Permanent weak module amenability of semigroup algebras

Abasalt Bodaghi · Massoud Amini · Ali Jabbari

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Abstract We employ the fact that $L^1(G)$ is $n$-weakly amenable for each $n \geq 1$ to show that for an inverse semigroup $S$ with the set of idempotents $E$, $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module with trivial left action. We study module amenability and weak module amenability of the module projective tensor products of Banach algebras.

Keywords Banach modules · module derivation · $n$-weak module amenability · inverse semigroup · module projective tensor product

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1 Introduction

A Banach algebra $A$ is amenable if $H^1(A, X^*) = \{0\}$ for every Banach $A$-module $X$, where $H^1(A, X^*)$ is the first Hochschild cohomology group of $A$ with coefficients in $X^*$. The notion is introduced by Johnson in [17]. Dales et al. introduced the notion of $n$-weak amenability of Banach algebras in [12]. A Banach algebra $A$ is $n$-weakly amenable if $H^1(A, A^{(n)}) = \{0\}$, where $A^{(n)}$ is $n$th dual space of $A$ (1-weak amenability is called...
A Banach algebra is called **permanently weakly amenable** if it is \( n \)-weakly amenable for each positive integer \( n \). It is well known that for any locally compact group \( G \), \( L^1(G) \) is \( n \)-weakly amenable whenever \( n \in \mathbb{N} \) (see [10], [12] and [18]). As for \( \mathcal{A} \hat{\otimes} \mathcal{B} \), for amenable Banach algebras \( \mathcal{A} \) and \( \mathcal{B} \), that is always amenable (see [17]). However, \( \mathcal{A} \hat{\otimes} \mathcal{B} \) is weakly amenable when \( \mathcal{A} \) and \( \mathcal{B} \) are commutative and weakly amenable (see [15]).

In [3], the second author and Bagha extended the notion of weak amenability for a Banach algebra \( \mathcal{A} \) to the case that there is an extra \( \mathfrak{A} \)-module structure on \( \mathcal{A} \) and showed that \( \ell_1(S) \) is weakly module amenable, as an \( \ell_1(E) \)-module, when \( S \) is a commutative inverse semigroup with the set of idempotents \( E \). The same is true for an arbitrary inverse semigroup with trivial left action (see [5]). Also Bodaghi et al. in [8] showed that \( \ell_1(S) \) is \( n \)-weakly module amenable as an \( \ell_1(E) \)-module (with trivial left action) when \( n \) is odd. It is proved in [4] that the module projective tensor product \( \ell_1(S) \hat{\otimes} \ell_1(E) \ell_1(S) \) is module amenable when \( S \) is amenable (the module contractibility case is shown in [6]).

In this paper, we investigate \( n \)-weak module amenability of semigroup algebras and using the fact that for a locally compact group \( G \), \( L^1(G) \) is \( n \)-weakly amenable for all \( n \in \mathbb{N} \), we show that the inverse semigroup algebra \( \ell_1(S) \) is \( n \)-weakly module amenable as an \( \ell_1(E) \)-module for all \( n \in \mathbb{N} \). We also investigate module amenability and weak module amenability of the module projective tensor product \( \mathcal{A} \hat{\otimes} \mathcal{B} \) (as \( \mathfrak{A} \)-module) under certain conditions. In particular, we show that \( \ell_1(S) \hat{\otimes} \ell_1(E) \ell_1(S) \) is weakly module amenable.

### 2 \( n \)-weak module amenability of semigroup algebras

Let \( \mathfrak{A} \) and \( \mathcal{A} \) be Banach algebras such that \( \mathcal{A} \) is a Banach \( \mathfrak{A} \)-bimodules with following compatible actions \( \alpha \cdot (ab) = (\alpha \cdot a)b, (ab) \cdot \alpha = a(b \cdot \alpha) \), for every \( a, b \in \mathcal{A}, \alpha \in \mathfrak{A} \).

If \( \mathfrak{A} \) is a unital Banach algebra, the Banach algebra \( \mathcal{A} \) is said to be a unital Banach \( \mathfrak{A} \)-module if \( e_{\mathfrak{A}} \cdot a = a \cdot e_{\mathfrak{A}} = a \) for every \( a \in \mathcal{A} \). Let \( \mathcal{A} \) and \( \mathcal{B} \) be Banach \( \mathfrak{A} \)-bimodules with compatible actions. An \( \mathfrak{A} \)-module map is a mapping \( \varphi : \mathcal{A} \rightarrow \mathcal{B} \) with following properties

1. \( \varphi(a \pm b) = \varphi(a) \pm \varphi(b) \) \( (a, b \in \mathcal{A}) \);
2. \( \varphi(\alpha \cdot a) = \alpha \cdot \varphi(a), \varphi(a \cdot \alpha) = \varphi(a) \cdot \alpha \) \( (a \in \mathcal{A}, \alpha \in \mathfrak{A}) \).

