Polaroid operators with SVEP and perturbations of property \((gaw)\)

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Abstract In this paper we establish for a bounded linear operator defined on a Banach space several sufficient and necessary conditions for which property \((gaw)\) holds. In this work, we consider commutative perturbations by algebraic operator and quasinilpotent operator for \(T \in B(X)\) such that \(T^*\) satisfies property \((gaw)\). We prove that if \(A\) is an algebraic and \(T \in PS(X)\) is such that \(AT = TA\), then \(f(T^* + A^*)\) satisfies property \((gaw)\) for every \(f \in H_c(\sigma(T + A))\). Moreover, we show that if \(Q\) is a quasi-nilpotent operator and \(T \in PS(X)\) is such that \(TQ = QT\), then \(f(T^* + Q^*)\) satisfies the property \((gaw)\) for every \(f \in H_c(\sigma(T + Q))\). At the end of this paper, we apply the obtained results to a number of subclasses of \(PS(X)\).

Keywords Weyl’s theorem · Weyl spectrum · polaroid operators · property \((gaw)\) · property \((aw)\)

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1 Introduction

An operator \(T \in B(X)\) has the single-valued extension property at a point \(\lambda_0 \in \mathbb{C}\), SVEP at \(\lambda_0\), if for every open disc \(U\) centered at \(\lambda_0\) the only analytic function \(f : U \to X\) satisfying \((T - \lambda I)f(\lambda) = 0\) is the function \(f \equiv 0\) The single valued extension property plays an important role in local spectral theory and Fredholm theory (see [1] and [38]). Evidently, every \(T\) has SVEP at points in the resolvent \(\rho(T) = \mathbb{C} \setminus \sigma(T)\) or the boundary \(\partial(T)\) of the spectrum \(\sigma(T)\) of \(T\). It is easily verified that SVEP is inherited by restrictions. We say that \(T\) has SVEP if it has SVEP at every \(\lambda \in \sigma(T)\).

An operator \(T \in B(X)\) is polaroid (see [27,31]) if \(\pi(T) = \{\lambda \in \mathbb{C}: \lambda \in \text{iso}(\sigma(T))\}\), where \(\pi(T)\) is the set of poles of the resolvent of \(T\) and \(\text{iso}(\sigma(T))\) is the set of isolated points of \(\sigma(T)\). A necessary and sufficient condition for \(\lambda \in \pi(T)\) is that \(a(T - \lambda I) = d(T - \lambda I) < \infty\). Although elements belonging to a number of the commonly considered classes of polaroid operators have SVEP, not every polaroid operator has SVEP. Thus, if \(X = H\) is a Hilbert space and \(R \in B(H)\) is the forward unilateral shift, then the backward unilateral shift \(R^*\) is polaroid (\(\text{iso}(R^*) = \emptyset\)) but does not

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have SVEP. Again, if $T \in \mathcal{B}(\mathcal{H})$ satisfies the growth condition $(G_m): \|T - \lambda I\|^{-1} \leq K/\text{dist}(\lambda, \sigma(T))^m$ for some scalar $K > 0$, natural number $m (m \in \mathbb{N})$ and all $\lambda \notin \sigma(T)$, then $T$ is polaroid: however, not every $(G_m)$-operator has SVEP (see [51]).

Let $PS(\mathcal{X})$ the class of all operators $T \in \mathcal{B}(\mathcal{X})$ such that $T$ is polaroid and $T$ has the SVEP. The class $PS(\mathcal{X})$ considered by Duggal in [28] is large. It contains most of the classes of operators studied in literature in connection with Weyl’s type theorems. The property (gaw) introduced and studied by [20] in connection with Weyl type theorems (see [50]).

In this note, we establish for a bounded linear operator defined on a Banach space several sufficient and necessary conditions for which property (gaw) holds. We also relate this property with generalized Weyl’s theorem and with another variant of it, generalized a-Weyl’s theorem. We show that generalized Weyl’s theorem, generalized a-Weyl’s theorem and property (gaw) for $T$ (respectively $T^*$) coincide whenever $T^*$ (respectively $T$) satisfies SVEP. We prove that if $T \in PS(\mathcal{X})$ then the property (gaw) holds for $f(T^*)$, for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ stands for the set of all analytic functions on a neighborhood of $\sigma(T)$. Recall that an operator $K \in \mathcal{A}(\mathcal{X})$ is algebraic if $h(K) = 0$ for some non-trivial polynomial $h(\cdot)$. Stability of Browder and Weyl’s theorems under perturbations for operators $T \in PS(\mathcal{X})$ is studied in [28]. However, preservation of property (gaw) under perturbations by finite rank and nilpotent operators for $T \in PS(\mathcal{X})$ is studied in [21]. In this work, we consider commutative perturbations by algebraic operator and quasinilpotent operator for $T \in PS(\mathcal{X})$ such that $T^*$ satisfies property (gaw). We prove that if $K \in \mathcal{A}(\mathcal{X})$ and $T \in PS(\mathcal{X})$ is such that $KT = TK$, then $f(T^* + K^*)$ satisfies property (gaw) for every $f \in H(\sigma(T + K))$. Moreover, we show that if $Q$ is a quasi-nilpotent operator and $T \in PS(\mathcal{X})$ is such that $TQ = QT$, then $f(T^* + Q^*)$ satisfies the property (gaw) for every $f \in H(\sigma(T + Q))$. This gives generalization of results proved in [21]. At the end of this paper, we apply the obtained results to a number of subclasses of $PS(\mathcal{X})$ which have attracted the attention of a number of authors (see [1–3, 21, 23, 28, 39]).

2 Terminology

Throughout this paper, $\mathcal{B}(\mathcal{X})$ denote the algebra of all bounded linear operators acting on a Banach space $\mathcal{X}$. For $T \in \mathcal{B}(\mathcal{X})$, let $T^*, \ker(T), \mathcal{R}(T), \sigma(T), \sigma_0(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of $T$. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of $T$ defined by $\alpha(T) := \dim \ker(T)$ and $\beta(T) := \text{codim} \mathcal{R}(T)$. If the range $\mathcal{R}(T)$ of $T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then $T$ is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator.

