Generalized outer synchronization of stochastic neural networks with time-varying delays

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Abstract In this paper, generalized outer synchronization between two different stochastic coupled complex dynamical networks with time-varying delays has been investigated. A novel controller is given and the stochastic invariance principle is applied. A stochastic disturbance which is described in term of a Brownian motion are considered in complex dynamical networks. Moreover, some sufficient conditions are derived to ensure generalized outer synchronization of stochastic neural networks. Surprisingly, it is found that complex networks with different structure can be synchronized.

Keywords complex networks · generalized outer synchronization · time-varying delays

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1 Introduction

The synchronization problem of complex networks has been successfully applied to physics, engineering, biology and medical science, etc. Recently, network synchronization has been extensively investigated see [5,6,13,14]. A complex network consists of a large number of nodes and the connections between them, while these nodes may have different meanings in different situations. However, the network structure facilitates and constrains the network dynamical behaviors. Chaos synchronization has played a significant role in nonlinear science. As a special kind of complex networks, there has been an increasing interest in artificial neural networks due to their fruitful applications in numerous areas [2-27].

Different concepts of synchronization, like complete synchronization, generalized synchronization, phase synchronization, lag synchronization and anticipated synchronization, have been widely investigated. The synchronization within one network is named “inner synchronization” [10], which is concerned with the synchronization among the nodes within a network. Different from the “inner synchronization”, the
synchronization between two or more complex networks is called “outer synchronization”, which was firstly studied by Li et al. [10]. In [10], the authors studied the complete outer synchronization problem for two complex networks with identical topological structures by using an open-plus-closed-loop controller (OPCL). Later on, the outer synchronization for two coupled complex networks was extended to the discrete time case (see [11]).

In real world, due to random uncertainties such as stochastic forces on the physical systems and noisy measurements caused by environmental uncertainties. Subsequently, the stochastic synchronization problem for complex networks has begun to attract some research interests. Stochastic synchronization is the process by which noisy phase synchronization of two periodic (or aperiodic) signals can occur. Stochastic resonance (SR) is an ubiquitous phenomenon found in both natural and man-made systems spanning a range of disciplines from physics and engineering to biology and medical science. The signal transfer with in complex networks could be perturbed randomly from the release of probabilistic causes such as neurotransmitters and packet dropouts (for example, the signal disturbance in the network control, the packet dropouts in the network transmission, see [22,26]). When analyzing the dynamical behaviors of complex networks, the obtained results are often largely affected by the stochastic disturbances. In [22,26], the stochastic synchronization problems have been intensively investigated for delayed complex networks with stochastic disturbances. Time delays usually exist in spreading due to the finite speeds of transmission as well as traffic congestion. Time-delay system is frequently encountered in many areas and a time-delay is often a source of instability and oscillators. However, to the best of our knowledge, there are few results on the stochastic neural networks with discrete and time-varying coupling delays, and the coupling terms of the models being studied are always linear (see [3,22,26]). Therefore, the stochastic synchronization problem for time-varying delayed neural networks has not been fully investigated, and there still exists much room for further research.

Motivated by the above discussion, this paper investigates the generalized outer problem of stochastic neural networks, which consider time-varying delays. A nonlinear coupling scheme is insteaded of the common linear coupling scheme. Adaptive controllers are designed for the global outer synchronization. Two networks may have nonidentical topological structures and different node dynamics.

## 2 Network modeling and preliminaries

In this paper, $x_i(t) = (x_{i1}(t),\ldots,x_{in}(t))^T \in \mathbb{R}^n$, $y_i(t) = (y_{i1}(t),\ldots,y_{in}(t))^T \in \mathbb{R}^n$ are state vectors, $f,g: \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable functions. $A, P, Q, M$ and $N$ are inner connection matrices between two connected nodes. $A = (a_{ij})_{N \times N}$, $B = (b_{ij})_{N \times N}$, $C = (c_{ij})_{N \times N}$ and $D = (d_{ij})_{N \times N}$ represent the coupling configurations of both networks, in which $a_{ij} > 0$, $b_{ij} > 0$, $c_{ij} > 0$ and $d_{ij} > 0$ if there is a link from node $j$ to node $i$ ($i \neq j$), and otherwise $(a_{ij} = 0), (b_{ij} = 0), (c_{ij} = 0)$ and $(d_{ij} = 0)$ $i \neq j$, and $a_{ii} = -\sum_{j=1,j \neq i}^{N} a_{ij}$, $b_{ii} = -\sum_{j=1,j \neq i}^{N} b_{ij}$, $c_{ii} = -\sum_{j=1,j \neq i}^{N} c_{ij}$, $d_{ii} = -\sum_{j=1,j \neq i}^{N} d_{ij}$.