One should note that \( \varphi \) is not linear, in general. Let \( X \) be a Banach \( \mathcal{A} \)-bimodule and a Banach \( \mathfrak{A} \)-bimodule with following compatible actions: \( \alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, a(\alpha \cdot x) = (a \cdot \alpha) \cdot x, (\alpha \cdot x) \cdot a = a \cdot (\alpha \cdot x), (a \cdot \alpha) \cdot x = a \cdot (\alpha \cdot x), (x \cdot \alpha) \cdot a = x \cdot (\alpha \cdot a), \) for every \( x \in X, a \in \mathcal{A}, \alpha \in \mathfrak{A} \). In this case, we say \( X \) is a Banach \( \mathcal{A} \hat{\otimes} \mathfrak{A} \)-module. If \( \alpha \cdot x = x \cdot \alpha \), for every \( x \in X \) and \( \alpha \in \mathfrak{A} \), then \( X \) is called a commutative Banach \( \mathcal{A} \hat{\otimes} \mathfrak{A} \)-module. Moreover, if \( a \cdot x = x \cdot a \), for every \( x \in X \) and \( a \in \mathcal{A} \), then \( X \) is called a bi-commutative Banach \( \mathcal{A} \hat{\otimes} \mathfrak{A} \)-module. It is clear that \( \mathcal{A} \) is a Banach \( \mathfrak{A} \)-module. Also, if \( \mathcal{A} \) is a commutative \( \mathfrak{A} \)-bimodule, then \( \mathcal{A} \) is a commutative \( \mathcal{A} \hat{\otimes} \mathfrak{A} \)-module, and so is the \( n \)-th dual of \( \mathcal{A} \). If moreover \( \mathcal{A} \) is a commutative Banach algebra, then it is a bi-commutative Banach \( \mathcal{A} \hat{\otimes} \mathfrak{A} \)-module, and the same holds for its \( n \)-th dual.

An \( \mathfrak{A} \)-module map \( D : \mathcal{A} \rightarrow X \) is called a module derivation if \( D(ab) = a \cdot D(b) + D(a) \cdot b, \) \( (a, b \in \mathcal{A}) \). A module derivation \( D \) is called bounded if there exists \( M > 0 \)
such that \( \|D(a)\| \leq M\|a\| \), for every \( a \in A \). Note that boundedness of \( D \) implies its norm continuity while \( D \) can be non-linear. We use the notations \( Z_A(A,X) \) and \( N_A(A,X) \) for the set of all continuous module derivations and continuous inner module derivations from \( A \) to \( X \) respectively. Also the quotient space \( Z_A(A,X)/N_A(A,X) \) (which we call the first \( A \)-module cohomology group of \( A \) with coefficients in \( X \)) is denoted by \( H^n(A,X) \). From now on, by a module derivation we mean a continuous module derivation.

The Banach algebra \( A \) is called module amenable (as an \( A \)-module) if for any commutative Banach \( A \)-module \( X \), each module derivation \( D : A \rightarrow X^* \) is inner [1], in other word, \( A \) is module amenable if \( H^n(A,X^*) = \{0\} \), for each commutative Banach \( A \)-module \( X \) (see [1]). It is proved in [5] that \( A \) is module amenable if \( H^n(A,X^*) = \{0\} \), for each commutative Banach \( A \)-module \( X \) (see [6, 21]). Also \( A \) is weakly module amenable (as an \( A \)-module) if for any subset \( Y \) of \( A^* \) which is \( A \)-submodule and commutative Banach \( A \)-submodule, each module derivation from \( A \) to \( Y \) is inner [5].

Let \( G \) be a locally compact and \( A \) be a Banach space. Suppose that \( G \) acts on \( X \) by homomorphisms, i.e., we have the continuous mappings \( (s, x) \mapsto s \cdot x \) from \( G \times X \) into \( X \) and \( (x, s) \mapsto x \cdot s \) from \( X \times G \) into \( X \) such that \( (s \cdot t) \cdot x = s \cdot (t \cdot x) \), \( s \cdot (x \cdot t) = (s \cdot x) \cdot t \), \( x \cdot (s \cdot t) = x \cdot s \cdot t \), \( (s,t) \in G , x \in X \). A map \( \delta : G \rightarrow X \) is called a \( G \)-derivation if \( \delta(st) = s \cdot \delta(t) + \delta(s) \cdot t \), for every \( s,t \in G \). The \( G \)-derivation \( \delta \) is called inner if there exists \( x \in X \) such that \( \delta(s) = s \cdot x - x \cdot s \), for every \( s \in S \). In this case we write \( \delta = ad_x \). A map \( \psi : G \rightarrow X \) is called a crossed homomorphism if \( \psi(st) = s \cdot \psi(t) \cdot s^{-1} + \psi(s) \), for every \( s,t \in G \), and \( \psi \) is called principal if there exists \( x \in X \) such that \( \psi(s) = s \cdot x \cdot s^{-1} - x \), for every \( s \in G \). Let \( \delta : G \rightarrow X \) be a \( G \)-derivation, and set \( \delta(s) = \delta(s) \cdot s^{-1} \), for \( s \in G \). Then \( \psi \) is a crossed homomorphism, and \( \psi \) is principal if \( \delta \) is inner. Conversely, Let \( \psi : G \rightarrow X \) be a crossed homomorphism. Set \( \delta(s) = \psi(s) \cdot s \) for \( s \in G \). Then \( \delta \) is a \( G \)-derivation, and \( \delta \) is inner if \( \psi \) is principal. Let \( D : \ell^1(G) \rightarrow X^* \) be continuous derivation. Set \( \delta(s) = D(\delta_s) \) for every \( s \in G \). Then \( \delta \) is a \( G \)-derivation, and it is clear that if \( D \) is an inner derivation then so is \( \delta \).

