The periods of the k-step Fibonacci and k-step Pell sequences in $D_{2n} \times \mathbb{Z}_2^i$ and $D_{2n} \times \varphi \mathbb{Z}_2^i$

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Abstract The direct product $D_{2n} \times \mathbb{Z}_2^i$ and the semidirect product $D_{2n} \times \varphi \mathbb{Z}_2^i$, for $n, i \geq 3$ are defined by the presentations

$$D_{2n} \times \mathbb{Z}_2^i = \langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = [x, z] = [y, z] = e \rangle$$

and

$$D_{2n} \times \varphi \mathbb{Z}_2^i = \langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = e, z^{-1} x z x = e, z^{-1} y z y = e \rangle,$$

respectively. In this paper, we obtain the periods of the $k$-nacci sequences and the generalized order-k Pell sequences in the direct product $D_{2n} \times \mathbb{Z}_2^i$ and the semidirect product $D_{2n} \times \varphi \mathbb{Z}_2^i$, for $n, i \geq 3$.

Keywords Period · sequence · direct product · semidirect product

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1 Introduction and preliminaries

The recurrence sequences in groups was firstly studied by Wall [18] who calculated the periods of the Fibonacci sequences in cyclic groups. As a natural generalization of the problem, Wilcox [19] investigated the Fibonacci lengths to abelian groups. In [3] the Fibonacci length of a 2-generator group is defined, thus extending the idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation first introduced by Wall in [18]. The theory has been expanded to nilpotent groups, see for example [1, 16]. Campbell et al. [2] examined the Fibonacci lengths of $D_{2n}$, $i > 1$ where $D_{2n}$ is the dihedral group of order $2n$. Knox [14] proved that the periods of the $k$-nacci ($k$-step Fibonacci) sequences in dihedral groups were equal to $2k+2$. Li and Wang [15] contributed to the study of the Wall number for the $k$-step
Fibonacci sequence. Falcon and Plaza \[10\] examined the periods \( k \)-Fibonacci sequences modulo \( m \). Deveci and Karaduman \[8\] calculated the periods of the \( k \)-nacci sequences in the semidirect product \( \mathbb{Q}_2 \times \mathbb{Z}_{2m} \). Deveci and Karaduman \[6\] expanded the concept to Pell sequences. Recently, many authors have studied some special linear recurrence sequences in groups; see for example, \[5,7,12,17\].

Let \( f_{n}^{(k)} \) denote the \( n \)th number of the \( k \)-step Fibonacci sequence defined as

\[
f_{n}^{(k)} = \sum_{j=1}^{k} f_{n-j}^{(k)} \quad \text{for } n > k \quad (1.1)
\]

with boundary conditions \( f_{i}^{(k)} = 0 \) for \( 1 \leq i < k \) and \( f_{k}^{(k)} = 1 \). Reducing this sequence by a modulus \( m \), we can get a repeating sequence, which we denote by

\[
f_{n}^{(k,m)} = f_{n}^{(k)} \pmod{m}
\]

where \( f_{n}^{(k,m)} = f_{n}^{(k)} \pmod{m} \). We then have that \( (f_{1}^{(k,m)}, f_{2}^{(k,m)}, \ldots, f_{k}^{(k,m)}) \) \((0, 0, \ldots, 0, 1)\) and it has the same recurrence relation as in \((1.1)\).

For more information see \[15\].

**Theorem 1.1** \( f_{n}^{(k,m)} \) is a periodic sequence \[15\].

Let \( h_{k}(m) \) denote the smallest period of \( f_{n}^{(k,m)} \), called the period of \( f_{n}^{(k,m)} \) or the Wall number of the \( k \)-step Fibonacci sequence modulo \( m \). It is important to note that \( h_{k}(2) = k + 1 \).

For more information see \[15\].

**Definition 1.2** A \( k \)-nacci sequence in a finite group is a sequence of group elements \( x_{0}, x_{1}, x_{2}, \ldots \) for which, given an initial (seed) set \( x_{0}, \ldots, x_{j-1} \), each element is defined by

\[
x_{n} = \begin{cases} 
x_{0}x_{1}\cdots x_{n-1} & \text{for } j \leq n < k, \\
x_{n-k}x_{n-k+1}\cdots x_{n-1} & \text{for } n \geq k.
\end{cases}
\]

We also require that the initial elements of the sequence, \( x_{0}, \ldots, x_{j-1} \), generate the group, thus forcing the \( k \)-nacci sequence to reflect the structure of the group. The \( k \)-nacci sequence of a group generated by \( x_{0}, \ldots, x_{j-1} \) is denoted by \( F_{k}(G; x_{0}, \ldots, x_{j-1}) \) \[14\].

