On the structure of le-semigroups

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Abstract In this paper, $(0,m)$-ideal elements and 0-minimal $(0,m)$-ideal elements in poe-semigroups are investigated. Then, for any positive integers $m$, $n$, relations $I_{n,m}$, $H_{n,m}$, $B_{n,m}$ and $Q_{n,m}$ on le-semigroups are introduced. We show that, in any le-semigroup, $B_{n,m} \subseteq Q_{n,m} \subseteq H_{n,m}$ and provide some sufficient conditions to get the equality. We, then, prove that either a $Q_{n,m}$-class (resp. $H_{n,m}$-class) is $(m,0)$ and $(0,n)$-regular or none of its element is $(m,0)$ and $(0,n)$-regular respectively. Next we show that either a $B_{n,m}$-class (resp. $Q_{n,m}$-class, $H_{n,m}$-class) is $(m,n)$-right weakly regular or none of its element is $(m,n)$-right weakly regular. Finally, we define a strong $(m,n)$-quasi-ideal element in an le-semigroup and provide some sufficient conditions on an $(m,n)$-quasi-ideal element to be a strong $(m,n)$-quasi-ideal element.

Keywords \(\forall e\)le-semigroup · $(m,n)$-ideal element · $(m,n)$-quasi-ideal element

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1 Introduction

In [14], S. Lajos introduced the concept of an $(m,n)$-ideal in a semigroup. Prompted by this, several authors studied $(m,n)$-ideals in various algebraic structures such as rings, semirings and ordered semigroups etc; for example, Akram et al. [1], Bussaban and Changphas [3], Changphas [4], Tilidetzke [18], Yaqoob and Chinram [19] and many others. In [2], Bhuniya and Kumbhakar studied bands and normal bands in le-semigroups and obtained several characterizations of regular and intra-regular le-semigroups. Kehayopulu [5] studied $(m,n)$-ideal elements and $(m,n)$-quasi-ideal elements in poe-semigroups and le-semigroups. In [6, 7, 10], Kehayopulu characterized intra-regular \(\forall e\)-semigroups and regular le-semigroups in terms of left (resp. right)-ideal elements, ideal-elements, bi-ideal elements and quasi-ideal elements of respective semigroups. In [8, 9], Kehayopulu studied left (resp. right) duo regular, left-regular and left duo poe-semigroups. In [13], Kehayopulu characterized idempotent ideal elements of le-semigroups in terms of
semisimple elements while intra-regular le-semigroup in terms of prime and semiprime ideal elements. The Green-Kehayopulu relations \( \mathcal{L}, \mathcal{R} \) and \( \mathcal{H} \) have been introduced and investigated by Kehayopulu in [11,12]. In [17], Petro and Pasku studied the property of Green-Kehayopulu relation \( \mathcal{H} \) and obtained some properties of Green-Kehayopulu relation \( \mathcal{H} \) on le-semigroup which differ from Green-relation \( \mathcal{H}_{\text{plain}} \) on plain semigroups. They also proved that an \( \mathcal{H} \)-class \( H \) of an le-semigroup satisfies Green's condition \((a, b \in H \text{ implies } ab \in H)\) if and only if it contains an idempotent element and provided several conditions under which an \( \mathcal{H} \)-class of an le-semigroup forms a subsemigroup. In [15], Pasku and Petro showed that, in an le-semigroup, \( B \subseteq H \) and provided some necessary and sufficient conditions for a \( B \)-class to be regular or intra-regular. They further investigated several conditions under which a \( B \)-class formed a subsemigroup.