\textbf{Theorem 2.1} Let \( G \) be a discrete group. Then \( \ell^1(G) \) is \( n \)-weakly amenable for all \( n \in \mathbb{N} \).

\textbf{Proof.} The odd case is proved in [12, Theorem 4.1] and [18]. Therefore it is sufficient to prove the result in the even case (compare with [10]). Let \( D : \ell^1(G) \rightarrow \ell^1(G)^{(2n)} \) be a bounded derivation. Since \( G \) is discrete, the group algebra \( \ell^1(G) \) has an identity. Hence \( \ell^1(G) = M(G) \). Put \( \Omega_0 = G \). On the other hand, \( \ell^\infty(G) \) is a commutative unital \( C^* \)-algebra so by Gelfand-Naimark Theorem we have \( \ell^\infty(G) \cong C(\Omega_1) \), where \( \Omega_1 \) is \( w^* \)-compact. Similarly \( \ell^\infty(G)^{**} \cong C(\Omega_1)^{**} \cong C(\Omega_2) \), for a \( w^* \)-compact space \( \Omega_2 \). Now if \( n \geq 2 \) and \( \ell^1(G)^{(2n-2)} \cong M(\Omega_{n-1}) \), for some \( w^* \)-compact space \( \Omega_{n-1} \), then \( \ell^1(G)^{(2n)} \cong (\ell^1(G)^{(2n-2)})^{**} \cong C(\Omega_{n-1})^{***} \cong C(\Omega_n)^* \cong M(\Omega_n) \), for some \( w^* \)-compact space \( \Omega_n \).
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Let us show that $G$ acts on $\Omega_1$ by homeomorphisms. Each $x \in G$ induces a map $\alpha_x : \Omega_1 \to \Omega_1$ defined by $\alpha_x(\omega)(f) = \omega(\ell_x f)$ for $f \in \ell^\infty(G)$, where $\ell_x$ is the left translation by $x$. It is clear that if $\omega$ is a continuous character on $\ell^\infty(G)$, so is $\omega_x$ and $\alpha_x$ is $w^*\cdot w^*$-continuous. Also $\alpha_x^{-1} = \alpha_{x^{-1}}$ and $\alpha_x$ is a homeomorphism. Similarly $G$ acts on $\Omega_n$ by homeomorphisms and $M(\Omega_n)$ is a Banach $\ell^1(G)$-module and the above isomorphism of Banach spaces is indeed an isomorphism of Banach $\ell^1(G)$-modules. By Johnson’s Theorem ([11, Theorem 5.6.39]), $D'(s) = D(\delta_s)$ defines a bounded crossed homomorphism $\psi$ from $G$ into $M(\Omega_n)$, and by Theorem 1.1 of [20], $\psi$ is principal, and therefore $D$ is inner. □

JOHNSON and RINGROSE in [19] showed that for a discrete group $G$, $H^1(\ell^1(G), \ell^1(G)) = \{0\}$. This is not true for arbitrary locally compact groups. Let $G$ be an infinite compact (which is unimodular), noncommutative group. Choose $x \in G$ which is not in the center of $G$, then the continuous derivation $D : \ell^1(G) \to \ell^1(G) ; f \mapsto \delta_x \ast f - f \ast \delta_x$ is not inner. Indeed, if $D = ad_g$ for some $g \in \ell^1(G)$, then take a local basis $U_a$ of identity element $e$ such that $xU_a$ is not equal to $(U_a)x$ ($x$ is not in the center), and let $f_a$ be the characteristic function of $U_a$ divided by Haar measure of $U_a$, then $f_a$ is a bounded approximate identity, and if we consider $D(f_a) = ad_g(f_a)$ and let $a \to \infty$ then the right hand side goes to zero, but the left hand side does not. However, if $D : \ell^1(G) \to \ell^1(G)$ is a bounded derivation then a similar argument as in the proof of the above theorem shows that $D = ad_\mu$ for some $\mu \in M(G)$ and $ad_\mu - ad_e$ is an inner derivation on $\ell^1(G)$ if and only if $\mu - \nu \in \ell^1(G)$, where $\ell^1(G)$ is considered as a closed ideal of $M(G)$. When $n \geq 1$, as in the above argument we get $\ell^1(G)^{(2n)} = M(\Omega_n)$ for some $G$-space $\Omega_n$ and Johnson’s Theorem ([11, Theorem 5.6.39]) implies that $D$ is inner. Summing up:

Theorem 2.2 Let $G$ be a locally compact group. Then $H^1(\ell^1(G), \ell^1(G)) = M(G)/\ell^1(G)$ and $H^1(\ell^1(G), \ell^1(G)^{(2n)}) = \{0\}$, for $n \geq 1$.