It is well-known that a sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence.

**Theorem 1.3** A \( k \)-nacci sequence in finite group is simply periodic \[14\].
In [14], Knox had denoted the period of the sequence $F_k(G; x_0, \ldots, x_{j-1})$ by $P_k(G; x_0, \ldots, x_{j-1})$.

In [13], Kiliç and Taşci defined the $k$ sequences of the generalized order-$k$ Pell numbers as follows:

for $n > 0$ and $1 \leq i \leq k$

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i,$$

(1.2)

with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 - k \leq n \leq 0$,

where $P_n^i$ is the $n$th term of the $i$th sequence. If $k = 2$, the generalized order-$k$ Pell sequence, $\{P_n^k\}$, is reduced to the usual Pell sequence, $\{P_n\}$.

Reducing the generalized order-$k$ Pell sequence by a modulus $m$, we can get a repeating sequence, denoted by

$$\{P_{n,m}^k\} = \{P_{1-k}^{k,m}, P_{2-k}^{k,m}, \ldots, P_{0}^{k,m}, P_{1}^{k,m}, P_{2}^{k,m}, \ldots, P_{n}^{k,m}, \ldots\},$$

where $P_{n,m}^k = P_n^k \mod m$. It has the same recurrence relation as in (1.2). For more information see [6].

**Theorem 1.4** $\{P_{n,m}^k\}$ is a periodic sequence [6].

Let the notation $hP_k^r(m)$ denotes the smallest period of $\{P_{n,m}^k\}$, called the period of the generalized order-$k$ Pell sequence modulo $m$. When $k = 2$, $hP_2^r(m)$ is the period of the Pell sequence modulo $m$ [6].

**Definition 1.5** A generalized order-$k$ Pell sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, \ldots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_0 x_1 \cdots (x_{n-1})^2 & \text{for } j \leq n < k, \\ x_{n-k} x_{n-k+1} \cdots (x_{n-1})^2 & \text{for } n \geq k. \end{cases}$$

It is required that the initial elements of the sequence, $x_0, \ldots, x_{j-1}$, generate the group, thus, forcing the generalized order-$k$ Pell sequence to reflect the structure of the group. We denote the generalized order-$k$ Pell sequence of a group $G$ generated by $x_0, \ldots, x_{j-1}$ by $Q_k(G; x_0, \ldots, x_{j-1})$ [6].

**Theorem 1.6** A generalized order-$k$ Pell sequence in a finite group is periodic [6].

In [6], Deveci and Karaduman had denoted the period of the sequence $Q_k(G; x_0, \ldots, x_{j-1})$ by $\text{Per}Q_k(G; x_0, \ldots, x_{j-1})$.

From the definitions it is clear that the periods of both the $k$-nacci sequence and the generalized order-$k$ Pell sequence in a group depend on the chosen generating set and the order in which the assignments of $x_0, x_1, x_2, \ldots, x_{j-1}$ are made.
2 Main results and proofs

We use the natural generating set for \( D_{2n} \), as in [4], defined as satisfying
\[ D_{2n} = \langle x, y : x^2 = y^n = (xy)^2 = e \rangle. \]
This is extended to direct product by using the following well known method of construction:
If \( G_1 = \langle A : R_1 \rangle \) and \( G_2 = \langle B : R_2 \rangle \), then \( G_1 \times G_2 = \langle A, B : R_1, R_2, [A, B] \rangle \)
where \( [A, B] = \{ [a, b] : a \in A, b \in B \} \), see [11].
The direct product \( D_{2n} \times \mathbb{Z}_{2^i} \), \( n, i \geq 3 \) is defined by the presentation
\[ D_{2n} \times \mathbb{Z}_{2^i} = \langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = [x, z] = [y, z] = e \rangle. \]

