Separation axioms via ideals

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Abstract The purpose of this paper is twofold. Firstly, conditions are given under which an I-compact set is closed in regular or normal spaces. Secondly, a new class of spaces called Is-regular spaces and Is-normal spaces have been introduced and their properties have been investigated. Is-regularity and Is-normality are separation axioms defined by using semi-open sets.

Keywords I-compact · Is-regular · Is-normal · semi-open set

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1 Introduction

Regularity and normality are important topological properties and hence it is of significance to obtain decompositions of regularity and normality in terms of weaker topological properties. In recent years many authors have studied several forms of regularity and normality. By using semi-open sets derived by Levine [8], Maheshwari and Prasad [9, 10] introduced s-regular and s-normal spaces. Garg and Sivaraj proved [1] that s-normality is preserved under continuous presemiclosed surjections.

The concept of ideals has been introduced by K. Kuratowski [7], Newcomb [11], Vaidyanathaswamy [13], Hamlet and Janković [3, 4, 5] applied topological ideals to generalize the most basic properties in general topology. By using the concept of ideals, Remukadevi and Sivaraj [12] introduced a new class of spaces called I-normal spaces which contains the class of all normal spaces and discussed some of its properties. I-regular spaces was introduced by Hamlett and Janković [4]. Remukadevi and Sivaraj [12] further investigated the topic.
The purpose of this paper is twofold. Firstly conditions are given under which an $I$-compact set is closed in regular or normal spaces. Further, new spaces called $Is$-regular and $Is$-normal have been introduced and their properties have been discussed. $Is$-normality and $Is$-regularity are separation axioms defined by using semi-open sets.

Throughout this paper $(X,\tau)$ and $(Y,\sigma)$ are topological spaces with no separation axioms assumed. For a subset $A$ of topological space, $\text{cl}(A)$ and $\text{int}(A)$ are denoted by closure and interior of $A$ respectively.

2 Preliminaries

In this section, the basic definitions of semi-open sets, $s$-regular, $s$-normal, $I$-regular, $I$-normal have been given.

**Definition 2.1** [8] Let $(X,\tau)$ be a topological space. A subset $A$ of $X$ is said to be semi-open set if $A \subseteq \text{cl}(\text{int}A)$ and a semi-closed set if $\text{int}(\text{cl}A) \subseteq A$.

**Definition 2.2** [8] The intersection of all semi closed sets containing a subset $A$ of a space $X$ is called semi-closure of $A$ and is denoted by $\text{scl}(A)$. Also $\text{scl}(A) = A \cup \text{int}(\text{cl}(A))$.

**Definition 2.3** [9] A space $X$ is said to be $s$-regular if for each closed set $F$ and a point $x \notin F$, there exist disjoint semi-open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

**Definition 2.4** [10] A space $X$ is said to be $s$-normal if for each pair of disjoint closed set $A$ and $B$, there exist disjoint semi-open sets $U$ and $V$ such that $A \subset U$ and $F \subset V$.

**Definition 2.5** [6] A function $f : X \rightarrow Y$ is called $s$-continuous if $f^{-1}(G)$ is open in $X$ for every semi-open set $G$ of $Y$.

**Definition 2.6** [4] A non empty collection $I$ of subsets on a topological space $(X,\tau)$ is called a topological ideal if it satisfies following two conditions:

(i) $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity).

(ii)$A \in I$ and $B \in I$ implies $A \cup B \in I$ (finite additivity).

We denote a topological space $(X,\tau)$ with an ideal $I$ by $(X,\tau,I)$.

**Proposition 2.7** [11] If $(X,\tau,I)$ is an ideal space, $(Y,\sigma)$ is a topological space and $f : (X,\tau,I) \rightarrow (Y,\sigma)$ is a map, then $f(I) = \{f(I_1) : I_1 \in I\}$ is an ideal of $Y$.

**Proposition 2.8** [11] If $I$ is ideal of subsets of $X$ and $Y$ is subset of $X$, then $I_Y = \{Y \cap I_1 : I_1 \in I\}$ is an ideal of subsets of $Y$. 

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Proposition 2.9 [11] If \( f : (X, \tau) \rightarrow (Y, \sigma, I) \) is an injection then \( f^{-1}(I) = \{ f^{-1}(B) : B \in I \} \) is an ideal on \( X \).