Recall that a left Banach $\mathcal{A}$-module $X$ is called a right essential $\mathcal{A}$-module if the linear span of $X \cdot \mathcal{A} = \{x \cdot a : a \in \mathcal{A}, x \in X\}$ is dense in $X$. Left essential $\mathcal{A}$-modules and (two-sided) essential $\mathcal{A}$-bimodules are defined similarly. The following result is proved in [8, Theorem 3.14] when $n$ is an odd number but essentially the same proof works for an even natural number. So we have:

Theorem 2.3 Let $n \in \mathbb{N}$. Let $\mathcal{A}/J$ has a left or right identity and $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left. If $\mathcal{A}$ is $n$-weakly module amenable, then $\mathcal{A}/J$ is $n$-weakly amenable. The converse is true if $\mathcal{A}$ is a right essential $\mathfrak{A}$-module.

A semigroup $S$ is called an inverse semigroup if for each $s \in S$ there exists unique $s^* \in S$ with $ss^*s = s$, $s^*ss^* = s^*$. More details on inverse semigroups may be found in [16]. The mapping $s \mapsto s^*$ is an involution on $S$, i.e. $s^{**} = s$ and $(st)^* = t^*s^*$ for all $s, t \in S$. We denote the set of idempotents in $S$ by $E$. Each idempotent of $S$ is self-adjoint, and $E$ is a commutative idempotent subsemigroup of $S$ and a semilattice. In particular, $\ell^1(E)$ is a subalgebra of $\ell^1(S)$. We can consider $\ell^1(S)$ as a Banach module over $\ell^1(E)$ with trivial left action and multiplication as the right action (see [1]), that is $\delta_e \cdot s = \delta_s$, $\delta_e^* = \delta_e$ for $s, t \in S$ and $e \in E$. To consider the inverse semigroup $S$.

Considering $\mathcal{A} = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$ in the above Theorem, $J$ is the closed ideal of $\ell^1(S)$ generated by $\{\delta_{st} - \delta_{st} : s, t \in S, e \in E_S\}$. We consider an equivalence relation on $S$ as follows: $s \approx t \iff \delta_s - \delta_t \in J$, for every $s, t \in S$. For an inverse semigroup $S$,
the quotient $S/ \approx$ is a discrete group (see [2] and [21]), and by Theorem 3.3 of [22], we have $\ell^1(S)/J \cong \ell^1(S/\approx)$.

**Corollary 2.4** Let $n \in \mathbb{N}$ and let $S$ be an inverse semigroup with the set of idempotents $E$. Then $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module with trivial left action.

**Proof.** The result is proved in Theorem 3.15 of [8] when $n$ is odd. In the above action of $\ell^1(E)$ on $\ell^1(S)$, $\ell^1(S)$ is a right essential $\ell^1(E)$-module (see the proof of [8, Theorem 3.15]). According to above statement, $S/\approx$ is a discrete group, so Theorem 2.1 implies that $\ell^1(S/\approx)$ is $2n$-weakly module amenable. Now by applying Theorem 2.3, $\ell^1(S)$ is $2n$-weakly module amenable. \( \square \)

It is well known that amenability of $\ell^1(S)$ implies amenability of $S$ (see [11, Theorem 5.6.1]). In general, $\ell^1(S)$ is not even weakly amenable if $S$ is amenable (a concrete example is bicyclic inverse semigroup). However, if $S$ is inverse semigroup, then $\ell^1(S)$ is amenable if and only if $S$ has only finitely many idempotents and every subgroup of $S$ is amenable (see [13]). In the case that $S$ is commutative:

(i) $\ell^1(S)$ is amenable if and only if $S$ is a finite semilattice of amenable groups (see [14]);

(ii) If every element of $S$ is idempotent, then $\ell^1(S)$ is spanned by its idempotents and so it is weakly amenable by [11, Proposition 2.8.72].

Let $S$ be a Clifford semigroup (an inverse semigroup whose idempotents are central). Then $H^1(\ell^1(S), \ell^\infty(S)) = \{0\}$ [9, Theorem 2.1], but $H^1_{\ell^1(E)}(\ell^1(S), \ell^1(S))$ is zero in general. We also have $H^1(\ell^1(S), \ell^\infty(S)) = \{0\}$ when $S$ has only finitely many idempotents [9, Theorem 3.2]. Consider $\ell^1(S)$ as a $\ell^1(E)$-module with the following action:

$$\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_e * \delta_e = \delta_{se}. \quad (2.1)$$

Since every idempotent commutes with the elements of $S$, the proof of [8, Theorem 3.15] shows that $\ell^1(S)$ is an essential $\ell^1(E)$-module. Now, it follows from the proof of [5, Theorem 3.14] that every module derivation $D$ from $\ell^1(S)$ into $\ell^\infty(S)$ is inner and thus we have the following result:

**Theorem 2.5** If $S$ is a Clifford semigroup, then $\ell^1(S)$ is weakly module amenable.

It is proved in [3, Theorem 3.1] that $\ell^1(S)$ is $\ell^1(E)$-weakly module amenable with the actions (2.1) when $S$ is commutative. Note that in the proof of this result the commutativity of $S$ is not needed and the same proof works if each idempotent is central.