In this paper, we first characterize \((0,m)\)-ideal elements and 0-minimal \((0,m)\)-ideal elements in poe-semigroups. Then, we investigate \((m,n)\)-regular le-semigroups and prove that an le-semigroup \( S \) is \((m,n)\)-regular if and only if \( a \wedge q = q^m a q^n \) for each \((m,n)\)-quasi-ideal element \( q \) and for each ideal-element \( a \) of \( S \) besides some more results on \((m,n)\)-regular le-semigroups. We, then, have defined the relations \( m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n, \mathcal{Q}_m^n \) and \( \mathcal{H}_m^n \) on le-semigroups and provided some sufficient conditions under which these relations are equal. We also study the \((m,0)\)-regularity [(0,\( n \))-regularity] and \((m,n)\)-right weakly regularity of \( \mathcal{B}_m^n \)-classes, \( \mathcal{Q}_m^n \)-classes and \( \mathcal{H}_m^n \)-classes respectively in le-semigroups. Finally, after introducing the concept of strong \((m,n)\)-quasi-ideal elements in le-semigroups, we prove that, on le-semigroups, the relations \( \mathcal{H}_m^n \) and \( \mathcal{Q}_m^n \) coincide if and only if each \((m,n)\)-quasi-ideal element is a strong \((m,n)\)-quasi-ideal element and provide some sufficient conditions so that \((m,n)\)-quasi-ideal elements are strong \((m,n)\)-quasi-ideal elements.

**Definition 1.1** Let \( S \) be a non-empty set. The triplet \((S, \cdot, \leq)\) is called an ordered semigroup (or poe-semigroup) if \((S, \cdot)\) is a semigroup and \((S, \leq)\) is a partially ordered set such that

\[
    a \leq b \Rightarrow ac \leq bc \text{ and } ca \leq cb
\]

for all \( a, b, c \in S \). A poe-semigroup with a greatest element “e” (i.e., for each \( a \in S, a \leq e \)) is said to be a poe-semigroup.

Let \( S \) be a poe-semigroup and let \( a \in S \). The element \( a \) is called a subsemigroup element if \( a^2 \leq a \) and a left (resp. right) ideal element of \( S \) if \( ea \leq a \) (resp. \( ae \leq a \)). It is called an ideal element of \( S \) if it is both a left and a right-ideal element of \( S \) and a bi-ideal element of \( S \) if \( aea \leq a \). Further \( a \) is called an idempotent element if \( a = a^2 \); and a quasi-ideal element if \( ae \wedge ea \) exists and \( ae \wedge ea \leq a \). An element \( z \in S \) is called a zero element of \( S \) if \( za = az = z \) and \( z \leq a \) for each \( a \in S \). The zero element, if exists, is always unique. We shall denote it, in whatever follows, by the symbol 0. A poe-semigroup \( S \) is called regular (left-regular, right-regular) if
a \leq aea \ (a \leq ea^2, a \leq a^2e) \text{ for each } a \in S \text{ and commutative if } ab = ba \text{ for all } a, b \in S.

**Definition 1.2** A poe-semigroup S is said to be a \(\lor\)-semigroup if it is an upper semilattice under \(\lor\) and
\[
c(a \lor b) = ca \lor cb \text{ and } (a \lor b)c = ac \lor bc
\]
for all \(a, b, c \in S\). A \(\lor\)-semigroup which is a lattice is said to be an \(le\)-semigroup.

It is well known that in an \(le\)-semigroup S,
\[
a \leq b \iff a \land b = a \text{ and } a \lor b = b
\]
for all \(a, b \in S\).

For simplicity, throughout the paper, for any element \(a\) of an ordered semigroup \(S\), we shall write \(a^n\) for \(aa \cdots a\) \((n - \text{copies of } a)\). Also the integers \(m, n\) will stand for positive integers throughout the paper until and unless otherwise specified.

**Definition 1.3** [5] Let \(S\) be a poe-semigroup and \(m, n\) be non-negative integers. An element \(a\) of \(S\) is called an \((m, n)\)-ideal element of \(S\) if \(a^m e a^n \leq a\).