In the sequel $SF_+(\mathcal{X})$ (resp. $SF_-(\mathcal{X})$) will denote the set of all upper (resp. lower) semi-Fredholm operators. If $T \in \mathcal{B}(\mathcal{X})$ is either upper or lower semi-Fredholm, then $T$ is called a semi-Fredholm operator, and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is a Fredholm operator. An operator $T$ is called Weyl if it is Fredholm of index zero.

Let $a := a(T)$ be the ascent of an operator $T$; i.e., the smallest nonnegative integer $p$ such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be descent of an operator $T$; i.e., the smallest nonnegative integer $s$ such that $\mathcal{R}(T^s) = \mathcal{R}(T^{s+1})$, and if such integer does not exist we put
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d(T) − ∞. It is well known that if a(T) and d(T) are both finite then a(T) = d(T) (see [35, Proposition 38.3]). Moreover, 0 < a(T − λI) = d(T − λI) < ∞ precisely when λ is a pole of the resolvent of T (see Heuser [35, Proposition 50.2]).

An operator T ∈ ℬ(XHR) is called Browder if it is Fredholm “of finite ascent and descent”. The Weyl spectrum of T is defined by σ_w(T) = {λ ∈ C : T − λI is not Browder}. For T ∈ ℬ(XHR), let SF^+_T(XR) = {T ∈ SF^+_T(XR) : ind(T) ≤ 0}. Then the upper Weyl spectrum of T is defined by σ^+_SF(T) = {λ ∈ C : T − λI ∉ SF^+_T(XR)}. Let

\[ Δ(T) = \sigma(T) \setminus σ_w(T) \text{ and } Δ_a(T) = σ_a(T) \setminus σ^+_SF(T). \]

Following Coburn [24], we say that Weyl’s theorem holds for T ∈ ℬ(XHR) (in symbols, T ∈ W) if Δ(T) = E^0(T), where E^0(T) = {λ ∈ isoσ(T) : 0 < a(T − λI) < ∞} and that Browder’s theorem holds for T (in symbols, T ∈ B) if σ_w(T) = (T), where (T) = {λ ∈ C : T − λI is not Browder}.

Here and elsewhere in this paper, for K ⊆ C, isoK is the set of isolated points of K. According to Rakočević [41], an operator T ∈ ℬ(XHR) is said to satisfy a-Weyl’s theorem (in symbols, T ∈ aW) if Δ_a(T) = E_a(T), where E_a(T) = {λ ∈ isoσ_a(T) : 0 < a(T − λI) < ∞}.

It is known [41] that an operator satisfying a-Weyl’s theorem satisfies Weyl’s theorem, but the converse does not hold in general. For T ∈ ℬ(XHR) and a nonnegative integer n define T_0 to be the restriction of T to ℓ(T^n) viewed as a map from ℓ(T^n) into ℓ(T^n) (in particular T_0 = T). If for some integer n the range space ℓ(T^n) is closed and T_n is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) semi- B-Fredholm operator. In this case the index of T is defined as the index of the semi-B-Fredholm operator T_{[n]} (see [13]). Moreover, if T_{[n]} is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T ∈ ℬ(XHR) is said to be a B- Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum σ_{BW} of T is defined by

\[ σ_{BW}(T) = \{λ ∈ C : T − λI is not a B-Weyl\}. \]

Given T ∈ ℬ(XHR). Let Δ^0(T) = σ(T) \setminus σ_{BW}(T). We say that the generalized Weyl’s theorem holds for T (and we write T ∈ gW) if Δ^0(T) = E(T), where E(T) is the set of all isolated eigenvalues of T, and that the generalized Browder’s theorem holds for T (in symbols, T ∈ gB) if Δ^0(T) = π(T), where π(T) is the set of all poles of T, see [17, Definition 2.13]. It is known [17,34] that gW ⊆ gB ∩ W and that gB ∪ W ⊆ B. Moreover, given T ∈ gB, it is clear that T ∈ gW if and only if E(T) = π(T). Generalized Weyl’s theorem has been studied in [8,14,15,17,18,43–47,50,51] and the references therein.

Let SBF^+_T(XR) be the class of all upper semi-B-Fredholm operators, SBF^+_T(XR) = \{T ∈ SBF^+_T(XR) : ind(T) ≤ 0\}. The upper B-Weyl spectrum of T, σ_{SBF^+_T}(T) = {λ ∈ C : T − λI ∉ SBF^+_T(XR)}. For T ∈ ℬ(XHR). Let Δ^0(T) = σ_a(T) \setminus σ_{SBF^+_T}(T). We say that T obeys generalized a-Weyls theorem (in symbols, T ∈ gaW), if Δ^0(T) = E_a(T), where E_a(T) is the set of all eigenvalues of T which are isolated in σ_a(T) ([17, Definition 2.13]). Generalized a-Weyls theorem has been studied in [16,17,22,43–47,50,51]. Recall from [15] that an operator T is Drazin invertible if it has a finite ascent and descent. The Drazin spectrum σ_D(T) = {λ ∈ C : T − λI is not a Drazin invertible}.
We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$. Define the set $LD(\mathcal{X})$ by $LD(\mathcal{X}) = \{T \in \mathcal{B}(\mathcal{X}) : a(T) < \infty$ and $\Re(Ta(T) + 1) \text{ is closed} \}$. Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is called left Drazin invertible if $T \in LD(\mathcal{X})$. The left Drazin spectrum is defined by $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(\mathcal{X})\}$. We will say that $\lambda \in \sigma_a(T)$ is a left pole of $T$ if $T - \lambda I$ is left Drazin invertible and that $\pi_a^0(T)$ is a left pole of $T$ of finite rank if $\lambda$ is a left pole of $T$ and $\alpha(T - \lambda I) < \infty$. We will denote by $\pi_a(T)$ the set of all left poles of $T$, and by $\pi_a^0(T)$ the set of all left poles of $T$ of finite rank. It follows from the preceding description that $\sigma_{LD}(T) = \sigma_a(T) \setminus \pi_a(T)$. According to [41], an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be satisfies $a$-Browder’s theorem (in symbol, $T \in aB$) if $\Delta_a(T) = \pi_a^0(T)$. Following [9], we say that $T$ satisfies generalized $a$-Browder’s theorem (in symbol, $T \in gaB$) if $\Delta_a(T) = \pi_a(T)$ or equivalently, $\sigma_{LD}(T) = \sigma_{SRF}^-(T)$.