Let $C$ be the bicyclic inverse semigroup generated by $a$ and $b$, that is $C = \{a^mb^n : m, n \geq 0\}, (a^mb^n)^* = a^nb^m, E_C = \{a^nb^n : n = 0, 1, ...\}$. It is shown in [2] that $\ell^1(C)$ is $\ell^1(E_C)$-module amenable, and so it is weakly module amenable. Now, it follows from [9, Theorem 3.6] and the above discussions that $H^1_{\ell^1(E)}(\ell^1(C), \ell^1(C)) = \{0\}.$

The Brandt inverse semigroup corresponding to group $G$ and non-empty set $I$ is denoted by $S = M(G, I)$ which is the collection of all $I \times I$ matrices $(g)_{jk}$ with $g \in G$ in the $(j,k)^{th}$ place and zero elsewhere and the $I \times I$ zero matrix 0. It is shown in [21, Example 3.2] that $\ell^1(S)$ is super module amenable (as $\ell^1(E)$-module) and so $H^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)) = \{0\}.$
Lemma 2.6 Let $A$ be a essential bi-commutative $A\mathfrak{A}$-module. Then $A$ is weakly module amenable (as an $\mathfrak{A}$-module) if and only if for each bi-commutative Banach $A\mathfrak{A}$-module $X$, all bounded module derivations from $A$ into $X$ are zero.

Proof. We follow the standard argument in [11, Theorem 2.8.63]. Assume that there exists $D \in Z_{\mathfrak{A}}(A, X)$ with $D \neq 0$. Since $\mathfrak{A}^2 = \mathfrak{A}$, there exists $a_0 \in A$ such that $D(a_0^2) \neq 0$. We have $a_0 \cdot D(a_0) \neq 0$ and thus $f \in X^*$ with $f(a_0 \cdot D(a_0)) = 1$. Set $R : X \to A^*$ defined by $R(x)(a) = f(a \cdot x)$ where $a \in A, x \in X$. It is easy to check that $R \circ D \in Z_{\mathfrak{A}}(A, A^*)$. We get $\langle R \circ D(a_0), a_0 \rangle = \langle f, a_0 \cdot D(a_0) \rangle = 1$, and so $R \circ D \neq 0$. This shows that $\mathfrak{A}$ is not weakly module amenable. The converse is clear. \hfill $\Box$

Theorem 2.7 Let $n \in \mathbb{N}$ and let $S$ be a commutative inverse semigroup with the set of idempotents $E$. Then $\ell^1(S)$ is $n$-weakly module amenable as an $\ell^1(E)$-module with the actions (2.1).

Proof. For any semigroup $S$, the semigroup algebra $\ell^1(S)$ is commutative if and only if $S$ is commutative. Since $\ell^1(S)$ is a bi-commutative Banach $\ell^1(S)$-$\ell^1(E)$-module, so is $\ell^1(S)^{(n)}$. By [3, Theorem 3.1], $\ell^1(S)$ is weakly module amenable as an $\ell^1(E)$-module. The semigroup algebra $\ell^1(S)$ is essential, in fact $\ell^1(S) = \ell^1(S)\ast \ell^1(E) \subseteq \ell^1(S)\ast \ell^1(S) \subseteq \ell^1(S)$ (see the proof of [8, Theorem 3.15]). Now, it follows from Lemma 2.6 that every module derivation from $\ell^1(S)$ into $\ell^1(S)^{(n)}$ is zero. This shows that $\ell^1(S)$ is $n$-weakly module amenable. \hfill $\Box$

Corollary 2.8 Let $S$ be a commutative inverse semigroup with the set of idempotents $E$. Then $H^1_{\ell^1(E)}(\ell^1(S), \ell^1(S)) = \{0\}$.

3 Module amenability and weak module amenability

Let $A$ and $B$ be Banach algebras and Banach $\mathfrak{A}$-bimodules. Consider the Banach space $A\otimes_{\mathfrak{A}} B = A \otimes B/N$, where $N$ is the closed linear span of $\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b \mid \alpha \in \mathfrak{A}, a \in A, b \in B\}$. By the following actions $A\otimes_{\mathfrak{A}} B$ becomes a Banach $\mathfrak{A}$-bimodule

$$\alpha \cdot (a \otimes \beta) = \alpha \cdot a \otimes \beta$$

and

$$(a \otimes \beta) \cdot \alpha = a \otimes \alpha \cdot \beta$$

for every $\alpha \in \mathfrak{A}, a \in A$ and $b \in B$. By the above actions $N$ becomes $\mathfrak{A}$-submodule of $A\otimes_{\mathfrak{A}} B$. We define multiplication on $A\otimes_{\mathfrak{A}} B$ by usual algebraic tensor product on rings and modules as follows:

$$(a \otimes b)(c \otimes d) = (a \otimes b + N)(c \otimes d + N) = ac \otimes bd + N = ac \otimes_{\mathfrak{A}} bd,$$

for every $a, c \in A$ and $b, d \in B$. Also, $A\otimes_{\mathfrak{A}} B$ is a Banach $A$-bimodule by following actions

$$a \cdot (a' \otimes b') = aa' \otimes b \quad \text{and} \quad (a' \otimes b') \cdot a = a'a \otimes b', \quad (3.1)$$

for every $a, a' \in A$ and $b, b' \in B$. Similarly we can see $A\otimes_{\mathfrak{A}} B$ as a Banach $B$-bimodule. We call $A\otimes_{\mathfrak{A}} B$ the module projective tensor product of $A$ and $B$.