**Remark 1.1** Throughout the paper, we shall use the convention \(a^0b = b a^0 = b\), for all \(a, b \in S\). In particular, for \(m = 0, n = 1\) (resp. \(m = 1, n = 0\) and \(m = 1 = n\)), each \((m, n)\)-ideal element \(a\) of \(S\) is a left-ideal element (resp. right-ideal element and bi-ideal element). Clearly each left-ideal element (resp. each right-ideal element and each bi-ideal element) is a \((0, n)\)-ideal element for each positive integer \(n\) (resp. \((m, 0)\)-ideal element for each positive integer \(m\) and \((m, n)\)-ideal element for each positive integers \(m, n\). Therefore the concept of a \((0, n)\)-ideal element (resp. \((m, 0)\)-ideal element and \((m, n)\)-ideal element) is a generalization of the concept of a left-ideal element (resp. a right-ideal element and a bi-ideal element).

**Remark 1.2** For each positive integers \(m\) and \(n\), any \((m, 0)\)-ideal element (resp. \((0, n)\)-ideal element) is an \((m, n)\)-ideal element as \(a^m e a^n \leq a\) (resp. \(a^m e a^n \leq a\)).

**Definition 1.4** [5] An element \(q\) of a poe-semigroup \(S\) is called an \((m, n)\)-quasi-ideal element of \(S\) if \(q^m e \land q^n\) exists and \(q^m e \land q^n \leq q\). Clearly every quasi-ideal element is an \((m, n)\)-quasi-ideal element for each positive integers \(m\) and \(n\) such that \(q^m e \land q^n\) exists, but the converse is not true in general.

**Example 1.1** Let \(S = \{a, b, c, d\}\). Define the binary operation \(\cdot\)' as follows:

\[
\begin{array}{cccc}
  & a & b & c & d \\
 a & a & a & a & a \\
b & b & b & b & b \\
c & c & c & c & c \\
d & a & a & b & a
\end{array}
\]

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Then the set $P(S)$ of all non-empty subsets of $S$ is a poe-semigroup under the binary operation of set product induced by the binary operation defined by the above table and partially ordered by set inclusion. The set $\{a, d\}$ is an $(m, n)$-ideal element and an $(m, n)$ quasi-ideal element for each integers $m, n \geq 2$, but $\{a, d\}$ is neither a bi-ideal element nor a quasi-ideal element of the poe-semigroup $P(S)$.

We denote by $(a)$, $<a>_{m,n}$ and $(a)_{m,n}$ the ideal-element, $(m, n)$-ideal element and $(m, n)$ quasi-ideal element of $S$ generated by the element $a$ of $S$ i.e., the least ideal-element, the least $(m, n)$-ideal element and the least $(m, n)$ quasi-ideal element of $S$ greater than the element $a$ and had been described by [5,13] as follows:

$$(a) = a \lor ea \lor ae;$$

$$<a>_{m,n} = a \lor a^m e a^n;$$

$$(a)_{m,n} = a \lor (a^m e \land e a^n).$$

Thus an element $a \in S$ is an ideal (resp. $(m, n)$-ideal, $(m, n)$ quasi-ideal) element if and only if $(a) = a$ (resp. $<a>_{m,n} = a$, $(a)_{m,n} = a$).

2 Characterization of $(0, m)$-ideal elements

Lemma 2.1 [5] Let $S$ be a $\lor e$-semigroup, $a \in S$ and $m, n, k \geq 0$ are integers. Then

1. $(a \lor a^m e a^k)^m e = a^m e$;
2. $e(a \lor a^k e a^n)^n = e a^n$;
3. $<a>_{m,n}$ exists and $<a>_{m,n} = a \lor a^m e a^n$.

Lemma 2.2 Let $S$ be an $le$-semigroup, $m \in \mathbb{Z}^+$ and $a$ be a subsemigroup element of $S$. Then $a$ is an $(1, m)$-ideal element of $S$ if and only if there exist a $(0, m)$-ideal element $c$ and a right-ideal element $b$ of $S$ such that $bc^m \leq a \leq b \land c$.