Following [42], we say that $T \in \mathcal{B}(\mathcal{X})$ possesses property $(w)$ if $\Delta_a(T) = E^0(T)$. The property $(w)$ has been studied in [4,42]. In [4, Theorem 2.8], it is shown that property $(w)$ implies Weyl’s theorem, but the converse is not true in general. According to [11], we say that $T \in \mathcal{B}(\mathcal{X})$ possesses property $(gw)$ if $\Delta^g_a(T) = E(T)$. Property $(gw)$ has been introduced and studied in [11]. Property $(gw)$ extends property $(w)$ to the context of $B$-Fredholm theory, and it is proved in [11] that an operator possessing property $(gw)$ possesses property $(w)$ but the converse is not true in general. In a recent paper [49] the author studied this property and their perturbations. According to [19], an operator $T \in \mathcal{B}(\mathcal{X})$ is said to possess property $(gb)$ if $\Delta^g_a(T) = \pi_a(T)$, and is said to possess property $(b)$ if $\Delta_a(T) = \pi_a(T)$. It is shown [19, Theorem 2.3] that an operator possessing property $(gb)$ possesses property $(b)$ but the converse is not true in general, see also [48]. Following [20], we say that $T \in \mathcal{B}(\mathcal{X})$ possesses property $(aw)$ if $\Delta(T) = E^0_a(T)$, and that $T \in \mathcal{B}(\mathcal{X})$ satisfies property $(gaw)$ if $\Delta^g(T) = E_a(T)$, and it is proved in [20, Theorem 3.3] that an operator possessing property $(gaw)$ possesses property $(aw)$ but the converse is not true in general. Following [20], we say that $T \in \mathcal{B}(\mathcal{X})$ possesses property $(ab)$ if $\Delta(T) = \pi_a^0(T)$, and that $T \in \mathcal{B}(\mathcal{X})$ satisfies property $(gab)$ if $\Delta^g(T) = \pi_a(T)$, and it is proved in [20, Theorem 2.2] that an operator possessing property $(gab)$ possesses property $(ab)$ but the converse is not true in general. For more details about these properties the author refer to [50,52].

3 Property $(gaw)$ and perturbations

**Theorem 3.1** Let $T \in \mathcal{B}(\mathcal{X})$. Then $T$ satisfies property $(gaw)$ if and only $T$ satisfies generalized Weyl’s theorem and $E(T) = E_a(T)$.

**Proof.** Assume that property $(gaw)$ holds for $T$, i.e., $\Delta^g(T) = E_a(T)$. Then we conclude from Corollary 2.7 and Theorem 3.5 of [20] that $T$ satisfies generalized Browder’s theorem and $\pi(T) = \pi_a(T) = E_a(T)$. Since $\pi(T) \subseteq E(T)$ holds for every operator $T$ with no restriction on $T$, we have $\pi(T) = \pi_a(T) = E(T) = E_a(T)$. Therefore, $T$ satisfies generalized Weyl’s theorem.

Conversely, assume that $T$ satisfies generalized Weyl’s theorem and $E(T) = E_a(T)$. Then $\Delta^g(T) = E(T)$ and $E(T) = E_a(T)$ which implies that $\Delta^g(T)E_a(T)$ and $T$ satisfies property $(gaw)$. □

The following example [4, Example 2.14] shows that generalized $a$-Weyls theorem and generalized Weyl’s theorem does not imply property $(gaw)$.
Example 3.1 Let \( R \in \ell^2(\mathbb{N}) \) be the unilateral right shift and let \( U \) defined by \( U(x_1, x_2, ...) = (0, x_2, x_3, ...) \). If \( T = R \oplus U \), then \( \sigma(T) = \sigma_{BW}(T) = D(0, 1) \) the closed unit disc in \( \mathbb{C} \), is \( \sigma(T) = \emptyset \) and \( \sigma_0(T) = C(0, 1) \cup \{0\} \) where \( C(0, 1) \) is unit circle of \( \mathbb{C} \). It follows from [4, Example 2.14] that \( \sigma_{SF_+}(T) = C(0, 1) \). This implies that \( \sigma_{SBF_+}(T) = C(0, 1) \) and \( \Delta_2^g(T) = \{0\} \) and \( \sigma(T) \setminus \sigma_{BW}(T) = \emptyset \). Moreover, we have \( E(T) = \emptyset \) and \( E_\alpha(T) = \{0\} \). Hence \( T \) satisfies generalized a-Weyl’s theorem and so \( T \) satisfies generalized Weyl’s theorem. But \( T \) does not satisfy property \( (gaw) \).

Remark 3.1 We note that Example 3.1 shows that property \( (gaw) \) is not intermediate between generalized Weyl’s theorem and generalized a-Weyl’s theorem.

Theorem 3.2 Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( E(T) = E_\alpha(T) \). Then \( T \) satisfies property \( (gw) \) if and only if

(i) \( T \) satisfies property \( (gaw) \);

(ii) \( \text{ind}(T - \lambda I) = 0 \) for all \( \lambda \in \Delta_2^g(T) \).

Proof. Assume that \( T \) satisfies property \( (gw) \), then it follows from [11, Theorem 2.4] that \( T \) satisfies generalized Weyl’s theorem and \( \text{ind}(T - \lambda I) = 0 \) for all \( \lambda \in \Delta_2^g \). Now it follows from the assumption that \( E(T) = E_\alpha(T) \) and hence \( T \) satisfies property \( (gaw) \) by Theorem 3.1.

