N + k(X, Y) JN,$$

where $N$ be a unit ($g(N, N) = -g(JN, JN) = 1$) local normal field on $M$.

The Weingarten formula is given by $\tilde{\nabla}_X N = -A_N X$, for all $X \in \chi(M)$, where $A_N$ is the Weingarten endomorphism associated to $N$. By using the Gauss and Weingarten formulae we get

$$(2.3) \quad h(X, Y) = g(A_N X, Y) \quad \text{and} \quad k(X, Y) = -g(JA_N X, Y),$$

where $JA_N$ is the Weingarten endomorphism associated to $JN$. The Gauss formula (2.1) reads $\tilde{\nabla}_X Y = \nabla_X Y + g(A_N X, Y) N - g(JA_N X, Y) JN$. In addition, it is easy to see from (2.2) and (2.3) that $JA_N = A_N J$.

**Theorem 1.** Let $R$ and $\tilde{R}$ be the Riemannian curvature tensor of $M$ and $\tilde{M}$, respectively. Then

$$\tilde{R}(X, Y) Z = R(X, Y) Z + g(A_N X, Z) A_N Y - g(A_N Y, Z) A_N X + g(JA_N Y, Z) JA_N X - g(JA_N X, Z) JA_N Y$$

$$+ \{g((\nabla_X A_N) Y, Z) - g((\nabla_Y A_N) X, Z)\} N$$

$$+ \{g((\nabla_Y JA_N) X, Z) - g((\nabla_X JA_N) Y, Z)\} JN,$$

for all $X, Y, Z \in \chi(M)$. From (2.4), the Gauss and Codazzi equations are given by

$$\tan \tilde{R}(X, Y) Z = R(X, Y) Z + g(A_N X, Z) A_N Y - g(A_N Y, Z) A_N X + g(JA_N Y, Z) JA_N X - g(JA_N X, Z) JA_N Y$$
and
\[
\text{nor} \tilde{R}(X, Y)Z = \{g((\nabla_X A_N)Y, Z) - g((\nabla_Y A_N)X, Z)\} N \\
+ \{g((\nabla_Y JA_N)X, Z) - g((\nabla_X JA_N)Y, Z)\} JN,
\]
respectively (see [4]).

Now we give following theorem due to Gauss equation:

**Theorem 2.** Let \( M \) be a holomorphic hypersurface of an anti-Kaehler manifold \( \tilde{M} \). The Ricci curvatures of \( \tilde{M} \) and \( M \) are given by \( \text{Ric} \) and \( \text{Ric} \), respectively. Then relations between the Ricci curvatures are given by the following an equation:

\[
\text{Ric}(X, Y) = \text{Ric}(X, Y) - (\text{trace} A_N) g(A_N X, Y) + 2g(A^2_N X, Y) \\
+ (\text{trace} JA_N) g(JA_N X, Y) + 2g(\tilde{R}(N, X)Y, N),
\]
for all \( X, Y \in \chi(M) \) and \( N \in \chi(M) \perp \) (see [4]).

### 3. Differential geometry of holomorphic hypersurfaces of an anti-Kaehler manifold with constant totally real sectional curvature

In the previous sections, we gave relations between curvatures of an anti-Kaehler manifold and a holomorphic hypersurface of its. In this section, the conditions for being an almost Einstein manifold and parallel Ricci tensor of a holomorphic hypersurface of an anti-Kaehler manifold with constant totally real sectional curvature are obtained. Finally, we find considerable a result that if a holomorphic hypersurface is an almost Einstein manifold, then it has parallel Ricci tensor.

**Theorem 3.** Let \( M \) be a holomorphic hypersurface of an anti-Kaehler manifold \( \tilde{M} \) (\( \dim \tilde{M} \geq 6 \)) with constant totally real sectional curvature. Then the Ricci curvature of \( M \) is

\[
\text{Ric}(X, Y) = 2(n - 1)(\tilde{\nu}g(X, Y) - \tilde{\nu}^* g(X, JY)) - 2g(A^2_N X, Y) \\
+ (\text{trace} A_N) g(A_N X, Y) - (\text{trace} JA_N) g(JA_N X, Y),
\]
for all \( X, Y \in \chi(M) \), where \( \tilde{\nu} \) and \( \tilde{\nu}^* \) are constant totally real sectional curvatures (see [4]).
**Theorem 4.** Let $M$ be a holomorphic hypersurface of an anti-Kaehler manifold $\tilde{M}$ with constant totally real sectional curvature. Then the Ricci tensor of $M$ is symmetric, i.e. $\text{Ric}(X, Y) = \text{Ric}(Y, X)$, for all $X, Y \in \chi(M)$.