The usual notation \( G_1 \times_\varphi G_2 \) is used for the semidirect product of the group \( G_1 \) by \( G_2 \), where \( \varphi : G_2 \to \text{Aut } (G_1) \) is a homomorphism such that \( b \varphi = \varphi_b \) and \( \varphi_b : G_1 \to G_1 \) is an element \( \text{Aut } (G_1) \).
The semidirect product \( D_{2n} \times_\varphi \mathbb{Z}_{2^i} \) \( n, i \geq 3 \) is defined by the presentation
\[ D_{2n} \times_\varphi \mathbb{Z}_{2^i} = \langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = e, z^{-1} xzx = e, z^{-1} yzy = e \rangle. \]

Where if \( \mathbb{Z}_{2^i} = \langle z \rangle \), then \( \varphi : \mathbb{Z}_{2^i} \to \text{Aut } (D_{2n}) \) is a homomorphism such that \( z \varphi = \varphi_z ; \varphi_z : D_{2n} \to D_{2n} \) is defined by \( x \varphi_z = x \) and \( y \varphi_z = y^{-1} \).
For more information see [9].

**Theorem 2.1** The periods of the \( k \)-nacci sequences in the direct product \( D_{2n} \times \mathbb{Z}_{2^i} \), \( n, i \geq 3 \) are as follows:
i. \( P_2(D_{2n} \times \mathbb{Z}_{2^i} ; x, y, z) = \text{lcm } [h_2 (2^i) , h_2 (n)] \), the least common multiple of the \( h_2 (2^i) \) and the \( h_2 (n) \).
ii. If \( k \geq 3 \), then \( P_k(D_{2n} \times \mathbb{Z}_{2^i} ; x, y, z) = \frac{\text{lcm } [2^i, n] (k+1)}{2} \).

**Proof.** We first note that in the group defined
\[ \langle x, y, z : x^2 = y^n = (xy)^2 = z^{2^i} = [x, z] = [y, z] = e \rangle, \]
\( xz = zx \) and \( yz = zy \).
i. If \( k = 2 \), we have the sequence
\[ x_0 = x, x_1 = y, x_2 = z, x_3 = yz, x_4 = z^2y, x_5 = z^3y^2, x_6 = z^5y^3, x_7 = z^8y^5, x_8 = z^{13}y^8, x_9 = z^{21}y^{13}, x_{10} = z^{34}y^{21}, \ldots. \]

From this sequence we obtain a subsequence as follow:
\[ a_1 = y, a_2 = z, a_3 = yz, a_4 = z^2y, a_5 = z^3y^2, a_6 = z^5y^3, a_7 = z^8y^5, a_8 = z^{13}y^8, a_9 = z^{21}y^{13}, a_{10} = z^{34}y^{21}, \ldots. \]

In fact it is easy to see that the 2-nacci sequence conforms to the following pattern:
\[ a_{t+1} = z^{f(2)} y^{f(2)}, \]
\[ a_{t+2} = z^{f(2)} y^{f(2)}. \]
We need the smallest $t$, satisfying $a_{t+1} = y$ and $a_{t+2} = z$. Letting
\[ \text{lcm} \left[ h_2(2^i), h_2(n) \right] = \alpha, \]
then we have $2^i \mid f_{a_{t+1}}^{(2)}, n \mid f_{a_{t+1}}^{(2)}, f_{a_{t+2}}^{(2)} \equiv 1 \mod 2^i, f_{a_{t+2}}^{(2)} \equiv 1 \mod n, f_{a_{t+2}}^{(2)} \equiv 1 \mod 2^i$ and $f_{a_{t+2}}^{(2)} \equiv 1 \mod n$ (where by $2^i \mid f_{a_{t+1}}^{(2)}$ we mean that $2^i$ divides $f_{a_{t+1}}^{(2)}$). If we choose $t = \alpha$, then we obtain $a_{\alpha+1} = y$ and $a_{\alpha+2} = z$. So we get $P_2(D_{2n} \times \mathbb{Z}_2; x, y, z) = \text{lcm} \left[ h_2(2^i), h_2(n) \right]$.