Conversely, assume that property \( (gaw) \) holds for \( T \) and \( \text{ind}(T - \lambda I) = 0 \) for all \( \lambda \in \Delta_2^g \). Then it follows from Theorem 3.1 that \( T \) satisfies generalized Weyl’s theorem. Hence it follows from [11, Theorem 2.4] that \( T \) satisfies property \( (gw) \). □

The following definition is due to Berkani and Koliha [17].

Definition 3.3 Let \( T \in \mathcal{B}(\mathcal{H}) \) and let \( s \in \mathbb{N} \). Then \( T \) has a uniform descent for \( n \geq s \) if \( \Re(T) + \ker(T^n) = \Re(T) + \ker(T^s) \), for all \( n \geq s \). If, in addition, \( \Re(T) + \ker(T^n) \) is closed then \( T \) is said to have a topological uniform descent for \( n \geq s \).

Theorem 3.4 Let \( T \in \mathcal{B}(\mathcal{H}) \) and \( E(T) = E_\alpha(T) \). The following assertions are equivalent:

(i) \( T \) satisfies property \( (gw) \);

(ii) \( T \) satisfies property \( (gaw) \).

Proof. Suppose that \( T \) satisfies property \( (gw) \). Then it follows from [11, Theorem 2.4] that \( T \) satisfies generalized Weyl’s theorem, i.e., \( \Delta^g(T) = E(T) \). From the assumption \( E(T) = E_\alpha(T) \), we have \( \Delta^g(T) = E_\alpha(T) \) and so \( T \) satisfies property \( (gaw) \). Conversely, assume that \( T \) satisfies property \( (gaw) \) and \( E(T) = E_\alpha(T) \), then \( T \) satisfies property \( (gaw) \) and \( E_\alpha(T) = \pi_\alpha(T) \). Thus we show that \( \sigma_{SBF_+}(T) = \sigma_0(T) \setminus E(T) = \sigma_0(T) \setminus \pi_\alpha(T) = \sigma_{LF}(T) \). As we have always \( \sigma_{SBF_+}(T) \subseteq \sigma_{LD}(T) \), we only need to show that \( \sigma_{LD}(T) \subseteq \sigma_{SBF_+}(T) \). So let \( \lambda_0 \notin \sigma_{SBF_+}(T) \). Then \( T - \lambda I \) is an upper semi-B-Fredholm operator with \( \text{ind}(T - \lambda_0 I) \leq 0 \). In particular \( T - \lambda_0 I \) is an operator of topological uniform descent. It follows from [16, Theorem 2.10] that if \( |\lambda - \lambda_0| \) is small enough and not equal to zero then \( T - \lambda I \) is an upper semi-B-Fredholm operator with \( \text{ind}(T - \lambda I) \leq 0 \) and \( T \) is also an operator of topological uniform descent for \( n \geq 0 \). Since property \( (gaw) \) holds for \( T \), then \( \lambda \) is a left pole of \( T \). So \( \sigma_{SBF_+}(T) = \sigma_{LD}(T) \), as desired. □
An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be a-polaroid if every isolated point of $\sigma_a(T)$ is a pole of the resolvent operator $(T-\lambda I)^{-1}$, or equivalently $0 < a(T-\lambda I) = d(T-\lambda I) < \infty$ (see [35, Proposition 50.2]). Clearly,

$$T \text{ is a-polaroid } \implies T \text{ is polaroid } \quad (3.1)$$

and the opposite implication is not generally true.

**Proposition 3.5** Suppose that $T \in \mathcal{B}(\mathcal{X})$ is a-polaroid. Then $E_a(T) = \pi(T)$.

**Proof.** If $\lambda \in E_a(T)$ then $\lambda$ is isolated in $\sigma_a(T)$ and hence $0 < a(T-\lambda I) = d(T-\lambda I) < \infty$. Hence $\lambda$ is a pole of the resolvent of $T$, i.e., $\lambda \in \pi(T)$ and since the other inclusion is always verified we have $E_a(T) = \pi(T)$. □

**Remark 3.2** Observe that if $T^*$ has SVEP then $\sigma(T) = \sigma_a(T)$ (see [1, Corollary 2.45]), so that

$$T^* \text{ has SVEP and } T \text{ is polaroid } \implies T \text{ is a-polaroid } \quad (3.2)$$

If $T$ is polaroid then $T^*$ is polaroid [7]. Moreover, if $T$ has SVEP then $\sigma(T) = \sigma_a(T^*)$ (see [1, Corollary 2.45]), hence

$$T \text{ has SVEP and } T \text{ is polaroid } \implies T^* \text{ is a-polaroid } \quad (3.3)$$

**Remark 3.3** If $T^*$ has SVEP, then it known (see [38, Page 35]) that $\sigma_a(T) = \sigma(T)$ and from [6, Theorem 2.9] we have $\sigma_{SBF^+}(T) = \sigma_{BW}(T)$. Consequently, $E_a(T) = E(T)$ and $\Delta^g(T) = E(T) = \Delta^g(T)$.

**Theorem 3.6** Suppose that $T \in \mathcal{B}(\mathcal{X})$. Then the following assertions hold:

(i) If $T$ is polaroid and $T^*$ has SVEP then property (gaw) holds for $T$.

(ii) If $T$ is polaroid and $T$ has SVEP then property (gaw) holds for $T^*$.

**Proof.** (i) Suppose that $T$ is a polaroid operator and $T^*$ has the SVEP. Then, from [10, Theorem 2.3], $T$ satisfies generalized Browder’s theorem. As $T$ is polaroid, then $E(T) = \pi(T)$. Hence, by Corollary 2.1 of [9] it then follows that $T$ satisfies generalized Weyl’s theorem, i.e., $\Delta^g(T) = E(T)$. Since $T^*$ has SVEP then it follows from Remark 3.9 that $E(T) = E_a(T)$. Therefore, $\Delta^g(T) = E_a(T)$. That is, property (gaw) holds for $T$.