In ideal space \((X, \tau, I)\), the collection \( \beta(I, \tau) = \{ U - I_1 : U \in \tau, I_1 \in I \} \) is a basis for a topology \( \tau(I) \) finer than \( \tau \) [5]. When no ambiguity is present, we denote \( \beta(I, \tau) \) by \( \beta \) and \( \tau(I) \) by \( \tau^* \). Let \((X, \tau, I)\) be an ideal space. Then \( A^*(I, \tau) = \{ x \in X : A \cap U \notin I \text{ for every } U \in \tau(x) \} \) where \( \tau(x) = \{ U \in \tau : x \in U \} \). When there is no ambiguity, we will write \( A^* \) for \( A^*(I, \tau) \) and call it the “local function of \( A \)”. The simplest ideals are \( \{ \phi \} \) and \( \varphi(X) = \{ A : A \subseteq X \} \). Observe that \( A^*(\{ \phi \}) = \text{cl}(A) \) and \( A^*(\varphi(X)) = \phi \) for every \( A \subseteq X \).

Definition 2.10 [5] Let \((X, \tau, I)\) be an ideal space and let \( A \) be a subset of \( X \). Then \( A \) is \( \tau^* \)-closed if and only if \( A^* \subseteq A \).

Definition 2.11 [4] An ideal space \((X, \tau, I)\) is said to be I-regular if for each closed sets \( F \) and a point \( p \notin F \), there exist disjoint open sets \( U \) and \( V \) such that \( p \in U \) and \( F - V \in I \).

Definition 2.12 [12] An ideal space \((X, \tau, I)\) is said to be I-normal if for every pair of disjoint closed sets \( A \) and \( B \) of \( X \), there exist disjoint open sets \( U \) and \( V \) such that \( A - U \in I \) and \( B - V \in I \).

Definition 2.13 [11] A subset \( A \) of a space \((X, \tau, I)\) is said to be I-compact or compact modulo an ideal if for every cover \( \{ U_\lambda : \lambda \in A \} \) of \( A \) by open sets of \( X \), there exists a finite subset \( A_0 \) of \( A \) such that \( A = \bigcup \{ U_\lambda : \lambda \in A_0 \} \in I \).

3 I-compact set with separation axioms

Garg and Goel [2] have taken up the question of closedness of a compact set in a regular or normal space. Here conditions are given under which an I-compact set is closed with respect to an ideal in regular or normal spaces.

Theorem 3.1 An I-compact subset \( K \) of a regular space \( X \) is \( \tau^* \)-closed if either \( K \) or \( X - K \) is a union of \( \tau \)-closed sets.

Proof. Let \( K \) be union of \( \tau \)-closed sets and \( K = \bigcup F_\alpha \) where \( K \) is I-compact and \( F_\alpha \), are \( \tau \)-closed in \( X \). Let \( x \notin K \), then \( x \notin F_\alpha \) for any \( \alpha \). Since \( X \) is regular, there exist disjoint open sets \( U_\alpha \) and \( V_\alpha \) such that \( x \in U_\alpha \) and \( F_\alpha \subseteq V_\alpha \) for each \( \alpha \). Here \( \{ V_\alpha \}_{\alpha \in K} \) is an open cover of \( K \) which is I-compact, therefore there exists a finite subfamily \( \{ V_{\alpha_i} \} \) where \( i = 1,...,n \) such that \( K - \bigcup_{i=1}^n V_{\alpha_i} \in I \). Let \( I_1 = K - \bigcup_{i=1}^n V_{\alpha_i} \), where \( I_1 \in I \). Then the set \( U = \bigcup_{i=1}^n U_{\alpha_i} - I_1 \), is \( \tau^* \)-open set containing \( x \) which is disjoint from \( K \). Hence \( K \) is \( \tau^* \)-closed.

Let \( X - K \) be union of \( \tau \)-closed sets and \( X - K = \bigcup F_\alpha \) where \( K \) is I-compact and \( F_\alpha \), are \( \tau \)-closed in \( X \). Then \( K = \bigcap F_\alpha^c \). Let \( x \in K \). Then \( x \notin F_\alpha \) for any \( \alpha \). Now for each fixed \( F_\alpha \) and \( x \in K \) there exists an open set
V_\alpha containing x such that clV_\alpha \subseteq (X - F_\alpha). Now \{V_\alpha\}_{\alpha \in \mathcal{K}} is an open cover of K which is I-compact. Therefore there exists a finite subfamily \{V_{\alpha_i}\}_{i=1}^n where i = 1,...,n such that K - \bigcup_{i=1}^n V_{\alpha_i} \in I. Let U_\alpha = X - \bigcup_{i=1}^n cl(V_{\alpha_i}). Then F_\alpha \subseteq U_\alpha as F_\alpha \subseteq X - clV_\alpha for each i. Let V_\alpha = U_\alpha - I_\alpha, where I_\alpha = K - \bigcup_{i=1}^n cl(V_{\alpha_i}) \subseteq K - \bigcup_{i=1}^n V_{\alpha_i}. Then V_\alpha is \tau^*\text{-open. Therefore } X - K = V_\alpha so X - K is \tau^*\text{-open and hence } K is \tau^*\text{-closed.}