**Proof.** Let $M$ be a holomorphic hypersurface of an anti-Kaehler manifold $\tilde{M}$ with constant totally real sectional curvature. Taking into account (3.1), we obtain the following equations

$$\text{Ric}(X, Y) = 2(n - 1)(\tilde{\nu}g(X, Y) - \tilde{\nu}^*g(X, JY)) - 2g(A_N^2X, Y)$$
$$+ (\text{trace } A_N)g(A_NX, Y) - (\text{trace } JA_N)g(JA_NX, Y)$$

and

$$\text{Ric}(Y, X) = 2(n - 1)(\tilde{\nu}g(Y, X) - \tilde{\nu}^*g(Y, JX)) - 2g(A_N^2Y, X)$$
$$+ (\text{trace } A_N)g(A_NY, X) - (\text{trace } JA_N)g(JA_NY, X).$$

From these equations, the Ricci tensor is symmetric, i.e. $\text{Ric}(X, Y) = \text{Ric}(Y, X)$, for all $X, Y \in \chi(M)$.

**Theorem 5.** Let $M$ be a holomorphic hypersurface of an anti-Kaehler manifold $\tilde{M}$ (dim $\tilde{M} \geq 8$) with constant totally real sectional curvature. If $M$ is connected and h-umbilical (holomorphic umbilical), then $M$ is an almost Einstein manifold.

**Proof.** Let $M$ be connected and h-umbilical. Due to the fact that $M$ is connected and h-umbilical, they are written by the equation

$$A_N = \lambda I - \mu J \quad \text{and} \quad JA_N = \mu I + \lambda J,$$

where $\lambda = \frac{\text{trace } A_N}{2n} = \text{const}$ and $\mu = \frac{\text{trace } JA_N}{2n} = \text{const}$. Taking into account the equations (3.1) and (3.2), we obtain the following equation

$$\text{Ric}(X, Y) = 2(n - 1)(\tilde{\nu}g(X, Y) - \tilde{\nu}^*g(X, JY)) - 2g(A_N^2X, Y)$$
$$+ (\text{trace } A_N)g(A_NX, Y) - (\text{trace } JA_N)g(JA_NX, Y)$$
$$= 2(n - 1)(\tilde{\nu}g(X, Y) - \tilde{\nu}^*g(X, JY)) - 2g((\lambda^2 - \mu^2)X - 2\lambda\mu JX, Y)$$
$$+ 2n\lambda g(JX + \lambda JX, Y) - 2n\mu g(\mu X + \lambda JX, Y)$$
$$= 2(n - 1)(\tilde{\nu}g(X, Y) - \tilde{\nu}^*g(X, JY)) - 2((\lambda^2 - \mu^2)g(X, Y) + 4\lambda\mu g(X, JY)$$
$$+ 2n\lambda^2 g(X, Y) - 2\lambda\mu g(JX, Y) - 2\mu^2 g(X, Y) - 2\lambda\mu g(JX, Y)$$
$$= 2(n - 1)(\tilde{\nu} + \lambda^2 - \mu^2)g(X, Y) + 2(1 - n)(\tilde{\nu}^* + 2\lambda\mu)g(JX, Y)$$
$$= ag(X, Y) + bg(X, JY),$$
for all $X, Y \in \chi(M)$, where $a = 2(n - 1)(\bar{\nu} + \lambda^2 - \mu^2)$ and $b = 2(1 - n)(\bar{\nu}^* + 2\lambda\mu)$. Then the holomorphic hypersurface $M$ is an almost Einstein manifold. \hfill \Box

**Theorem 6.** Let $M$ be a holomorphic hypersurface of an anti-Kaehler manifold $\tilde{M}$ $(\dim \tilde{M} \geq 6)$ with constant totally real sectional curvature. If $M$ is totally geodesic, then $M$ is an almost Einstein manifold.

**Proof.** Let $M$ be totally geodesic. Due to the fact that $M$ is totally geodesic, they are written by the equation

\begin{equation}
A_N = 0, \quad JA_N = 0, \quad \text{trace } A_N = 0 \quad \text{and } \quad \text{trace } JA_N = 0.
\end{equation}

Taking into account the equations (3.1) and (3.3), we obtain the following equation $Ric(X, Y) = 2(n - 1)(\bar{\nu}g(X, Y) - \bar{\nu}^*g(X, JY))$, for all $X, Y \in \chi(M)$, where $\bar{\nu}$ and $\bar{\nu}^*$ are constant totally real sectional curvature. Then the holomorphic hypersurface $M$ is an almost Einstein manifold. \hfill \Box

**Theorem 7.** Let $M$ be a holomorphic hypersurface of an anti-Kaehler manifold $\tilde{M}$ $(\dim \tilde{M} \geq 6)$ with constant totally real sectional curvature. Then the derivation of Ricci tensor of a holomorphic hypersurface $M$ is

\begin{equation}
(\nabla_Z Ric)(X, Y) = \nabla_Z (\text{trace } A_N)g(A_N X, Y)
- \nabla_Z (\text{trace } JA_N)g(A_N X, JY)
+ (\text{trace } A_N)g((\nabla_Z A_N)X, Y)
- (\text{trace } JA_N)g((\nabla_Z A_N)X, JY)
- 2g((\nabla_Z A_N^2)X, Y),
\end{equation}

for all $X, Y, Z \in \chi(M)$.