Module amenability of module projective tensor product of Banach algebras and semigroup algebras are studied in [4] and [7], respectively. We say that $X$ is a Banach $A\mathfrak{B}$-$\mathfrak{A}$-module if $X$ is a Banach $A\mathfrak{A}$-module and a Banach $B\mathfrak{A}$-module.
Theorem 3.1 Let $A$ and $B$ be unital Banach $\mathfrak{A}$-modules. If $A$ and $B$ are module amenable then every continuous module derivation from $A \widehat{\otimes}_\mathfrak{A} B$ into $X^*$ is inner, where $X$ is a Banach $A-B-\mathfrak{A}$-module.

Proof. Let $D : A \widehat{\otimes}_\mathfrak{A} B \to X^*$ be a continuous module derivation. Set $A_1 = A \widehat{\otimes}_\mathfrak{A} e_B$ and $B_1 = e_A \widehat{\otimes}_\mathfrak{A} B$. Then the map $D_1 : A \to X^*$ defined by $D_1(a) = D(a \otimes e_B + N)$ is a continuous module derivation, hence there exists $f \in X^*$ such that $D(a \otimes e_B + N) = (a \otimes e_B + N) \cdot f - f \cdot (a \otimes e_B + N) = D_f(a \otimes e_B + N), (a \in A)$.

It is clear that $D - D_f|_{A_1} = 0$. Therefore for every $a \in A$ and $b \in B$ we have

$$(D - D_f)(a \otimes b + N) = (D - D_f)((a \otimes e_B + N)(e_A \otimes b + N))$$

$$= (a \otimes e_B + N) \cdot (D - D_f)(e_A \otimes b + N)$$

$$+ (D - D_f)(a \otimes e_B + N) \cdot (e_A \otimes b + N)$$

$$= (a \otimes e_B + N) \cdot (D - D_f)(e_A \otimes b + N)$$

$$= (D - D_f)((e_A \otimes b + N)(a \otimes e_B + N))$$

$$= (e_A \otimes b + N) \cdot (D - D_f)(a \otimes e_B + N)$$

$$+ (D - D_f)(e_A \otimes b + N) \cdot (a \otimes e_B + N)$$

$$= (D - D_f)(e_A \otimes b + N) \cdot (a \otimes e_B + N).$$

Thus, (3.2) and (3.3) imply that $(a \otimes e_B + N) \cdot (D - D_f)(e_A \otimes b + N) = (D - D_f)(e_A \otimes b + N) \cdot (a \otimes e_B + N)$, for every $a \in A$ and $b \in B$. Let $Y$ be the annihilator of $(D - D_f)(e_A \otimes b + N)$ in $X$ then $Y$ is a $B$-submodule. By relations (3.2) and (3.3) we have $((a \otimes e_B + N) \cdot y - y \cdot (a \otimes e_B + N), (D - D_f)(e_A \otimes b + N)) = 0$, for every $y \in Y, a \in A$ and $b \in B$. By a similar argument $X/Y$ is a Banach $\mathfrak{A}$-$\mathfrak{A}$-bimodule. Since the restriction of $D - D_f$ to $B_1$ defines a continuous module derivation from $B$ into $Y^* \subseteq X^*$ and since $B$ is module amenable, there is $g \in Y^*$ such that $D - D_f = D_g$ and $D - D_f - D_g|_{B_1} = 0$. Since $(A \widehat{\otimes}_{\mathfrak{A}} e_B) \cup (e_A \widehat{\otimes}_{\mathfrak{A}} B)$ generates $A \widehat{\otimes}_{\mathfrak{A}} B$, $D - D_f - D_g|_{A \widehat{\otimes}_{\mathfrak{A}} B} = 0$. This shows that $D = D_f + D_g$, and so $D$ is inner. □

Corollary 3.2 Let $A$ and $B$ be unital commutative Banach algebras and unital Banach $\mathfrak{A}$-modules. If $A$ and $B$ are module amenable then $A \widehat{\otimes}_{\mathfrak{A}} B$ is module amenable.

Proof. Let $D : A \widehat{\otimes}_{\mathfrak{A}} B \to X^*$ be a continuous module derivation, where $X$ is a Banach $A \widehat{\otimes}_{\mathfrak{A}} B-\mathfrak{A}$-module. Since $A$ and $B$ are commutative, $X$ is a Banach $A-B-\mathfrak{A}$-module. Similar to the proof of Theorem 3.1, set $A_1 = A \widehat{\otimes}_{\mathfrak{A}} e_B$ and $B_1 = e_A \widehat{\otimes}_{\mathfrak{A}} B$. Then the mapping $D_1 : A \to X^*$ defined by $D_1(a) = D(a \otimes e_B + N)$ is a continuous module derivation. Then $D|_{A_1} = 0$. Therefore for every $a \in A$ and $b \in B$ we have