(ii) As $T$ has the SVEP, then $T$ satisfies generalized Browder’s theorem. As $T$ is polaroid, then $E(T) = \pi(T)$. Hence $T$ satisfies generalized Weyl’s theorem. As we have $\sigma(T) = \sigma(T^*)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*)$, $\pi(T) = \pi(T^*)$, $E(T) = E(T^*) = E_a(T^*)$, then $T^*$ satisfies generalized Weyl’s theorem too. Therefore, $T^*$ satisfies property (gaw). □

**Theorem 3.7** Let $T \in \mathcal{B}(\mathcal{X})$. If $T^*$ has SVEP, then the following are equivalent:

(i) Property (gw) holds for $T$;

(ii) generalized Weyl’s theorem holds for $T$;

(iii) generalized $a$-Weyl’s theorem holds for $T$;

(iv) Property (gaw) holds for $T$. 

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Proof. Suppose that $T^*$ has the SVEP, then it follows from Remark 3.3 that $\sigma(T) = \sigma_a(T), \sigma_{BW}(T) = \sigma_{SBF}(T)$, $E(T) = E_a(T)$ and $\Delta^g(T) = \Delta_0^g(T)$. This makes (i) $\iff$ (ii) $\iff$ (iii) and (ii) $\iff$ (iv) evident. \(\Box\)

Remark 3.4 If $T$ has the SVEP, then from [38, Page 35] we have $\sigma(T) = \sigma(T^*), \sigma_{BW}(T) = \sigma_{BW}(T^*) = \sigma_{SBF_{\pi}}(T)$ and $E(T^*) = E_a(T^*)$.

Theorem 3.8 Let $T \in \mathcal{B}(X)$. If $T$ has SVEP, then the following are equivalent:

(i) Property (gw) holds for $T$;
(ii) generalized Weyl’s theorem holds for $T^*$;
(iii) generalized a-Weyl’s theorem holds for $T^*$;
(iv) Property (gaw) holds for $T^*$.

Let $H(\sigma(T))$ (respectively $H_c(\sigma(T))$) denote the class of functions $f$ which are (defined and) analytic on an open neighborhood $U$ of $\sigma(T)$ (respectively $f \in H(\sigma(T))$ such that $f$ is non-constant on each connected component of $U$).

Theorem 3.9 Let $T \in \mathcal{B}(X)$ be a polaroid operator.

(i) If $T^*$ has SVEP, then property (gaw) holds for $f(T)$ for every $f \in H(\sigma(T))$.
(ii) If $T$ has SVEP, then property (gaw) holds for $f(T^*)$ for every $f \in H(\sigma(T))$.

Proof. (i) Suppose that $T^*$ has the SVEP, then $f(T^*) = f(T)^*$ has the SVEP, see [1, Theorem 2.22], which in turn implies that generalized Browder’s theorem holds for $f(T)$, that is, $\Delta^g(f(T)) = \pi(f(T))$. Since $T$ is polaroid and $T^*$ has SVEP, then $T$ is $a$-polaroid. Hence it follows from Proposition 3.5 that $\pi(f(T)) = E_a(f(T))$. Therefore, $f(T)$ satisfies property (gaw).

(ii) Suppose that $T$ has the SVEP, then $f(T)$ has the SVEP (see [1]), which in turn implies that generalized Browder’s theorem holds for $f(T^*)$, that is, $\Delta^g(f(T^*)) = \pi(f(T^*))$. Since $T$ is polaroid, then it follows from [7, Theorem 2.5] that $T^*$ is polaroid. By implication 3.2 it then follows that $T^*$ is $a$-polaroid and hence $\pi(f(T^*)) = E_a(f(T^*))$. That is, $f(T^*)$ satisfies property (gaw). \(\Box\)

The class $PS(X)$ considered by Duggal in [28] is large. It contains, amongst others, the class of $H(p)$ operators considered in [1] and [40]. The class $PS(X)$ contains also the class of totally hereditarily normaloid operators (see [28]). Recall, [31], that an operator $T \in \mathcal{B}(X)$ is said to be totally hereditarily normaloid $T \in THN$, if every part and also $T_p^{-1}$ for every invertible part $T_p$, of $T$ is normaloid. Here a part of an operator is its restriction to an invariant subspaces and an operator is normaloid if its spectral radius is equals to its norm.

Lemma 3.10 Let $T \in \mathcal{B}(X)$. If $T \in PS(X)$, then property (gaw) holds for $f(T^*)$ for every $f \in H(\sigma(T))$.

Proof. If $T \in PS(X)$. Then $T$ is polaroid and has the SVEP. Hence the result follows from Theorem 3.9. \(\Box\)

Theorem 3.11 Let $T \in PS(X)$ and $A \in \mathcal{A}(X)$. Then $f((T + A)^*)$ satisfies property (gaw) for every $f \in H(\sigma(T + A))$. 279
Proof. Assume that $T$ is polaroid and has the SVEP. Then, by [28, Lemma 3.7] $T + A$ is polaroid and from [28, Lemma 3.4] $T + A$ has the SVEP. Thus $T + A \in PS(\mathcal{X})$. By Lemma 3.10, we conclude that $f((T + A)^*)$ satisfies property (gaw), for every $f \in H(\sigma(T + A))$. □

For perturbations by finite rank operator we have the following result.

**Corollary 3.12** Let $T \in PS(\mathcal{X})$ and let $F \in \mathcal{B}(\mathcal{X})$ such that $F^n$ is finite dimensional for some positive integer $n$. If $FT = TF$, then $f((T + F)^*)$ satisfies property (gaw) for every $f \in H(\sigma(T + F))$.

**Proof.** Operators $F \in \mathcal{B}(\mathcal{X})$ such that $F^n$ is finite dimensional for some $n \in \mathbb{N}$ are algebraic, since there exists a polynomial $p(.)$ such that $p(F^n) = 0$. So the result follows from Theorem 3.11. □

To study perturbations under quasi-nilpotent operator we need the following Lemma.