If $k = 3$, then we have the sequence
\[
\begin{align*}
&x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \quad x_4 = z^2x, \\
&x_5 = z^4y^{-1}, \quad x_6 = z^7y^{-2}, \quad x_7 = z^{13}y^3x, \quad x_8 = z^{24}x, \\
&x_9 = z^{44}y, \quad x_{10} = z^{81}y^4, \quad x_{16} = z^{3136}x, \\
&x_{17} = z^{5768}y, \quad x_{18} = z^{10609}y^8, \quad x_{24} = z^{410744}x, \\
&x_{25} = z^{755476}y, \quad x_{26} = z^{13865657}y^{12}, \quad x_{32} = z^{53798080}x, \\
&x_{33} = z^{9895096y}, \quad x_{34} = z^{181997601}y, \quad \ldots.
\end{align*}
\]

Using the above, the 3-nacci sequence becomes:
\[
x_{a_8} = z^{2^{4a+1}}x, x_{a_8+1} = z^{2^{4a+1}+1}y, x_{a_8+2} = z^{2^{4a+2}+1}y^4, \ldots.
\]

Where $t$ is odd and $\alpha \in \mathbb{N}$ such that $2^\alpha \cdot t$ and $u_\alpha$, $u_{\alpha+1}$, $u_{\alpha+2} \in \mathbb{N}$ such that $\text{lcm} \left[ u_\alpha, u_{\alpha+1}, u_{\alpha+2} \right] = 1$. We need the smallest $\alpha$, satisfying $x_{a_8} = x$, $x_{a_8+1} = y$ and $x_{a_8+2} = z$. If we choose $\alpha = \frac{\text{lcm}[2^i, n]}{4}$, then we obtain $x_{2\text{lcm}[2^i, n]} = x$, $x_{2\text{lcm}[2^i, n]+1} = y$ and $x_{2\text{lcm}[2^i, n]+2} = z$. So we get $P_k(D_{2n} \times \mathbb{Z}_2; x, y, z) = 2\text{lcm} \left[ 2^i, n \right]$.

If $k \geq 4$, we have the sequence
\[
\begin{align*}
&x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \\
&x_4 = z^2, \quad \ldots, \quad x_k = z^{2^{k-3}}, \quad x_{k+1} = z^{2^{k-2}}, \\
&x_{k+2} = z^{2^{k-1}+1}y^{-1}, \quad x_{k+3} = z^{2^{k-1}+1}y^{-2}, \quad x_{k+4} = z^{2^{k-1}+3}y^3x, \\
&x_{k+5} = z^{2^{k+2}+2}, \quad x_{k+6} = z^{2^{k+3}+2}, \quad \ldots, \\
&x_{2k+1} = z^{2^{2k-2}+2}, \quad x_{2k+2} = z^{2^{2k-1}+1}y^{-1}, \\
&x_{2k+3} = z^{2^{2k-1}+2}x, \quad x_{2k+4} = z^{2^{2k+1}+2}(k-1)y, \quad \ldots.
\end{align*}
\]

Using the above, the $k$-nacci sequence becomes:
\[
\begin{align*}
x_{a(2k+2)-k+3} = z^{2^{4a+3}}, \quad x_{a(2k+2)-k+4} = z^{4a+4}, \\
x_{a(2k+2)-1} = z^{2^{4a+k+3}}, \quad x_{a(2k+2)} = z^{2^{4a+k}x}, \\
x_{a(2k+2)+1} = z^{2^{4a+1}+1}y, \quad x_{a(2k+2)+2} = z^{4a+2+1}y^4a, \quad \ldots,
\end{align*}
\]

where $t$ is a positive odd integer and $\alpha$ is a positive integer such that $\alpha = 2^\tau \cdot t$,
\[
\tau_\alpha, \tau_{\alpha+1}, \ldots, \tau_{\alpha+k-4}, \varepsilon_\alpha, \varepsilon_{\alpha+1}, \varepsilon_{\alpha+2} \in \mathbb{N}
\]
and
\[
\text{lcm} \left[ \tau_\alpha, \tau_{\alpha+1}, \ldots, \tau_{\alpha+k-4}, \varepsilon_\alpha, \varepsilon_{\alpha+1}, \varepsilon_{\alpha+2} \right] = 1.
\]
We need the smallest $\alpha$, satisfying $x_{\alpha(2k+2)} = e$, $x_{\alpha(2k+2)+1} = y$ and $x_{\alpha(2k+2)+2} = z$, where $k - 3 \leq l \leq 1$. If we choose $\alpha = \frac{\lcm[2^i, n]}{4}$, then we obtain
\[
\begin{align*}
x_{\frac{\lcm[2^i, n]}{2}(k+1) - l} &= e, \quad x_{\frac{\lcm[2^i, n]}{2}(k+1)} = x, \\
x_{\frac{\lcm[2^i, n]}{2}(k+1)+1} &= y \text{ and } x_{\frac{\lcm[2^i, n]}{2}(k+1)+2} = z.
\end{align*}
\]
So we get $P_k(D_{2n} \times \mathbb{Z}_2; x, y, z) = \frac{\lcm[2^i, n](k+1)}{2}$. \hfill \Box