Proof. Let $a$ be any $(1, m)$-ideal element of $S$. Then $a \lor e a^m$ and $a \lor e a$ are $(0, m)$-ideal element and right-ideal element of $S$ respectively. Let $b = a \lor e a$ and $c = a \lor e a^m$. Then

$$bc^m = (a \lor e a)(a \lor e a^m)^m$$

$$= a(a \lor e a^m)^m \lor e a(a \lor e a^m)^m$$

$$= (a^2 \lor a e a^m)(a \lor e a^m)^{m-1} \lor a e a^m \quad \text{(by Lemma 2.1)}$$

$$\leq (a^2 \lor a)(a \lor e a^m)^{m-1} \lor a \quad \text{(as } a e a^m \leq a \text{)}$$

$$= a(a \lor e a^m)^{m-1} \lor a \quad \text{(as } a^2 \leq a \text{)}$$

$$= (a^2 \lor a e a^m)(a \lor e a^m)^{m-2} \lor a$$

$$\leq (a^2 \lor a)(a \lor e a^m)^{m-2} \lor a \quad \text{(as } a e a^m \leq a \text{)}$$

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= a(a \lor ea^m)^{m-2} \lor a \quad (a^2 \leq a)
= \cdots
= a.

As \ a \leq b \land c, we have \ b^m \leq a \leq b \land c, as required.

Conversely, assume that \ b \ is a right-ideal element and \ c \ is a \ (0,m)-ideal element of \ S \ such that \ b^m \leq a \leq b \land c. As \ a \leq b \land c, \ b \ is a right-ideal element and \ b^m \leq a, we have \ ace^m \leq (b \land c)e(b \land c)^m \leq bec^m \leq be^m \leq a. Therefore \ a \ is an \ (1,m) \ ideal element of \ S.

\qed

Definition 2.3 Let \ S \ be a poe-semigroup and \ a \ be any left-ideal (right-ideal) element of \ S. Then \ a \ is said to be a minimal left-ideal (right-ideal) element of \ S \ if for every left-ideal (right-ideal) element \ b \ of \ S, \ b \leq a \ implies \ b = a. Further any non-zero left-ideal (right-ideal) element \ a \ of a poe-semigroup \ S \ with \ 0 \ is said to be a 0-minimal if for each left-ideal (right-ideal) element \ b \ of \ S, \ b \leq a \ implies either \ b = 0 \ or \ b = a.

Similarly we may define a minimal and a 0-minimal \ (m,n)-ideal \ (resp. \ (m,0)-ideal, \ (0,n)-ideal) element for each positive integers \ m \ and \ n.

Lemma 2.4 Let \ S \ be a poe-semigroup with zero, \ m \in \mathbb{Z}^+ \ and let \ a \ be any 0-minimal left-ideal element of \ S. Then a subsemigroup element \ b \ of \ S \ is a \ (0,m)-ideal element of \ S \ smaller than \ a \ if and only if either \ b^m = 0 \ or \ b = a.

Proof. Let \ S \ be a poe-semigroup with 0 and \ b \ be a subsemigroup element as well as a \ (0,m)-ideal element of \ S \ smaller than the 0-minimal left-ideal element \ a \ of \ S. As \ eb^m \ is a left-ideal element of \ S \ and \ eb^m \leq b \leq a, \ so by minimality of the left-ideal element \ a \ of \ S, \ eb^m = 0 \ or \ eb^m = a. If \ eb^m = a, \ then \ a = eb^m \leq b. Therefore \ b = a. In the other case when \ eb^m = 0, as \ eb^m = 0 \leq b^m, \ b^m \ is a left-ideal element of \ S \ smaller than \ a \ (as \ b \ is a subsemigroup element and, so, \ b^m \leq b). Now, by 0-minimality of the left-ideal element \ a \ of \ S, \ we have \ b^m = 0 \ or \ b^m = a. Since \ b \ is a subsemigroup element, \ we have \ b^m \leq b. Therefore \ a = b^m \leq b. Hence in both of the cases, \ either \ b^m = 0 \ or \ b = a, \ as required.

The converse is obvious.

\qed

Lemma 2.5 Let \ S \ be a poe-semigroup with zero, \ m \in \mathbb{Z}^+ \ and let \ a \ be any subsemigroup element of \ S. If \ a \ is a 0-minimal \ (0,m)-ideal element of \ S, \ then \ a^m = 0 \ or \ a \ is a 0-minimal left-ideal element of \ S.