$$(D(a \otimes b + N) = D((a \otimes e_B + N)(e_A \otimes b + N))$$

$$= (a \otimes e_B + N) \cdot D(e_A \otimes b + N)$$

$$+ D(a \otimes e_B + N) \cdot (e_A \otimes b + N)$$

$$= (a \otimes e_B + N) \cdot D(e_A \otimes b + N)$$

$$= D(e_A \otimes b + N)(a \otimes e_B + N))$$

$$= (e_A \otimes b + N) \cdot D(a \otimes e_B + N)$$

$$+ D(e_A \otimes b + N) \cdot (a \otimes e_B + N)$$

$$= D(e_A \otimes b + N) \cdot (a \otimes e_B + N).$$
Then (3.4) and (3.5) imply that \((a \otimes e_B + N) \cdot D(e_A \otimes b + N) = D(e_A \otimes b + N) \cdot (a \otimes e_B + N)\) for every \(a \in A\) and \(b \in B\). Let \(Y\) be the annihilator of \(D(e_A \otimes b + N)\) in \(X\) then \(Y\) is a \(B\)-submodule. By relations (3.4) and (3.5) we have \(\langle \langle (a \otimes e_B + N) \cdot y - y \cdot (a \otimes e_B + N), D(e_A \otimes b + N) \rangle \rangle = 0\), for every \(y \in Y\), \(a \in A\) and \(b \in B\). Since \(X/Y\) is a Banach \(B\)-module, and the restriction of \(D\) to \(B_{1}\) defines a continuous module derivation from \(B\) into \(Y^{*}\), we have \(D|_{B_{1}} = 0\). Therefore \(D|_{A \otimes A B} = 0\). □

Replacing \(A \otimes A B\) by \(X\) in the Corollary 3.2, and viewing \(A \otimes A B\) as a Banach \(A\)-\(B\)-\(A\)-module (relations (3.1)), we have the following result:

**Corollary 3.3** Let \(A\) and \(B\) be unital commutative Banach algebras and unital Banach \(A\)-modules. If \(A\) and \(B\) are weakly module amenable then \(A \otimes A B\) is weakly module amenable.

Next we study weak module amenability of module projective tensor products of Banach algebras.

**Theorem 3.4** Let \(A\) be a unital Banach \(A\)-bimodule and \(\varphi : A \to A\) be a continuous surjective map. Suppose that \(ab = \varphi(a) \cdot b\), for each \(a, b \in A\). Then \(A\) is weakly module amenable.

**Proof.** Let \(D : A \to A^{*}\) be a continuous module derivation. Then
\[
\varphi(a) \langle c, Db \rangle = \langle c, D(ab) \rangle = \langle c, a \cdot D(b) + D(a) \cdot b \rangle = \langle ca, Db \rangle + \langle bc, Da \rangle = \varphi(c) \langle a, Db \rangle + \varphi(b) \langle c, Da \rangle,
\]
for each \(a, b, c \in A\). Let \(\lambda \in A^{*}\), and let \(\delta_{\lambda} : A \to A^{*}\) be the inner derivation specified by \(\lambda\). Hence
\[
\langle b, \delta_{\lambda}(a) \rangle = \langle b, a \cdot \lambda \cdot \lambda \cdot a \rangle = \langle ba, \lambda \rangle - \langle ab, \lambda \rangle = \varphi(b) \langle a, \lambda \rangle - \varphi(a) \langle b, \lambda \rangle,
\]
for each \(a, b \in A\). Choose \(a_{0} \in A\) with \(\varphi(a_{0}) = e_{A}\), and set \(\lambda(a) = \langle a_{0}, Da \rangle\) for each \(a \in A\). Clearly \(\lambda\) is linear. Using (3.6) and (3.7) we have \(\langle b, \delta_{\lambda}(a) \rangle = \varphi(b) \langle a, \lambda \rangle - \varphi(a) \langle b, \lambda \rangle = \varphi(b) \langle a_{0}, Da \rangle - \varphi(a) \langle a_{0}, Db \rangle = \varphi(a_{0}) \langle b, Da \rangle = \langle b, Da \rangle\), therefore \(D = \delta_{\lambda}\), and so \(A\) is weakly module amenable. □

**Example 3.1** Let \(\mathbb{N}\) be set of the positive integers. We can see it as a semigroup by denoting the product of two elements to be their maximum. The resulting semigroup, which we denote by \(\mathbb{N}\), is a semilattice. The semilattice \(\mathbb{N}\), is a commutative semigroup in which every element is idempotent. If we denote the set of idempotent elements of \(\mathbb{N}\) by \(E(\mathbb{N})\), then \(E(\mathbb{N}) = \mathbb{N}\). We may then form the \(\ell^{1}\)-convolution algebra \(\ell^{1}(\mathbb{N})\). For every \(t \in \mathbb{N}\) we denote the point mass concentrated at \(t\) by \(\delta_{t}\). The definition of multiplication in \(\ell^{1}(\mathbb{N})\) ensures that \(\delta_{n} \ast \delta_{m} = \delta_{\max\{n,m\}}\) for all \(m, n \in \mathbb{N}\). Consider \(\ell^{1}(\mathbb{N})\) as a Banach \(\ell^{1}(\mathbb{N})\)-bimodule. Define \(\varphi : \ell^{1}(\mathbb{N}) \to \ell^{1}(\mathbb{N})\) by \(\varphi(\delta_{n}) = \delta_{1}\). Then \(\delta_{n} \ast \delta_{m} = \delta_{m} \ast \delta_{n} = \varphi(\delta_{n}) \ast \delta_{\beta}\), where \(\alpha = \min\{m, n\}\) and \(\beta = \max\{m, n\}\). Then by Theorem 3.4, \(\ell^{1}(\mathbb{N})\) is weakly module amenable.