In [21], Tang, Chen, Lu and Tse studied the adaptive synchronization problem between two coupled complex networks. The two complex dynamical networks can be
expressed as follows:

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} P x_j(t), \quad i = 1, 2, \ldots, N, \quad (2.1)
\]

\[
\dot{y}_i(t) = f(y_i(t)) + \sum_{j=1}^{N} d_{ij} Q y_j(t) + u_i(t), \quad i = 1, 2, \ldots, N. \quad (2.2)
\]

And following \( u_i \) is the controller designed for node \( i \):

\[
u_i = \sum_{j=1}^{N} b_{ij} P y_j - g_i e_i, \quad \dot{g}_i = k_i |e_i|^2, \quad \dot{b}_{ij} = -e^T_i A y_j, \quad (2.3)
\]

where \( e_i = y_i - x_i \) is the synchronization error for node \( i \) between drive network and response network.

In [24], the generalized outer synchronization of the following networks with non-identical topological structures was investigated

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} P x_j(t), \quad i = 1, 2, \ldots, N, \quad (2.4)
\]

\[
\dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^{N} d_{ij} Q y_j(t) + u_i(t), \quad i = 1, 2, \ldots, N, \quad (2.5)
\] is the controller designed for node \( i \).

**Definition 1.** Let \( \Phi_i : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( i = 1, 2, \ldots, N \) be continuously differentiable function. A network (2.5) is said to achieve generalized outer synchronization with network (2.4). If

\[
\lim_{t \to \infty} \|y_i(t) - \Phi_i(x_i(t))\| = 0, \quad i = 1, 2, \ldots, N. \quad (2.6)
\]

In [18], the generalized outer synchronization between two delay-coupled complex dynamical networks was investigated, which may have nonidentical topological structures and different node dynamics. The complex networks are described by

\[
\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} P x_j(t - \tau), \quad i = 1, 2, \ldots, N, \quad (2.7)
\]

\[
\dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^{N} d_{ij} Q y_j(t - \tau) + u_i(t)
+ \sigma_i(e_i, e_i(t - \tau), t) W, \quad i = 1, 2, \ldots, N, \quad (2.8)
\]

where \( \sigma_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^{n \times m} \) is called the noisy intensity matrix, \( W = (w_1, \ldots, w_m)^T \) be an m-dimensional Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, P)\) with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). \( \dot{W} \) is an m-dimensional white noise.
Furthermore, the authors in [18] consider the dynamics of neural networks with delays. It is well known that the delays in neural networks are usually time-varying in the real world. Unfortunately, few authors have considered the generalized outer synchronization of neural networks with time-varying delays. Motivated by the above, in this present paper, we shall consider the generalized outer synchronization of the following stochastic neural networks with time-varying delays.

\[ \dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} P x_j(t - \tau(t)) + \sum_{j=1}^{N} a_{ij} M f_j(x_j(t - \tau(t))), i = 1, 2, \ldots, N, \tag{2.9} \]

\[ \dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^{N} d_{ij} Q y_j(t - \tau(t)) + \sum_{j=1}^{N} b_{ij} N g_j(y_j(t - \tau(t))) + u_i(t) + \sigma_i(e_i, e_i(t - \tau(t)), t) \hat{W}, i = 1, 2, \ldots, N. \tag{2.10} \]

And following \( u_i \) is the adaptive controller designed for node \( i \):

\[ u_i = D\dot{\Phi}_i(x_i) \dot{x}_i - g(\Phi_i(x_i)) - \sum_{j=1}^{N} d_{ij} Q \Phi_j(x_j(t - \tau(t))) - \sum_{j=1}^{N} b_{ij} N g_j(\Phi_j(x_j(t - \tau(t)))) - ke_i, \tag{2.11} \]

where \( D\dot{\Phi}_i(x_i) \) is the Jordan matrix of the vector function \( \dot{\Phi}_i(x_i) \), \( e_i = y_i(t) - \Phi_i(x_i(t)) \) is the synchronization error, and \( k \) is a sufficiently large positive constant.