**Lemma 3.13** Let $T \in PS(\mathcal{X})$ and $Q \in \mathcal{B}(\mathcal{X})$ a quasi-nilpotent operator which commutes with $T$, then $E(T) = E(T + Q) = E((T + Q)^*) = E_a(T^*) = E_a((T + Q)^*) = \pi(T + Q) = \pi(T)$.

**Proof.** Observe that SVEP implies that $\sigma(T + Q) = \sigma(T^* + Q^*) = \sigma_a(T^* + Q^*)$, $E_a(T^* + Q^*) = E(T^* + Q^*)$. By [12, Theorem 3.12] $T + Q$ is polaroid and hence $T^* + Q^*$ is polaroid. Therefore, $E(T^* + Q^*) = \pi(T^* + Q^*) = \pi(T + Q) = E(T + Q)$. Hence, by Lemma 3.7 of [12], the result follows. □

**Theorem 3.14** Let $T \in PS(\mathcal{X})$ and $Q \in \mathcal{B}(\mathcal{X})$ a quasi-nilpotent operator which commutes with $T$, then $(T + Q)^*$ satisfies property (gaw).

**Proof.** To prove that $(T + Q)^*$ satisfies property (gaw), it suffices to show that $\Delta^g((T + Q)^*) = E_a((T + Q)^*)$. Since $T \in PS(\mathcal{X})$, then by Lemma 3.10, $T^*$ satisfies property (gaw). That is $\Delta^g(T^*) = E_a(T^*)$. Since, by [10, Remark 3.8], $\Delta^g((T + Q)^*) = \Delta^g(T^*)$, then we conclude the result by Lemma 3.13. □

**Theorem 3.15** Let $T \in PS(\mathcal{X})$ and $Q \in \mathcal{B}(\mathcal{X})$ a quasi-nilpotent operator which commutes with $T$, then $f((T + Q)^*)$ satisfies property (gaw), for every $f \in H(\sigma(T + Q))$.

**Proof.** Suppose that $T \in PS(\mathcal{X})$, then by Corollary 3.13 of [12], $T + Q \in PS(\mathcal{X})$. Hence by Lemma 3.10, $f((T + Q)^*)$ satisfies property (gaw), for every $f \in H(\sigma(T + Q))$. □

The quasi-nilpotent part $H_0(T - \lambda I)$ and the analytic core $K(T - \lambda I)$ of $T - \lambda I$ are defined by $H_0(T - \lambda I) = \{ x \in \mathcal{X} : \lim_{n \to \infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0 \}$ and $K(T - \lambda I) = \{ x \in \mathcal{X} : \exists \text{ a sequence } \{ x_n \} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which } x = x_0, (T - \lambda I)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\|, \text{ for all } n \in \mathbb{N} \}$.

We note that $H_0(T - \lambda I)$ and $K(T - \lambda I)$ are generally non-closed hyper-invariant subspaces of $T - \lambda I$ such that $(T - \lambda I)^{-p}(0) \subseteq H_0(T - \lambda I)$ for all $p = 0, 1, \cdots$ and $(T - \lambda I)K(T - \lambda I) = K(T - \lambda I)$. Recall that if $\lambda \in \text{iso}\sigma(T)$, then $H_0(T - \lambda I) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in \mathcal{X}$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \to \mathcal{X}$ that satisfies $(T - \mu I)f(\mu) = x$, for all $\mu \in \mathbb{C} \setminus \{\lambda\}$ (see [1]).
A part of an operator is its restriction to a (closed) invariant subspace. An operator $T \in \mathcal{B}(\mathcal{H})$ is hereditarily polaroid, $T \in \mathcal{H}P$, if every part of $T$ is polaroid. The class of $\mathcal{H}P$ operators is substantial: it contains, amongst other classes (see examples in Section 4), the class $\mathcal{H}(p)$ operators considered by OUDGHIRI [40], the class $\mathcal{C}HN$ of completely hereditarily normaloid operators considered by DUGGAL [26].

A more satisfactory version of Theorem 3.15 holds for algebraic operators $A$ which commute with an $\mathcal{H}P$ operator. But before proving this, we introduce some (further) terminology. An operator $T \in \mathcal{B}(\mathcal{H})$ is isoloid (respectively $a$-isoloid) if $\lambda \in \text{iso}(T) \subseteq E(T)$ (respectively $\lambda \in \text{iso}_{a}(T) \subseteq E(T)$). Evidently, polaroid operators are isoloid and left polaroid operators are $a$-isoloid.

**Theorem 3.16** Let $T \in \mathcal{B}(\mathcal{H})$ be a polynomially $\mathcal{H}P$ operator (such that $g(T) \in \mathcal{H}P$) and let $A \in \mathcal{B}(\mathcal{H})$ be an algebraic operator which commutes with $T$. Then $f((T + A)^*)$ satisfies property $(gaw)$, for every $f \in H_{c}(\sigma(T + A))$.