**Theorem 2.2** The periods of the $k$-nacci sequences in the semidirect product $D_{2n} \times \varphi \mathbb{Z}_{2^i}$, $(n, i \geq 3)$ are as follows:

i. $P_{2,3}(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = h_{2,3}(2^i)$

ii. $P_{4,5}(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = \frac{\lcm[2^i, n](k+1)}{2}$.

iii. Let $k \geq 6$.

i'. If there is no $\omega \in [3, k - 3]$ such that $\omega$ is an odd factor of $n$, then $P_k(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = \frac{\lcm[2^i, n](k+1)}{2}$.

ii'. Let $\lambda$ be the biggest odd factor of $n$ in $[3, k - 3]$, then two cases occur:

1. If $\lambda 3^j \notin [3, k - 3]$ for $j \in \mathbb{N}$, then

\[
P_k(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = \frac{\lcm[2^i, n](k+1) \lambda}{2}.
\]

2. If $\mu$ is the biggest odd number which is in $[3, k - 3]$ and $\mu = \lambda 3^j$ for $j \in \mathbb{N}$, then

\[
P_k(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = \frac{\lcm[2^i, n](k+1) \mu}{2}.
\]

**Proof.** We first note that in the group defined
\[
D_{2n} \times \varphi \mathbb{Z}_2 = \langle x, y, z : x^2 = y^n = (xy)^2 = z^2, z^{-1}xzx = e, z^{-1}yzy = e \rangle,
\]

$xy = y^{-1}x$, $xz = zx$ and $yz = zy^{-1}$.

i. If $k = 2$, we have the sequence
\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = yz, \quad x_4 = z^2y^{-1}, \\
x_5 &= z^3y^{-2}, \quad x_6 = z^4y^{-1}, \quad x_7 = z^5y, \quad x_8 = z^{13}, \quad x_9 = yz^{21}, \ldots
\end{align*}
\]
The 2-nacci sequence can be said to form layers of length 6. Using the above, the 2-nacci sequence becomes:

\[
x_{60+1} = z^{f_{60+1}(2)}y, \quad x_{60+2} = z^{f_{60+2}(2)}y, \ldots
\]

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We need the smallest \( \alpha \), satisfying \( x_{6\alpha+1} = y \) and \( x_{6\alpha+2} = z \). If we choose \( \alpha = 2^{i-2} \), we obtain that

\[
x_{6,2^{i-2}+1} = z^{f(2)}(2^{i-2}) y, \quad x_{6,2^{i-2}+2} = z^{f(2)}(2^{i-2}+1) y.
\]

Since \( h_2 \left( 2^i \right) = 2^i \), \( h_2 \left( 2^i \right) = 2^i \), \( 3 = 6.2^{i-2} \), we have \( 2^i \mid f(2) \) and \( f(2) \equiv 1 \ mod \ 2^i \). Thus, \( x_{6,2^{i-2}+1} = y \) and \( x_{6,2^{i-2}+2} = z \) that is \( P_2(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = h_2 \left( 2^i \right) = 2^{i-1} \).

If \( k = 3 \), we have the sequence

\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \quad x_4 = z^2xy^{-2}, \\
x_5 &= z^2y^{-1}, \quad x_6 = z^2y^{-2}, \quad x_7 = yx^3, \quad x_8 = z^4x, \\
x_9 &= z^4y, \quad x_{10} = z^8, \quad x_{11} = xy^2z^{149}, \ldots
\end{align*}
\]

The 3-nacci sequence can be said to form layers of length 8. Using the above, the 3-nacci sequence becomes:

\[
x_{8\alpha} = z^{f(3)}(3), x_{8\alpha+1} = z^{f(3)}(3), x_{8\alpha+2} = z^{f(3)}(3), \ldots
\]

We need the smallest \( \alpha \), satisfying \( x_{8\alpha} = x \), \( x_{8\alpha+1} = y \) and \( x_{8\alpha+2} = z \). If we choose \( \alpha = 2^{i-2} \), we obtain that

\[
x_{8,2^{i-2}} = z^{f(3)}(3), x_{8,2^{i-2}+1} = z^{f(3)}(3), x_{8,2^{i-2}+2} = z^{f(3)}(3), \ldots
\]