To establish our main results, it is necessary to make the following assumptions:

(A1): For function \( g(x) \) there exists a positive constant \( l \) such that

\[ |x(t) - y(t)|^2 |g(x(t)) - g(y(t))| \leq l|x(t) - y(t)|^2|x(t) - y(t)|, x, y \in \mathbb{R}^n. \]

(A2): The noise intensity function \( \sigma_i(x, y, t)(i = 1, 2, \ldots, N) \) satisfies the Lipschitz condition and there exists positive constants \( p, q \) such that

\[ \text{trace}(\sigma_i^T \sigma_i) \leq px^T x + qy^T y. \]

(A3): Each function \( g_i : \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable functions with a constant \( c > 0 \), i.e.

\[ |g_i(u_i) - g_i(v_i)| \leq c|u_i - v_i|, \forall u_i, v_i \in \mathbb{R}^n, i = 1, 2, \ldots, n. \tag{2.12} \]

Moreover \( \sigma(0, 0, t) \equiv 0 \)

(A4): \( \tau(t) \) is a bounded differential function of time \( t \), and the following conditions are satisfied:

\[ r = \max_{t \in \mathbb{R}} \{\tau(t)\}, 0 \leq \dot{\tau}(t) \leq h, \]
where \( r \) and \( h \) are positive constants.

Consider a general \( n \)-dimensional stochastic differential delay equation:

\[
dx(t) = \Phi(t, x(t), x(t - \tau)) \, dt + \psi(t, x(t), x(t - \tau)) \, dw
\]  

(2.13)
on \( t \geq 0 \) with initial condition \( \xi \in C^b_{F_0}([-\tau, 0]; \mathbb{R}^n) \), where \( w(t) = (w_1(t), \ldots, w_n(t))^T \) is an \( n \)-dimensional Brownian motion defined on the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions. By assuming that \( \Phi \) and \( \psi \) both satisfy the local Lipschitz condition and the linear growth condition, MAO [16] established an invariance principle for stochastic differential delay equation (2.13), as follows:

**Lemma 1.** Assume that there are functions \( V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n; \mathbb{R}_+), \gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+) \), and \( \omega_1, \omega_2 \in C(\mathbb{R}^n; \mathbb{R}_+) \) such that

\[
\mathcal{L}V(t, x, y) := V_t(t, x) + V_x(t, x)\Phi(t, x, y) + \frac{1}{2} \text{trace}[\Psi^T(t, x, y)V_{xx}(t, x)\Psi(t, x, y)] \\
\leq \gamma(t) - \omega_1(x) + \omega_2(y), (t, x, y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \omega_1(x) \geq \omega_2(x), x \in \mathbb{R}^n
\]  

(2.14)

and \( \lim_{\|x\| \to \infty} \inf_{0 \leq t < \infty} V(t, x) = \infty. \) Then, for every \( \xi \in C^b_{F_0}([-\tau, 0]; \mathbb{R}^n), \lim_{t \to \infty} [\omega_1(x(t; \xi)) - \omega_2(x(t; \xi))] = 0, \ a.s. \)

Moreover, if \( Ker(\omega_1 - \omega_2) = \{0\}, \) then for every \( \xi \in C^b_{F_0}([-\tau, 0]; \mathbb{R}^n), \lim_{t \to \infty} x(t; \xi) = 0, \ a.s \)

### 3 Time-varying stochastic synchronization of neural networks

In this section, new criteria are presented for the generalized outer synchronization of the time-varying delays stochastic neural networks defined by (2.9) and (2.10).

**Theorem 1.** Let \( (A_1), (A_2), (A_3) \) and \( (A_4) \) hold, and under adaptive controller (11), there exists a sufficiently large positive constant \( k \) such that the response network (2.10) can almost surely achieve generalized outer synchronization with the drive network (2.9).