**Proof.** The operator $A$ being algebraic, $\sigma(A) = \{\mu_{1}, \mu_{2}, \ldots, \mu_{n}\}$ for some scalars $\mu_{i}$, $1 \leq i \leq n$. Let $A_{i} = A|_{\text{H}(A_{i} - \mu_{i}I)}$, $1 \leq i \leq n$ and $T_{i} = T|_{\text{H}(T_{i} - \mu_{i}I)}$, $1 \leq i \leq n$. The commutativity of $A$ with $T$ then implies that $A_{i}$ commutes with $T_{i}$, for all $1 \leq i \leq n$ (for the reason that the projection $H_{0}(A - \mu_{i}I)$ corresponding to $\mu_{i}$ commutes with $T$, for all $1 \leq i \leq n$), $T = \bigoplus_{i=1}^{n} T_{i}$ and $T + A = \bigoplus_{i=1}^{n} (T_{i} + A_{i})$. Since $A_{i} - \mu_{i}I$ is nilpotent, for all $1 \leq i \leq n$, the upper triangular operator $\bigoplus_{i=1}^{n} (T_{i} + A_{i} - \mu_{i}I) = \bigoplus_{i=1}^{n} T_{i}$, with entries $A_{i} - \mu_{i}I$ along the main diagonal, is nilpotent. Hence $T + A - \mu_{i}I$ and $T$ are quasi-nilpotent equivalent. The hypothesis $g(T) \in \mathcal{H}P$ implies $(g(T)$ has SVEP implies $T$ has SVEP; hence $T + A - \mu_{i}I$, equivalently $T + A$, has SVEP (so that both $T + A$ and $T^{*} + A^{*}$ satisfy generalized $a$-Browder’s theorem). Again, the hypothesis $g(T) \in \mathcal{H}P$ implies that $T \in \mathcal{H}P$ (see Example 2.5 of [31]). Arguing as in the proof of [29, Lemma 6] it is now seen that $H_{0}(T + A - \mu_{i}I) = \ker(T + A - \mu_{i}I)^{m}$, for some $m \in \mathbb{N}$, at every $\mu \in \text{iso}(T + A)$. (We remark here that [29, Lemma 6] is proved for CHN operators of Example 2.2 of [31].) CHN operators are simply $\mathcal{H}P$, but the simply polaroid property in the proof of [29, Lemma 6] is at best incidental, and the lemma holds just as well for $\mathcal{H}P$ operators.) Thus $T + A$ satisfies generalized Weyl’s theorem [30, Theorem 3.8], i.e., $\sigma(T + A) \setminus E(T + A) = \sigma_{BW}(T + A)$. The operator $T + A$ being isoloid, a familiar argument [1, Lemma 3.89] shows that $f(\sigma(T + A) \setminus E(T + A)) = f'(T + A))$ for every $f \in H_{c}(\sigma(T + A))$. (Note here that the hypothesis that the isolated eigenvalues have finite multiplicity in [1, Lemma 3.89] is immaterial to our case.)

Since $f(\sigma_{BW}(T + A)) = \sigma_{BW}(f(T + A))$ [13, Theorem 3.4], for every $f \in H_{c}(\sigma(T + A))$,

$$\sigma(f(T + A)) \setminus E(f(T + A)) = \sigma_{BW}(f(T + A)).$$

Observe that SVEP implies that $\sigma(T + Q) = \sigma(T^{*} + Q^{*}) = \sigma_{a}(T^{*} + Q^{*})$, $E_{a}(T^{*} + Q^{*}) = E(T^{*} + Q^{*})$ and the polaroid property of $T + Q$, and therefore of $T^{*} + Q^{*}$, implies that $E(T^{*} + Q^{*}) = \pi(T^{*} + Q^{*}) = \pi(T + Q) = E(T + Q)$. Recall from the proof of Proposition 3.2 of [31] that $\sigma_{SBE_{a}}((T + A)^{*}) = \sigma_{BW}((T + A)^{*}) = \sigma_{BW}(T + A)^{*}$. Hence $\sigma((T + A)^{*}) \setminus E_{a}(T + A)^{*} = \sigma_{BW}((T + A)^{*})$. That is, $(T + A)^{*}$ satisfies property $(gaw)$. Since $T^{*} + A^{*} = (T + A)^{*}$ is $a$-isoloid, $f((T + A)^{*})$ satisfies property $(gaw)$, for every $f \in H_{c}(\sigma(T + A))$. □

Let $m$ be a positive integer. Following [32], we say that $T \in loc(G_{m})$ (or, $T$ satisfies a local growth condition of order $m$) if for every closed set $F \subset \mathbb{C}$ and every $x \in X_{T}(F)$
there exists an analytic function $f : \mathbb{C} \setminus F \to \mathcal{X}$ such that $(T - \lambda I)f(\lambda) \equiv x$ and $\|f(\lambda)\| \leq M[\text{dist}(\lambda, F)]^{-m} \|x\|$, for some $M > 0$ (independent of $F$ and $x$). Hypo-normal operators are $\text{loc}(G_1)$ [53] and spectral operators of type $m - 1$ are $\text{loc}(G_m)$ [33, Theorem XV.6.7]. Evidently, $T \in \text{loc}(G_m) \implies T \in (G_m)$. It is known, [36, Proposition 2], that if the Banach space $\mathcal{X}$ is reflexive (in particular, a Hilbert space), then operators $T \in \text{loc}(G_m)$ satisfy Dunford’s condition (C). Hence $\text{loc}(G_m)$ operators $T \in \mathcal{B}(\mathcal{X})$ such that $\mathcal{X}$ is reflexive have SVEP, which implies that both $T$ and $T^*$ satisfy $a$-Browder’s theorem. If $T \in \text{loc}(G_m) \cap \mathcal{B}(\mathcal{X})$, $\mathcal{X}$ is reflexive. Rashid [51] proved that $f(T)$ satisfies generalized Weyl’s theorem and $f(T^*)$ satisfies generalized $a$-Weyl’s theorem for every function $f \in \mathcal{H}(\sigma(T))$. The hereditarily polaroid property of $T$ in the proof of Theorem 3.16 is used only to deduce SVEP (and the polaroid property): the theorem holds for polynomially polaroid operators with SVEP. Consequently:

**Corollary 3.17** If the Banach space $\mathcal{X}$ is reflexive (in particular, if it is a Hilbert space), and $T \in \mathcal{B}(\mathcal{X})$ is a polynomially $\text{loc}(G_m)$ operator which commutes with the algebraic operator $A \in \mathcal{B}(\mathcal{X})$, then $f((T + A)^*)$ satisfies property (gow), for every $f \in H_c(\sigma(T + A))$.

**Proof.** Since $g(T) \in \text{loc}(G_m)$ for some polynomial $g$ implies $g(T) \in (G_m)$ (i.e., $g(T)$ satisfies a growth condition of order $m$), and since $(G_m)$ operators are polaroid (see [32]), $T$ is polaroid. If $\mathcal{X}$ is reflexive, then $g(T)$ has SVEP (see [36, Proposition 2]). Hence $T$ has SVEP. □

### 4 Applications

In this section, we give examples of classes of operators which have attracted the attention of a number of authors in connection with Weyl’s theorem and its variants.