\square

**Theorem 3.2** An I-compact subset K of a normal space X is \tau^*\text{-closed if K as well as its compliment are union of } \tau\text{-closed sets.}

**Proof.** Let K be an I-compact subset of a normal space X and let K = \bigcup F_\alpha, X - K = \bigcup F_\beta, where F_\alpha and F_\beta are \tau\text{-closed sets in X. Then for each fixed } F_\beta \subseteq X - K and for each F_\alpha \subseteq K, there exist disjoint open sets U_\alpha and V_\alpha open in X such that F_\alpha \subseteq U_\alpha and F_\beta \subseteq V_\alpha. Then the family \{U_\alpha\} is an open cover of K and since K is I-compact, there exists a finite subfamily \{U_{\alpha_i}\}_{i=1}^n such that K - \bigcup\{U_{\alpha_i}\}_{i=1}^n \in I. Let U_\beta = X - cl(\bigcup_{i=1}^n U_{\alpha_i}) and I_1 = K - \bigcup_{i=1}^n cl(U_{\alpha_i}) \subseteq K - \bigcup_{i=1}^n U_{\alpha_i}, where I_1 \in I. Then V_\beta = U_\beta - I_1 is a \tau^*\text{-open set disjoint from K containing } F_\beta and X - K = V_\beta is a \tau^*\text{-open set. Hence } K is \tau^*\text{-closed in } X.

\square

4 \textit{s}-regular spaces and \textit{s}-normal spaces via ideals

The aim of this section is to introduce and study new classes of spaces called \textit{s}-regular spaces and \textit{s}-normal spaces. \textit{s}-regularity and \textit{s}-normality are separation axioms in term of semi-open sets.

**Definition 4.1** An ideal space \((X, \tau, I)\) is said to be \textit{s}-regular if for each closed set F and a point \(x \notin F\), there exist disjoint semi-open sets U and V such that \(x \in U\) and \(F - V \in I\).

**Remark 4.1** \textit{s}-regular implies \textit{s}-regular, but converse is not true as shown in following example.

**Example 4.1** Let \(X = \{a, b, c, d\}\) with topology \(\tau = \{\phi, X, \{c\}, \{a, b\}, \{a, b, c\}\}\) and ideal \(I = \{\phi, \{a\}, \{d\}, \{a, d\}\}\). Let the closed set be \(F = \{a, b, d\}\) and a point \(c \notin F\). Here \(U = \{c\}\) and \(V = \{a, b\}\) are two disjoint semi-open sets such that \(c \in U\) and \(F - V = \{d\} \in I\) therefore \((X, \tau, I)\) is regular. But \(F\) is not subset of \(V\) so \(X\) is not \(s\)-regular.

The following theorem 4.2 is a characterization of \(s\)-regular space which is an analogue of characterization for regular space in general topological space.

**Theorem 4.2** Let \((X, \tau, I)\) be an ideal space. Then the following are equivalent:

\begin{align*}
& \text{\(X, \tau, I\) is normal,} \\
& \text{\(X, \tau, I\) is \(s\)-normal.}
\end{align*}
(a) \( X \) is Is-regular.
(b) For each \( x \in X \) and open set \( U \) containing \( x \), there is a semi-open set \( V \) containing \( x \) such that \( scl(V) - U \in I \).
(c) For each \( x \in X \) and closed set \( A \) not containing \( x \), there is a semi-open set \( V \) containing \( x \) such that \( scl(V) \cap A \in I \).

Proof. (a) \( \Rightarrow \) (b) Let \( x \in X \) be arbitrary and \( U \) be an open set containing \( x \). Since \( X \) is Is-regular, there exist disjoint semi-open sets \( V \) and \( W \) such that \( x \in V \) and \( (X - U) - W \in I \). Let \( (X - U) - W = I_1 \in I \), then \( (X - U) \subset W \cup I_1 \). Now \( V \cap W = \emptyset \), implies that \( V \subset (X - W) \) and \( scl(V) \subset scl(X - W) = X - W \). Hence \( scl(V) - U \subset (X - W) \cap (W \cup I_1) = (X - W) \cap I_1 \). Here \( (X - W) \cap I_1 \subset I_1 \in I \). Therefore \( scl(V) - U \in I \).