**Proof.** The synchronization error between networks (2.9) and (2.10) can be written as:

\[
\dot{e}(t) = g(y(t)) - g(\Phi(t, x)) + \sum_{j=1}^{N} d_{ij} Q e_j(t - \tau(t)) \\
+ \sum_{j=1}^{N} b_{ij} N \{g(y(t - \tau(t))) - g(\Phi(t, x)) - g(\Phi(t, y(t - \tau(t))))\} \\
- ke_i + \sigma_i(e_i, e_i(t - \tau(t)), t)W, \quad i = 1, 2, \ldots, N.
\]  

(3.1)
It can be observed that the stability of the zero solution of synchronization error (3.1) implies the generalized outer synchronization of networks (2.9) and (2.10). Let \( e = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T \). Consider the following Lyapunov functional:

\[
V = \sum_{i=1}^{2} V_i(t),
\]

where

\[
V_1 = e^T e = \sum_{i=1}^{N} e_i^T e_i,
\]

\[
V_2 = \int_{t-\tau(t)}^{t} e^T(s)e(s)ds.
\]

Thus the diffusion operator \( \mathcal{L}V \) defined in (2.14) onto the function \( V \) along the error system (3.1) gives

\[
\mathcal{L}V_1 = 2\sum_{i=1}^{N} e_i^T \{ g(y_i(t)) - g(\Phi_i(x_i)) \} + 2\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} e_i^T Q e_j(t-\tau(t))
- 2k \sum_{i=1}^{N} e_i^T e_i; 2 \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} e_i^T N \{ g_j(y_j(t-\tau(t))) \}
- g_j(\Phi_j(x_j(t-\tau(t)))) + \sum_{i=1}^{N} \text{trace}(\sigma_i^T \sigma_i).
\]

From assumptions \( (A_1), (A_2) \) and \( (A_3) \), we have

\[
\mathcal{L}V_1 \leq (2l + p - 2k) e^T(t)e(t) + 2e^T(t)(D \otimes Q)e(t-\tau(t))
+ 2ce^T(t)(B \otimes N)e(t-\tau(t)) + qe^T(t-\tau(t))e(t-\tau(t)).
\]

Noting that

\[
2e^T(t)(D \otimes Q)e(t-\tau(t)) \leq e^T(t)\mathcal{E}e(t) + e^T(t-\tau(t))e(t-\tau(t)),
\]

where \( \mathcal{E} = D \otimes Q \), and

\[
2e^T(t)(B \otimes N)e(t-\tau(t)) \leq e^T(t)\mathcal{F}e(t) + e^T(t-\tau(t))e(t-\tau(t)),
\]

where \( \mathcal{F} = B \otimes N \). Thus

\[
\mathcal{L}V_1 \leq [2l + p + \lambda_{\max}(\mathcal{M}_1) + c \cdot \lambda_{\max}(\mathcal{M}_2) - 2k] e^T(t)e(t)
+ (1 + c + q)e^T(t-\tau(t))e(t-\tau(t)).
\]

Where \( \mathcal{M}_1 = \mathcal{E}e^T \) and \( \mathcal{M}_2 = \mathcal{F}e^T \).

With the assumption \( (A_4) \), we get

\[
\mathcal{L}V_2 = e^T(t)e(t) - [1 - \hat{\tau}(t)] e^T(t-\tau(t))e(t-\tau(t)) \leq e^T(t)e(t).
\]
From (3.9) and (3.10), it is obvious that
\[
\mathcal{L}V \leq -\left[2k - 2l - p - \lambda_{\text{max}}(M_1) - c \cdot \lambda_{\text{max}}(M_2) - 1\right]e^T(t)e(t) \\
+ (1 + c + q)e^T(t - \tau(t))e(t - \tau(t)) \\
\triangleq -\omega_1(e(t)) + \omega_2(e(t - \tau(t))).
\] (3.11)

If
\[
k > l + 1 + \frac{1}{2}\left[2k - 2l - p - \lambda_{\text{max}}(M_1) - c \cdot \lambda_{\text{max}}(M_2)\right],
\] (3.12)
then we have
\[
\omega_1(e) > \omega_2(e), \forall e \neq 0,
\] (3.13)
which thus implies Ker\{\omega_1(e) - \omega_2(e)\} = 0. Moreover,
\[
\lim_{\|e\| \to \infty} \inf_{0 \leq t < \infty} V(t,e) = \infty.
\]

By Lemma 1, then for every \(\xi \in C^b_{\tau_0}([-\tau,0];\mathbb{R}^n)\), we have
\[
\lim_{t \to \infty} e(t) \to 0, \text{ a.s.}
\] (3.14)

Hence the almost sure generalized outer synchronization between the drive network (2.9) and the response network (2.10) can be realized using the adaptive controller (2.11). This completes the proof of Theorem 1.