**Theorem 3.5** Let \(A\) and \(B\) be unital commutative Banach algebras and unital Banach \(A\)-modules, \(\varphi : A \to A\) and \(\psi : B \to A\) be continuous surjective maps. Suppose that \(ab = \varphi(a) \cdot b\), \(cd = \psi(c) \cdot d\), for each \(a, b \in A\) and \(c, d \in B\). Then \(A \otimes A B\) is weakly module amenable.
Proof. Let \( e_A \) and \( e_B \) be the unit elements of \( A \) and \( B \), respectively. Let \( D : A \hat{\otimes}_\mathbb{Q} B \rightarrow (A \hat{\otimes}_\mathbb{Q} B)^* \) be a continuous module derivation. For each \( a, c, e \in A \) and \( b, d, f \in B \) we have

\[
\langle c \otimes d + N, D(\lambda D e + N) + (e \otimes f + N) \rangle = \langle c \otimes d + N, D((a \otimes b + N)(e \otimes f + N)) \rangle
\]

\[
= \langle e \otimes d + N, (a \otimes b + N) \cdot D(e \otimes f + N) \rangle
\]

\[
+ \langle e \otimes d + N, D(a \otimes b + N) \cdot (e \otimes f + N) \rangle
\]

\[
= \langle ca \otimes db + N, D(e \otimes f + N) \rangle + \langle ec \otimes fd + N, D(a \otimes b + N) \rangle
\]

\[
= \varphi(c)\psi(d)(a \otimes b + N, D(e \otimes f + N)) + \varphi(e)\psi(f)(c \otimes d + N, D(a \otimes b + N))
\]

\[
= \varphi(a)\psi(b)(c \otimes d + N, D(e \otimes f + N)).
\]

Fix \( b_0 \in B \) with \( \psi(b_0) = e_\mathbb{Q} \). Then by (3.8), we may write

\[
\langle e_A \otimes e_B + N, D(a \otimes b + N) \rangle = \langle e_A \otimes e_B + N, D((a \otimes b_0 + N)(e_A \otimes b + N)) \rangle
\]

\[
= \varphi(a)\psi(b_0)(e_A \otimes e_B + N, D(e_A \otimes b + N))
\]

\[
= \langle a \otimes b_0 + N, D(e_A \otimes b + N) \rangle.
\]

for each \( a \in A \) and \( b \in B \). Hence there exists \( \lambda \in (A \hat{\otimes}_\mathbb{Q} B)^* \) such that

\[
\langle a \otimes b + N, \lambda \rangle = \langle a \otimes b_0 + N, D(e_A \otimes b + N) \rangle,
\]

(3.10) for each \( a \in A \) and \( b \in B \). Let \( \delta_\lambda : A \hat{\otimes}_\mathbb{Q} B \rightarrow (A \hat{\otimes}_\mathbb{Q} B)^* \) be the inner module derivation induced by \( \lambda \). Take \( a \in A \) and \( b, c \in B \). Then \( \langle a \otimes c + N, (e_A \otimes b + N) \cdot \lambda - \lambda \cdot (e_A \otimes b + N) \rangle = \langle a \otimes c + N, (e_A \otimes b + N) \cdot \lambda - (a \otimes c + N, \lambda \cdot (e_A \otimes b + N) \rangle = \langle a \otimes b_0 + N, \psi(c)D(e_A \otimes b + N) \rangle - \langle a \otimes b_0 + N, \lambda \cdot (e_A \otimes b + N) \rangle = \langle a \otimes b_0 + N, \psi(c)D(e_A \otimes b + N) \rangle - \langle a \otimes b_0 + N, \lambda \cdot (e_A \otimes b + N) \rangle = \langle a \otimes b_0 + N, \psi(c)D(e_A \otimes b + N) \rangle - \langle a \otimes b_0 + N, \lambda \cdot (e_A \otimes b + N) \rangle = \langle a \otimes b_0 + N, \psi(c)D(e_A \otimes b + N) \rangle
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It is shown in [1, Lemma 3.1] that if \( \ell^1(E) \) acts on \( \ell^1(S) \) by multiplication from right and trivially from left, then \( \ell^1(S) \hat{\otimes} \ell^1(E) \ell^1(S) \cong \ell^1(S \times S)/I \), where \( I \) is the closed ideal of \( \ell^1(S \times S) \) generated by the set of elements of the form \( \delta_{(st,u)} - \delta_{(st,u)} \), where \( s, t, u \in S \) and \( e \in E \).

**Corollary 3.6** Let \( S \) be commutative and unital inverse semigroup with the set of idempotents \( E \). Then \( \ell^1(S) \hat{\otimes} \ell^1(E) \ell^1(S) \) is weakly module amenable.

**Proof.** The result follows from [3, Theorem 3.1] and Corollary 3.3. \( \square \)

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**References**