Since

\[
h_3 \left( 2^i \right) = 2^i, h_3 \left( 2^i \right) = 2^{i-1} \cdot 4 = 8.2^{i-2},
\]

we have \( 2^i \mid f(3) \) \( 2^i \) \( f(3) \) \( 2^i \) \( f(3) \) \( 2^i \) \( f(3) \) \( 2^i \) \( f(3) \) \( 2^i \). Thus, \( x_{8,2^{i-2}} = x \), \( x_{8,2^{i-2}+1} = y \) and \( x_{8,2^{i-2}+2} = z \) that is \( P_3(D_{2n} \times \varphi \mathbb{Z}_2; x, y, z) = h_3 \left( 2^i \right) = 2^{i-1} \).

ii. The proof is similar to the proof of Theorem 2.1.ii. and is omitted.

iii. If \( k = 6 \), we have the sequence

\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \\
x_4 &= z^2y^{-2}, \quad x_5 = z^4y^{-4}, \quad x_6 = z^6y^{-8}, \ldots, \\
x_k &= z^{2^{k-3}}y^{-2^{k-3}}, \quad x_{k+1} = xy^{-2^{k-2}}z^{2^{k-2}}, \quad x_{k+2} = z^{2^{k-1}}y^{-1}, \\
x_{k+3} &= z^{2^{k-1}}y^{-2}, \quad x_{k+4} = xz^{2k+1-3}y, \\
x_{k+5} &= z^{2^{k+2}}y^{-24}, \quad x_{k+6} = z^{2^{k+3}}y^{-24}z^{2^{k+2}}, \ldots, \\
x_{2k+1} &= z^{2^{2k-2}2^{i-3}k}y^{4d}, \quad x_{2k+2} = z^{2^{2k-1}2^{i-3}k}y^{4d}x, \\
x_{2k+3} &= z^{2^{2k-2}2^{i-1}(k+2)}y, \quad x_{2k+4} = z^{2^{2k+1}2^{i}+3}y^{4d}z, \
\end{align*}
\]
where \( \vartheta_1, \vartheta_2 \in \mathbb{N} \). Using the above, the \( k \)-\textit{nacci} sequence becomes:

\[
\begin{align*}
& x_{\alpha(2k+2)-k+3} = z^{2^4\beta_1}y^{4\alpha}, \\
& x_{\alpha(2k+2)-k+4} = z^{2^4\beta_{\alpha+1}}y^{8\alpha^2+4\alpha}, \\
& x_{\alpha(2k+2)-k+5} = z^{2^4\beta_{\alpha+2}}y^{12\alpha^2+6\alpha}, \\
& x_{\alpha(2k+2)-k} = z^{2^4\beta_{\alpha+k-4}}y^{4\alpha+k}, \\
& x_{\alpha(2k+2)+1} = z^{2^4\beta_{\alpha+k-3}}xy^{4\alpha+k-5}, \\
& x_{\alpha(2k+2)+2} = z^{2^4\beta_{\alpha+k-2}}y, \\
& x_{\alpha(2k+2)+3} = z^{2^4\beta_{\alpha+k-1}}z, \\
\end{align*}
\]

where \( t \) is a positive odd integer and \( \alpha \) is a positive integer such that \( \alpha = 2^\sigma t \),

\[
\begin{align*}
& \beta_{\alpha}, \beta_{\alpha+1}, \ldots, \beta_{\alpha+k-1}, \ v_1, \ v_2, \ldots, v_{\alpha+k-5} \in \mathbb{N}, \\
& \text{lcm} \left[ \beta_{\alpha}, \beta_{\alpha+1}, \ldots, \beta_{\alpha+k-1} \right] = 1 \\
\text{and} \\
& \text{lcm} \left[ v_1, v_2, \ldots, v_{\alpha+k-5} \right] = 1.
\end{align*}
\]

We need the smallest \( \alpha \), satisfying \( x_{\alpha(2k+2)-l} = e \), \( x_{\alpha(2k+2)} = x \), \( x_{\alpha(2k+2)+1} = y \) and \( x_{\alpha(2k+2)+2} = z \), where \( k - 3 \leq l \leq 1 \).

\( i' \). The proof is similar to the proof of Theorem 2.2.ii. and is omitted.