**Proof.** Let $M$ be a holomorphic hypersurface of an anti-Kaehler manifold $\tilde{M}$ with constant totally real sectional curvature. By using equations

\begin{equation}
(\nabla_Z Ric)(X, Y) = \nabla_Z (Ric(X, Y)) - Ric(\nabla_Z X, Y) - Ric(X, \nabla_Z Y)
\end{equation}

and (3.1), we obtain

\begin{equation}
(\nabla_Z Ric)(X, Y) = \nabla_Z (\text{trace } A_N)g(A_N X, Y)
- \nabla_Z (\text{trace } JA_N)g(A_N X, JY)
+ (\text{trace } A_N)\{g(\nabla_Z (A_N X), Y) - g(A_N (\nabla_Z X), Y)\}
- \text{trace}(JA_N)\{g(\nabla_Z (A_N X), JY) - g(A_N (\nabla_Z X), JY)\}
- 2\{g(\nabla_Z (A_N^2 X), Y) - g(A_N^2 \nabla_Z X, Y)\}
\end{equation}
for all $X, Y, Z \in \chi(M)$. Hence, substituting the following equations

$$g(\nabla_Z (A_N X), Y) = g((\nabla_Z A_N) X, Y) + g(A_N \nabla_Z X, Y),$$
$$g(\nabla_Z (A_N X), JY) = g((\nabla_Z A_N) X, JY) + g(A_N \nabla_Z X, JY)$$

and

$$g(\nabla_Z (A^2_N X), Y) = g((\nabla_Z A^2_N) X, Y) + g(A^2_N \nabla_Z X, Y)$$

into (3.5), we get the equation (3.4).

**Theorem 8.** Let $M$ be a holomorphic hypersurface of an anti-K"ahler manifold $\tilde{M}$ $(\dim \tilde{M} \geq 8)$ with constant totally real sectional curvature. If $M$ is connected and h-umbilical, then $M$ has parallel Ricci tensor.

**Proof.** Let $M$ be connected and h-umbilical. Taking into account the equations (3.2) and (3.4), we obtain the following equation

$$(\nabla_Z \text{Ric})(X, Y) = \nabla_Z (2\lambda n)g(\lambda X - \mu JX, Y) - \nabla_Z (2\mu n)g(\lambda X - \mu JX, JY)$$
$$+ 2n\lambda g(\nabla_Z (\lambda I - \mu J)X, Y) - 2n\mu g(\nabla_Z (\lambda I - \mu J)X, JY)$$
$$- 2g(\nabla_Z ((\lambda^2 - \mu^2)I - 2\lambda \mu J)X, Y) = 0,$$

for all $X, Y, Z \in \chi(M)$, i.e. $\nabla \text{Ric} = 0$. Thus $M$ has parallel Ricci tensor.

**Theorem 9.** Let $M$ be a holomorphic hypersurface of an anti-K"ahler manifold $\tilde{M}$ $(\dim \tilde{M} \geq 6)$ with constant totally real sectional curvature. If $M$ is totally geodesic, then $M$ has parallel Ricci tensor.

**Proof.** Let $M$ be totally geodesic. Taking into account the equations (3.3) and (3.4), we obtain the following equation $\nabla \text{Ric} = 0$. Thus $M$ has parallel Ricci tensor.

We can give following corollary:

**Corollary 1.** Let $M$ be a holomorphic hypersurface of an anti-K"ahler manifold $\tilde{M}$ $(\dim \tilde{M} \geq 8)$ with constant totally real sectional curvature. If $M$ is an almost Einstein manifold, then $M$ has parallel Ricci tensor.

**Proof.** Let $M$ be an almost Einstein manifold. Then the Ricci curvature of a holomorphic hypersurface $M$ is

$$\text{Ric}(X, Y) = 2(n - 1)(vg(X, Y) - v^*g(X, JY)),$$
for all $X, Y \in \chi(M)$. Due to the fact that $v, v^*$ are constant totally real sectional curvature and $g$ is a metric tensor, the derivation of Ricci tensor is equivalent to zero, i.e., $\nabla \text{Ric} = 0$. Thus $M$ has parallel Ricci tensor.

Acknowledgement. This work was supported by Scientific Research Projects Coordination Unit of Süleyman Demirel University under project 2338-YL-10.

REFERENCES


