On the Stanley depth of a special class of Borel type ideals

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Abstract We give sharp bounds for the Stanley depth of a special class of monomial ideals of Borel type.

Keywords monomial ideals · ideals of Borel type · Stanley depth

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Introduction

Let \( K \) be a field and \( S = K[x_1, \ldots, x_n] \) the polynomial ring over \( K \). Let \( M \) be a \( \mathbb{Z}^n \)-graded \( S \)-module. A Stanley decomposition of \( M \) is a direct sum \( \mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i] \) of \( \mathbb{Z}^n \)-graded \( K \)-vector spaces, where \( m_i \in M \) is homogeneous with respect to \( \mathbb{Z}^n \)-grading, \( Z_i \subset \{ x_1, \ldots, x_n \} \) such that \( m_i K[Z_i] = \{ um_i : u \in K[Z_i] \} \subset M \) is a free \( K[Z_i] \)-submodule of \( M \). We define \( \text{sdepth}(\mathcal{D}) = \min_{1 \leq i \leq r} |Z_i| \) and \( \text{sdepth}(M) = \max \{ \text{sdepth}(\mathcal{D}) | \mathcal{D} \text{ is a Stanley decomposition of } M \} \). The number \( \text{sdepth}(M) \) is called the Stanley depth of \( M \).

Stanley conjectured in [17] that \( \text{sdepth}(M) \geq \text{depth}(M) \) for any \( M \). The conjecture was disproved in [9] for \( M = S/I \), where \( I \subset S \) is a monomial ideal, but remains open in the case \( M = S/I \). Herzog, Vladoiu and Zheng showed in [13] that \( \text{sdepth}(M) \) can be computed in a finite number of steps if \( M = I/J \), where \( J \subset I \subset S \) are monomial ideals. In [16], Rinaldo gave a computer implementation for this algorithm, in the computer algebra system CoCoA [8]. For an introduction in the thematic of Stanley depth, we refer the reader to [10].

We say that a monomial ideal \( I \subset S \) is of Borel type, see [12], if it satisfies the following condition: \( (I : x_j^n) = (I : (x_1, \ldots, x_j)^{\infty}) \), \( (\forall) 1 \leq j \leq n \). The Mumford-Castelnuovo regularity of \( I \) is the number \( \text{reg}(I) = \max \{ j - i : \beta_{ij}(I) \neq 0 \} \), where \( \beta_{ij} \)'s are the graded Betti numbers.

The regularity of the ideals of Borel type was extensively studied, see for instance [12], [1] and [5]. We study the invariant \( \text{sdepth}(I) \), for an ideal of Borel
type. In the general case, we note some bounds for \( \text{sdepth}(I) \), see Proposition 1.2 and we give some tighter ones, when \( I \) has a special form, see Theorem 1.6.

1 Main results

First, we recall the construction of the sequential chain associated to a Borel type ideal \( I \subset S \), see [12] for more details. Assume that \( \text{Ass}(S/I) = \{ P_0, \ldots , P_m \} \) with \( P_i = (x_1, \ldots , x_m) \), where \( n \geq n_0 > n_1 > \cdots > n_m \geq 1 \). Also, assume that \( I = \bigcap_{i=0}^m Q_i \), is the reduced primary decomposition of \( I \), with \( P_i = \sqrt{Q_i} \), for all \( 0 \leq i \leq m \).

We define \( I_k := \bigcap_{j=k}^m Q_j \), for all \( 0 \leq k \leq m \). One can easily check that \( I_i = (I_{i-1} : x_{n_{i-1}}^\infty) \), for all \( 1 \leq i \leq m \). The sequence of ideals \( I = I_0 \subset I_1 \subset \cdots \subset I_m \subset I_{m+1} := S \) is called the sequential sequence of \( I \). Let \( I_i \) be the monomial ideal generated by \( G(I_i) \) in \( S_i := K[x_1, \ldots , x_n] \), for all \( 0 \leq i \leq m \). Then, the saturation \( I_i^{sat} = (I_i : (x_1, \ldots , x_n)^\infty) = J_i + S_i \), for all \( 0 \leq i \leq m \), where \( J_{m+1} := S_{m+1} \). One has \( I_{i+1} / I_i \cong (J_{i+1}^{sat} / J_i)[x_{n+i+1}, \ldots , x_n] \). If \( M = \bigoplus_{i \geq 0} M_i \) is an Artinian graded \( S \)-module, we denote \( s(M) = \max \{ t : M_t \neq 0 \} \). We recall the following result.