(b) \( \Rightarrow \) (c) Let \( A \) be any closed set in \( X \) such that \( x \notin A \). Then by (b), there exists a semi-open set \( V \) containing \( x \) such that \( scl(V) - (X - A) \in I \), implies \( scl(V) \cap A \in I \). Therefore (c) holds

(c) \( \Rightarrow \) (a) Let \( A \) be any closed set in \( X \) such that \( x \notin A \). Then, there exists a semi-open set \( V \) containing \( x \) such that \( scl(V) \cap A \in I \). Therefore \( A - (X - scl(V)) \in I \). Here \( V \) and \( (X - scl(V)) \) are disjoint semi-open sets such that \( x \in V \) and \( A - (X - scl(V)) \in I \). Hence \( X \) is Is-regular.

\[ \square \]

**Theorem 4.3** Let \( (X, \tau, I) \) be Is-regular space and \( Y \) be any subset of \( X \), then \( Y \) is \( I_Y \)-s-regular.

Proof. Let \( (X, \tau, I) \) be Is-regular space and \( Y \) be open subset of \( X \). Let \( x \in Y \) and \( F \) be a closed subset of \( Y \) such that \( x \notin F \). Then there is a closed set \( A \) in \( X \) with \( F = Y \cap A \) and \( x \notin A \). Since \( X \) is Is-regular, there exist disjoint semi-open sets \( G \) and \( H \) such that \( x \in G \) and \( A - H \in I \). Here \( Y \cap G \) and \( Y \cap H \) are semi-open sets in \( Y \). Also \( x \in Y \) and \( x \in Y \) implies \( x \in Y \cap G \) and \( A - H = I \). Let \( I_1 \in I \) implies \( A \subseteq I_1 \cup H \). Therefore \( (Y \cap A) \subseteq Y \cap (I_1 \cup H) = (Y \cap I_1) \cup (Y \cap H) \) which implies \( F - (Y \cap H) \in I_Y \). Also \( (Y \cap G) \cap (Y \cap H) = \emptyset \). Hence \( Y \) is \( I_Y \)-s-regular.

\[ \square \]

**Theorem 4.4** If \( f : (X, \tau) \rightarrow (Y, \sigma, I) \) is a s-continuous closed injection and \( Y \) is Is-regular, then \( X \) is \( f^{-1}(I) \)-regular.

Proof. Let \( x \in X \) and \( F \) be closed subset of \( X \) such that \( x \notin F \), since \( f \) is closed injection, \( f(x) \) and \( f(F) \) are disjoint in \( Y \). Since \( Y \) is Is-regular, there exist disjoint semi-open sets \( U \) and \( V \) such that \( f(x) \in U \) and \( f(F) - V \in I \). Since \( f \) is s-continuous, \( f^{-1}(U) \) and \( f^{-1}(F) - V \) are disjoint open sets in \( X \). Here \( x \in f^{-1}(U) \) and \( f^{-1}(f(F) - V) \in f^{-1}(I) \) implies \( f - f^{-1}(V) \in f^{-1}(I) \). Therefore, \( X \) is \( f^{-1}(I) \)-regular.

\[ \square \]
Theorem 4.5 If \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is a continuous pre-semi open bijection and \( X \) is \( s \)-regular, then \( Y \) is \( f(I)s \)-regular.

Proof. Let \( y \in Y \) and \( A \) be a closed subset of \( Y \) such that \( y \notin A \), since \( f \) is continuous, \( f^{-1}(A) \) is closed set in \( X \) such that \( f^{-1}(y) \notin f^{-1}(A) \). Since \( X \) is \( s \)-regular, there exist disjoint semi-open sets \( U \) and \( V \) such that \( f^{-1}(y) \in U \) and \( f^{-1}(A) - V \in I \). Here \( y \in f(U) \) and \( f(f^{-1}(A) - V) \in f(I) \) implies \( A - f(V) \in f(I) \). Since \( f \) is pre-semi open, \( f(U) \) and \( f(V) \) are disjoint semi-open sets in \( Y \). Therefore, \( Y \) is \( f(I)s \)-regular.

\( \square \)

Theorem 4.6 Let \( (X, \tau, I) \) be \( s \)-regular space. Then for every nonempty set \( G \) and a closed set \( H \) in \( X \) with \( G \cap H = \emptyset \), there exist disjoint semi-open subsets \( U \) and \( V \) of \( X \) such that \( G \cap U \neq \emptyset \) and \( H - V \in I \).

Proof. Suppose \( X \) is \( s \)-regular. Let \( H \) be closed in \( X \) and \( G \) be nonempty set with \( G \cap H = \emptyset \). For \( x \in G \), there exist disjoint semi-open subsets \( U \) and \( V \) such that \( x \in U \) and \( H - V \in I \). for \( x \in G \), \( G \cap U \neq \emptyset \).