\( \text{ii'} \). Let \( \lambda \) be the biggest odd factor of \( n \) is \( [3, k - 3] \), then two cases occur:

1. If \( \lambda 3^j \notin [3, k - 3] \) for \( j \in \mathbb{N} \), then we obtain

\[
\begin{align*}
& x_{\text{lcm}[2^i, n] \lambda_{(k+1)\lambda}} = e, \ x_{\text{lcm}[2^i, n] \lambda_{(k+1)\lambda}} = x, \\
& x_{\text{lcm}[2^i, n] \lambda_{(k+1)\lambda}+1} = y \text{ and } x_{\text{lcm}[2^i, n] \lambda_{(k+1)\lambda}+2} = z
\end{align*}
\]

for \( \alpha = \frac{\text{lcm}[2^i, n] \lambda}{4} \). So we get \( P_k(D_{2n} \times \mathbb{Z}_{2^j}; x, y, z) = \frac{\text{lcm}[2^i, n] \lambda_{(k+1)\lambda}}{2} \).

2. If \( \mu \) is the biggest odd number which is in \( [3, k - 3] \) and \( \mu = \lambda 3^j \) for \( j \in \mathbb{N} \), then we obtain

\[
\begin{align*}
& x_{\text{lcm}[2^i, n] \lambda_{(k+1)\mu}} = e, \ x_{\text{lcm}[2^i, n] \lambda_{(k+1)\mu}} = x, \\
& x_{\text{lcm}[2^i, n] \lambda_{(k+1)\mu}+1} = y \text{ and } x_{\text{lcm}[2^i, n] \lambda_{(k+1)\mu}+2} = z
\end{align*}
\]

for \( \alpha = \frac{\text{lcm}[2^i, n] \mu}{4} \). So we get \( P_k(D_{2n} \times \mathbb{Z}_{2^j}; x, y, z) = \frac{\text{lcm}[2^i, n] \lambda_{(k+1)\mu}}{2} \). \( \square \)

**Theorem 2.3** The periods of the generalized order-\( k \) Pell sequences in the direct product \( D_{2n} \times \mathbb{Z}_{2^i} \), \( n, i \geq 3 \) are as follows:

\( i \). \( \text{Per}_{Q_2}(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \text{lcm} \left[ hP_2 \left( 2^i \right), hP_2 \left[ n \right] \right] \).

\( ii \). If \( k \geq 3 \), then \( \text{Per}_{Q_k}(D_{2n} \times \mathbb{Z}_{2^i}; x, y, z) = \frac{\text{lcm}[2^i, n] hP_2(2)}{2} \).
Theorem 2.4 The periods of the generalized order-k Pell sequences in the semidirect product $D_{2n} \times \varphi Z_{2^i}$, $(n, i \geq 3)$ are as follows:

i. $\text{Per}_{Q_{2,3}}(D_{2n} \times \varphi Z_{2^i}; x, y, z) = hP_{2,3}(2^i)$

ii. $\text{Per}_{Q_{4,5}}(D_{2n} \times \varphi Z_{2^i}; x, y, z) = \frac{\text{lcm}(2^i, n)hP_{k}(2)}{2}$

iii. Let $k \geq 6$.

i’. If there is no $\omega \in [3, k-3]$ such that $\omega$ is an odd factor of $n$, then

$$\text{Per}_{Q_k}(D_{2n} \times \varphi Z_{2^i}; x, y, z) = \frac{\text{lcm}(2^i, n)hP_{k}(2)}{2}.$$ 

ii’. Let $\lambda$ be the biggest odd factor of $n$ in $[3, k-3]$, then two cases occur:

1. If $\lambda 3^j \notin [3, k-3]$ for $j \in \mathbb{N}$, then $\text{Per}_{Q_k}(D_{2n} \times \varphi Z_{2^i}; x, y, z) = \frac{\text{lcm}(2^i, n)hP_{k}(2)\lambda}{2}$.

2. If $\mu$ is the biggest odd number which is in $[3, k-3]$ and $\mu = \lambda 3^j$ for $j \in \mathbb{N}$, then

$$\text{Per}_{Q_k}(D_{2n} \times \varphi Z_{2^i}; x, y, z) = \frac{\text{lcm}(2^i, n)hP_{k}(2)\mu}{2}.$$ 

The proofs of the Theorem 2.3 and Theorem 2.4 are similar to the proofs of Theorem 2.1 and Theorem 2.2, respectively and are omitted.

References


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