Proposition 1.1 [12, Corollary 2.7] \( \text{reg}(I) = \max \{ s(J_0^{sat} / J_0), \ldots , s(J_m^{sat} / J_m) \} + 1 \).

Proposition 1.2 With the above notations, the following assertions hold:

1. \( \text{sdepth}(S/I) = \text{depth}(S/I) = n - n_0 \), for all \( 0 \leq i \leq m \).
2. \( \text{sdepth}(I_0) \leq \text{sdepth}(I_1) \leq \cdots \leq \text{sdepth}(I_m) \).
3. \( \text{depth}(I_i) = n - n_i + 1 \leq \text{sdepth}(I_i) \leq \text{sdepth}(P_i) = n - \left\lfloor \frac{n_i}{2} \right\rfloor \), \( \forall 0 \leq i \leq m \).

Proof: (1) From [13, Lemma 3.6] it follows that \( \text{sdepth}(S/I) = \text{sdepth}(S_i/J_i) + n - n_i \). Also, we have \( \text{sdepth}(S_i/J_i) \leq \text{depth}(S_i/J_i) + n - n_i \).

Since \( P_i S_i = (x_1, \ldots , x_n) S_i \subset \text{Ass}(S_i/J_i) \), it follows that \( \text{depth}(S_i/J_i) = 0 \) and thus, by [6, Theorem 1.4] or [10, Proposition 18], we get \( \text{sdepth}(S_i/J_i) = \text{depth}(S_i/J_i) = 0 \).

(2) Since \( I_i = (I_{i-1} : x_{n_{i-1}}^\infty) \), by [14, Proposition 1.3] (see arXiv version), we get \( \text{sdepth}(I_{i-1}) \leq \text{sdepth}(I_i) \), for all \( 1 \leq i \leq m \).

3. Since \( I_i = J_i S_i \), by [13, Lemma 3.6], it follows that \( \text{sdepth}(I_i) = n - n_i + \text{sdepth}_S(J_i) \geq n - n_i + 1 \). Since \( P_i \in \text{Ass}(I_i) \), it follows that there exists a monomial \( v \in S_i \) such that \( P_i = (I_i : v) \). Therefore, by [14, Proposition 1.3] (see arXiv version), it follows that \( \text{sdepth}(P_i) \geq \text{sdepth}(I_i) \). On the other hand, \( P_i \) is generated by variables. Thus, by [13, Lemma 3.6] and [2, Theorem 1.1], it follows that \( \text{sdepth}(P_i) = n - \left\lfloor \frac{n_i}{2} \right\rfloor = n - \left\lfloor \frac{n_i}{2} \right\rfloor \).

\[ \square \]

Lemma 1.3 Let \( r \leq n \) and \( a_1, \ldots , a_r \) be positive integers. If \( Q = (x_1^{a_1}, \ldots , x_r^{a_r}) \subset S \), then \( \text{reg}(Q) = a_1 + \cdots + a_r - r + 1 \).

Proof. Let \( \bar{Q} := Q \cap S' \subset S' \), where \( S' = K[x_1, \ldots , x_r] \). As a particular case of Proposition 1.1, we get \( \text{reg}(Q) = \text{reg}(\bar{Q}) = s(S' / \bar{Q}) + 1 = a_1 + \cdots + a_r - r + 1 \).

We recall the following result from [1].
Proposition 1.4 [1, Corollary 3.17] If \( I \subset S \) is an ideal of Borel type with the irredundant irreducible decomposition \( I = \bigcap_{j=1}^{r} C_j \), then \( \text{reg}(I) = \max\{\text{reg}(C_i) : 1 \leq i \leq r\} \).

Let \( n \geq n_0 > n_1 > \cdots > n_m \geq 1 \) be some integers. Let \( a_{ij} \) be some positive integers, where \( 0 \leq i \leq m \) and \( 1 \leq j \leq n_i \). We consider the monomial irreducible ideals \( Q_i = (x_{i1}^{a_{i1}}, \ldots, x_{in_i}^{a_{in_i}}) \), for \( 0 \leq i \leq m \). Let \( I_i := \bigcap_{j=1}^{n_i} Q_j \) and denote \( I = I_0 \). Since \( P_i = (x_1, \ldots, x_{n_i}) = \sqrt{Q_i} \) for all \( 0 \leq i \leq m \), by [11, Proposition 5.2] or [5, Corollary 1.2], it follows that \( I \) is an ideal of Borel type. As a direct consequence of Lemma 1.3 and Proposition 1.4, we get the following corollary.