\( \square \)

The concept of \( s \)-normal spaces has been introduced by using ideals and \( I \)-normal spaces defined by Maheshwari and Prasad [10].

Definition 4.7 An ideal space \( (X, \tau, I) \) is said to be \( s \)-normal if for every pair of disjoint closed sets \( A \) and \( B \) of \( X \), there exist disjoint semi-open sets \( U \) and \( V \) such that \( A - U \in I \) and \( B - V \in I \).

Theorem 4.8 Let \( (X, \tau, I) \) be an ideal space. Then the following are equivalent:

(a) \( X \) is \( s \)-normal,

(b) For every closed set \( F \) and open set \( G \) containing \( F \), there is a semi-open set \( V \) such that \( F - V \in I \) and \( scl(V) - G \in I \).

(c) For each pair of disjoint closed sets \( A \) and \( B \), there exists a semi-open set \( U \) such that \( A - U \in I \) and \( scl(U) \cap B \in I \).

Proof. (a) \( \Rightarrow \) (b) Let \( F \) be closed and \( G \) be open set such that \( F \subset G \). Then \( X - G \) is a closed set such that \( (X - G) \cap F = \emptyset \). Since \( X \) is \( s \)-normal, there exist disjoint semi-open sets \( U \) and \( V \) such that \( (X - G) - U \in I \) and \( F - V \in I \). Now \( U \cap V = \emptyset \) implies that \( V \subset X - U \), therefore \( scl(V) \subset scl(X - U) = (X - U) \) and \( (X - G) \cap scl(V) \subset (X - G) \cap (X - U) \) implies \( scl(V) - G \subset (X - G) - U \in I \). Therefore, \( scl(V) - G \in I \).

(b) \( \Rightarrow \) (c) Let \( A \) and \( B \) be disjoint closed subsets of \( X \). Then by (b) there exists a semi-open set \( U \) such that \( A - U \in I \) and \( scl(U) - (X - B) \in I \). This implies \( A - U \in I \) and \( scl(U) \cap B \in I \).

(c) \( \Rightarrow \) (a) Let \( A \) and \( B \) be disjoint closed subsets in \( X \). Then there exists a semi-open set \( U \) such that \( A - U \in I \) and \( scl(U) \cap B \in I \). Now \( scl(U) \cap B \in I \)
implies that $B - (X - \text{scl}(U)) \in I$. If $V = (X - \text{scl}(U))$, then $V$ is semi-open set such that $B - V \in I$ and $U \cap V = U \cap (X - \text{scl}(U)) = \emptyset$. Hence $X$ is $Is$-normal.

\[ \square \]

**Theorem 4.9** If $f : (X, \tau) \rightarrow (Y, \sigma, I)$ is a $s$-continuous closed injection and $Y$ is $Is$-normal, then $X$ is $f^{-1}(I)$-normal.

**Proof.** Let $A$ and $B$ be disjoint closed sets of $X$, since $f$ is closed injection, $f(A)$ and $f(B)$ are disjoint closed sets of $Y$. Since $Y$ is $Is$-normal, there exist disjoint semi-open sets $U$ and $V$ such that $f(A) - U \in I$ and $f(B) - V \in I$. Since $f$ is $s$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in $X$. Here $f^{-1}(f(A) - U) \in f^{-1}(I)$ and $f^{-1}(f(B) - V) \in f^{-1}(I)$ implies $A - f^{-1}(U) \in f^{-1}(I)$ and $B - f^{-1}(V) \in f^{-1}(I)$. Therefore, $X$ is $f^{-1}(I)$-normal.

\[ \square \]

**Theorem 4.10** If $f : (X, \tau, I) \rightarrow (Y, \sigma, I)$ is a continuous pre-semi open surjection and $X$ is $Is$-normal, then $Y$ is $f(I)s$-normal.

**Proof.** Let $A$ and $B$ be disjoint closed sets in $Y$. Since $f$ is continuous surjection, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in $X$. Given $X$ is $Is$-normal, there exist disjoint semi-open sets $U$ and $V$ such that $f^{-1}(A) - U \in I$ and $f^{-1}(B) - V \in I$. Let $f(f^{-1}(A) - U) \in f(I)$ and $f(f^{-1}(B) - V) \in f(I)$ implies $A - f(U) \in f(I)$ and $B - f(V) \in f(I)$. Here $f(U)$ and $f(V)$ are disjoint semi-open sets in $Y$, because $f$ is pre-semi open. Therefore, $Y$ is $f(I)s$-normal.

\[ \square \]

**References**


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