Where \(\sigma_i\) is a zero matrix, the networks (2.10) turn out to be the following complex network without stochastic perturbation.
\[
\dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^{N} d_{ij}Qy_j(t - \tau(t)) + \sum_{j=1}^{N} b_{ij}Ng_j(y_j(t - \tau(t))) \\
+ u_i(t), i = 1, 2, \ldots, N.
\] (3.15)

**Corollary 1.** Let \((A_1), (A_3)\) and \((A_4)\) hold, and under adaptive controller (2.11), there exists a sufficiently large positive constant \(k\) such that the response network (3.15) can almost surely achieve generalized outer synchronization with the drive network (2.9).

**Proof.** If
\[
k > l + 1 + \frac{1}{2}\left[p + c + q + \lambda_{\text{max}}(M_1) + c \cdot \lambda_{\text{max}}(M_2)\right]
\] (3.16)
the (3.14) holds, and the network (3.15) are generalized outer synchronization.

**Corollary 2.** Let \((A_1), (A_2), (A_3)\) and \((A_4)\) hold. If networks (2.9) and (2.10) have identical dynamics, namely, \(f = g\), then the two nonidentical networks can achieve complete outer synchronization under the following control scheme:
\[
u_i = -ke_i + \sum_{j=1}^{N} [c_{ij}P - d_{ij}Q]x_j(t - \tau(t)) \\
+ \sum_{j=1}^{N} [a_{ij}M - b_{ij}N]f_j(x_j(t - \tau(t))), i = 1, 2, \ldots, N,
\] (3.17)

By Lemma 1, then for every \(\xi \in C^b_{\tau_0}([-\tau,0];\mathbb{R}^n)\), we have
\[
\lim_{t \to \infty} e(t) \to 0, \text{ a.s.}
\] (3.14)

Hence the almost sure generalized outer synchronization between the drive network (2.9) and the response network (2.10) can be realized using the adaptive controller (2.11). This completes the proof of Theorem 1.

Where \(\sigma_i\) is a zero matrix, the networks (2.10) turn out to be the following complex network without stochastic perturbation.
\[
\dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^{N} d_{ij}Qy_j(t - \tau(t)) + \sum_{j=1}^{N} b_{ij}Ng_j(y_j(t - \tau(t))) \\
+ u_i(t), i = 1, 2, \ldots, N.
\] (3.15)

**Corollary 1.** Let \((A_1), (A_3)\) and \((A_4)\) hold, and under adaptive controller (2.11), there exists a sufficiently large positive constant \(k\) such that the response network (3.15) can almost surely achieve generalized outer synchronization with the drive network (2.9).

**Proof.** If
\[
k > l + 1 + \frac{1}{2}\left[p + c + q + \lambda_{\text{max}}(M_1) + c \cdot \lambda_{\text{max}}(M_2)\right]
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\[
u_i = -ke_i + \sum_{j=1}^{N} [c_{ij}P - d_{ij}Q]x_j(t - \tau(t)) \\
+ \sum_{j=1}^{N} [a_{ij}M - b_{ij}N]f_j(x_j(t - \tau(t))), i = 1, 2, \ldots, N,
\] (3.17)
where
\[ k > l + 1 + \frac{1}{2} [ p + c + q + \lambda_{\text{max}}(M_1) + c \cdot \lambda_{\text{max}}(M_2) ]. \] 

(3.18)

**Corollary 3.** Let \((A_1), (A_2), (A_3)\) and \((A_4)\) hold. If networks (2.9) and (2.10) have the same topological structures and uniform inner-coupling, i.e., \(A = B, C = D, P = Q, M = N\), and also have identical node dynamics, namely, \(f = g\), then the two networks can achieve complete outer synchronization under the following control scheme:

\[ u_i = -ke_i, i = 1, 2, \ldots, N, \] 

(3.19)

where
\[ k > l + 1 + \frac{1}{2} [ p + c + q + \lambda_{\text{max}}(M_1) + c \cdot \lambda_{\text{max}}(M_2) ]. \] 

(3.20)

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