**Example 4.1** Let $P(\mathcal{X})$ the class of all operators $T \in \mathcal{B}(\mathcal{X})$ such that for every complex number $\lambda$ there exists an integer $p := p(\lambda) \in \mathbb{N}$ for which the following condition holds $\ker(T - \lambda I)^p = \ker(T - \lambda I)^q$. The class $P(\mathcal{X})$ contains the classes of generalized scalar, sub-scalar, algebraically totally paranormal and transaloid operators on a Banach space, and multipliers of commutative semi-simple Banach algebra, and $*$-totally paranormal, $M$-hyponormal, $p$-hyponormal $(0 < p < 1)$ and log-hyponormal, algebraically $w$-hyponormal, algebraically $(p, k)$-hyponormal, algebraically $wF(p, q, r)$ operators on a Hilbert space (see [23], [39], [40], [43, 7, 7, 47]). Every $T \in P(\mathcal{X})$ is polaroid and has SVEP, hence $P(\mathcal{X}) \subseteq PS(\mathcal{X})$.

**Example 4.2** An operator $T \in \mathcal{B}(\mathcal{X})$ is totally hereditarily normaloid, $T \in THN$, if every part, and $T_p^{-1}$ for every invertible part $T_p$ of $T$ is normaloid (i.e., the norm of the part equals its spectral radius); $T$ is completely hereditarily normaloid, $T \in CHN$, if either $T \in THN$ or $T - \lambda I$ is normaloid for every $\lambda \in \mathbb{C}$. $CHN$ operators are simply hereditarily polaroid, i.e., the poles of every part of the operator are simple (or order one) [26, Proposition 2.1]. In particular, paranormal operator ([35, page 229]) are simply $HP$ operators.

**Example 4.3** Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is hereditarily normaloid, $T \in HN$, if every part of $T$ is normaloid; a part of $T$ is its restriction to an invariant subspace.
We say that $T \in HN$ is totally hereditarily normaloid if also every invertible part of $T$ is normaloid, and $T$ is completely totally hereditarily normaloid, $T \in CHN$, if either $T$ is totally hereditarily normaloid or $T - \lambda I \in HN$ for every $\lambda \in \mathbb{C}$. The class of operators $CHN$ is independent of the class $P(\mathcal{X})$ with a non empty intersection (see [26]). Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is paranormal, if $\|Tx\|^2 \leq \|T^2x\|$ for each unit vector $x \in \mathcal{X}$. Paranormal operators are example of $CHN$ operators which are not in $P(\mathcal{X})$ (see [5]). $CHN$ operators are polaroid (see [26, Proposition 2.1]) and has SVEP (see [28]). Hence $CHN \in PS(\mathcal{X})$.

**Example 4.4** A Hilbert space operator $T \in \mathcal{B}(\mathcal{H})$ is $(p,k)$-quasihyponormal, $T \in (p,k) - Q$, for some integer $k \in \mathbb{N}$ and $0 < p < 1$, if $T^{*k}([|T|^{2p} - |T^*|^{2p}]T^k \geq 0$ (see [37,54]). The restriction of a $(p,k) - Q$ operator to an invariant subspace is again $(p,k) - Q$ (see [37]) and algebraically $(p,k) - Q$ if there exists a non-constant complex polynomial $p(.)$ such that $p(T)$ is $(p,k) - Q$. Since algebraically $(p,k) - Q$ operators are polaroid (see [47, Proposition 2.10]). The class algebraically $(p,k) - Q$ is independent of class $P(\mathcal{X})$ and $CHN$. Operators in algebraically $(p,k) - Q$ has SVEP (see [47, Corollary 2.9]). Hence algebraically $(p,k) - Q \in PS(\mathcal{X})$.

**Example 4.5** An operator $T \in \mathcal{B}(\mathcal{H})$ is a 2-isometry (or, a 2-isometric operator) if $T^{*2}T^2 - 2T^*T + I = 0$. Every 2-isometric operator is left invertible, if $T$ is not invertible, then $\sigma(T)$ is the closed unit disc (iso$\sigma(T) = \emptyset$), and if $T$ is invertible, then it is a unitary. Evidently, the restriction of a 2-isometry to an invariant subspace is a 2-isometry. Hence, 2-isometric operators are polaroid and has SVEP and hence operators 2-isometry are in $PS(\mathcal{X})$.

**Example 4.6** Operators in $(G_m)$ are polaroid (see, for example, [25, Lemma 3]), $loc(G_m)$ operators are polaroid. However, $loc(G_m)$ operators are not HP. Moreover, operators in $loc(G_m)$ has SVEP (see [32,51]). Hence, if $T \in loc(G_m)$, then $T \in PS(\mathcal{X})$.

**Example 4.7** An operator $T \in \mathcal{B}(\mathcal{X})$ is polynomially $PS(\mathcal{X})$ if there exists a non-trivial polynomial $h$ such that $h(T) \in PS(\mathcal{X})$. polynomially $PS(\mathcal{X})$ are $PS(\mathcal{X})$. Note that polynomially $PS(\mathcal{X})$ operators $T$ retains many of the property of $h(T)$. In particular, $T$ is polaroid and has the SVEP (see [28]). Thus if $T$ is polynomially $PS(\mathcal{X})$ then $T \in PS(\mathcal{X})$.

**Remark 4.1** If $T$ is in $P(\mathcal{X})$, $CHN$, $HP$, 2-isometric, algebraically $(p,k) - Q$, $loc(G_m)$ or polynomially $PS(\mathcal{X})$ then from 3.9 property (gaw) holds for $f(T^*)$, for every $f \in H(\sigma(T))$. Also, if $A$ is an algebraic operator, then by Theorem 3.11 the property (gaw) holds for $f(T^* + A^*)$, for every $f \in H(\sigma(T + A))$, and if $Q$ is quasinilpotent, then from Theorem 3.15 the property (gaw) holds for $f(T^* + Q^*)$, for every $f \in H(\sigma(T + Q))$.

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