Corollary 1.5 If \( a_{ij} \geq a_{i+1,j} \) for all \( j \leq n_{i+1} \) and \( i \leq m \), then \( \text{reg}(I_i) = \text{reg}(Q_i) = a_{i1} + a_{i2} + \cdots + a_{in_i} - n_i + 1 \), for all \( 0 \leq i \leq m \).

Theorem 1.6 If \( a_{ij} \geq a_{i+1,j} \) for all \( j \leq n_{i+1} \) and \( i \leq m \), then for all \( 0 \leq i \leq m \), it holds that

\[
\text{sdepth}(I_i) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i.
\]

Proof: The first inequality follows from Proposition 1.2(3). In order to prove the second one, let \( i \leq m \). If \( i = m \), then \( I_m = Q_m \) is an irreducible ideal, and therefore, by [7, Theorem 1.3], \( \text{sdepth}(I_m) = n - \left\lfloor \frac{n_m}{2} \right\rfloor = n + \left\lceil \frac{n_m}{2} \right\rceil - n_m \).

Assume \( i < m \). We can write \( Q_i = U_i + V_i \), where \( U_i = (x_{i1}^{a_{i1}}, \ldots, x_{in_i}^{a_{in_i}}) \) and \( V_i = (x_{n_{i+1}}^{a_{i+1}}, \ldots, x_{n_i}^{a_{ni}}) \). Since \( a_{ij} \geq a_{i+1,j} \) for all \( j \leq n_{i+1} \), it follows that \( U_i \subset Q_{i+1} \).

Therefore, \( I_i = (U_i + V_i) \cap I_{i+1} = (U_i \cap I_{i+1}) + (V_i \cap I_{i+1}) \). Note that \( J := U_i \cap I_{i+1} = U_i \cap I_{i+2} \) is a Borel type ideal with the irredundant irreducible decomposition \( J = U_i \cap Q_{i+2} \cap \cdots \cap Q_m \), and, therefore, of the same class as \( I_{i+1} \). Thus, by induction hypothesis, it follows that \( \text{sdepth}(J) \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_{i+1} \).

On the other hand, by [4, Remark 1.3] and the induction hypothesis, \( \text{sdepth}(V_i \cap I_{i+1}) \geq \text{sdepth}(V_i) + \text{sdepth}(I_{i+1}) - n = \text{sdepth}(I_{i+1}) - \left\lceil \frac{n_{i+1}}{2} \right\rceil \).

Let \( \tilde{V} \subset S' = K[x_{n_{i+1}}, \ldots, x_{n_i}] \) be the monomial ideal generated by \( G(V_i) \) and let \( J \subset S'' = K[x_1, \ldots, x_{n_{i+1}}, x_{n_i+1}, \ldots, x_{n_i}] \) be the monomial ideal generated by \( G(J) \).

Since \( J \subset I_{i+1} \), it follows that \( I_i = (J \cap_K (S' / \tilde{V})) \oplus (V_i \cap I_i) \). By [3, Proposition 2.10] and [15, Lemma 2.2], we get:

\[
\text{sdepth}(I_i) \geq \min \{ \text{sdepth}(J) - n_i + n_{i+1}, \text{sdepth}(I_{i+1}) - \left\lceil \frac{n_i - n_{i+1}}{2} \right\rceil \} \geq n + \left\lceil \frac{n_m}{2} \right\rceil - n_i,
\]
as required. \( \square \)

Question: What can we say about the case when the condition \( a_{ij} \geq a_{i+1,j} \) is removed? Of course, the method used in the proof of the Theorem 1.6 does not work. However, our computer experiments in \texttt{Cocoa} [8] suggested that the conclusion of the Theorem 1.6 might be true. Unfortunately, we are not able to give either a proof, or a counterexample.

The next example shows that the bounds given in Theorem 1.6 are sharp.
Example 1.1 Let $I = Q_0 \cap Q_1$, where $Q_0 = (x_1^3, x_2^2, x_3^2, x_4, x_5)$, $Q_1 = (x_1, x_2, x_3, x_4)$ are ideals in $S = K[x_1, \ldots, x_5]$. Then $I_1 = Q_1$ and $sdepth(I_1) = 5 - \left\lfloor \frac{n}{2} \right\rfloor = 3$. Also $n = 5$, $n_0 = 5$ and $n_1 = 4$. Using CoCoa, we get $sdepth(I) = 2 = n - \left\lfloor \frac{n_0}{2} \right\rfloor - n_0$. Let $Q' = (x_1^2, x_2^2, x_3, x_4, x_5) \subseteq S$ and $I' = Q'_0 \cap Q_1$. Using CoCoa [8], we get $sdepth(I') = 3 = n - \left\lfloor \frac{n_0}{2} \right\rfloor$.

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