Proof. Let \ S \ be a poe-semigroup with 0 and \ a \ be any subsemigroup 0-minimal \ (0,m)-ideal element of \ S. Therefore \ a^m \leq a \ and \ e(a^m)^m = e^{a^m}a^m\cdots a^m \leq aa\cdots a = a^m, \ it follows that \ a^m \ is (0,m)-ideal element
of \( S \). As \( a^m \leq a \), by minimality of a 0-minimal \((0, m)\)-ideal element \( a \) of \( S \), \( a^m = 0 \) or \( a^m = a \). Suppose \( a^m = a \). Now \( ea = ea^m \leq a \) implies \( a \) is left-ideal element of \( S \). It remains to show that \( a \) is a 0-minimal left-ideal element of \( S \). So, let \( b \) be any left-ideal element of \( S \) such that \( b \leq a \). As \( b \) is a \((0, m)\)-ideal element of \( S \), \( a \) is a 0-minimal \((0, m)\)-ideal element of \( S \) and \( b \) \( \leq a \), we have either \( b = 0 \) or \( b = a \). Hence \( a \) is a 0-minimal left-ideal element of \( S \).

\[ \square \]

**Lemma 2.6** Let \( S \) be an le-semigroup, \( m \in \mathbb{Z}^+ \) and \( a \) be any subsemigroup element of \( S \). Then \( a \) is a \((0, m)\)-ideal element of \( S \) if and only if \( ba \leq a \) for some \((0, m-1)\)-ideal element \( b \) \((a \leq b)\) of \( S \).

**Proof.** Suppose \( a \) is a \((0, m)\)-ideal element of \( S \). By Lemma 2.1, \( e(a \lor ea^{m-1})^{m-1} = ea^{m-1} \). So it follows that \( e(a \lor ea^{m-1})^{m-1} = ea^{m-1} \leq a \lor ea^{m-1} \) i.e., \( a \lor ea^{m-1} \) is a \((0, m-1)\)-ideal element of \( S \). Let \( b = a \lor ea^{m-1} \). Then, as \( a \) is a subsemigroup and \((0, m)\)-ideal element of \( S \), \( ba = (a \lor ea^{m-1})a = a^2 \lor ea^m \leq a^2 \lor a = a \), as required.

Conversely assume that \( b \) is a \((0, m-1)\)-ideal element of \( S \) and \( a \) be an element of \( S \) such that \( ba \leq a \) with \( a \leq b \). Now \( ea^m \leq eb^{m-1}a \leq ba \leq a \). Therefore \( a \) is a \((0, m)\)-ideal element of \( S \).

\[ \square \]

**Lemma 2.7** Let \( S \) be a poe-semigroup and let \( a \in S \). Then \( a \) is a minimal \((m, m-1)\)-ideal element \( (m \in \mathbb{Z}^+, m \geq 2) \) if and only if \( a \) is a minimal bi-ideal element of \( S \).

**Proof.** Let \( S \) be a poe-semigroup and \( a \) be a minimal \((m, m-1)\)-ideal element of \( S \). Then, by definition, \( a^m ea^{m-1} \leq a \), we have \((a^m ea^{m-1})^m \leq a^m ea^{m-1} \). Therefore \( a^m ea^{m-1} \) is a \((m, m-1)\)-ideal element of \( S \) such that \( a^m ea^{m-1} \leq a \). So, by minimality of \( a \), we have \( a^m ea^{m-1} = a \).

Now

\[ a = (a^m ea^{m-1})(a^m ea^{m-1}) \leq a^m ea^{m-1} = a \]

and

\[ a = a^m ea^{m-1} \leq a^m ea^{m-1} = a. \]

This implies that \( a \) is a bi-ideal element of \( S \). Now we show that \( a \) is a minimal bi-ideal element of \( S \). So take any bi-ideal element \( b \) of \( S \) such that \( b \leq a \). As \( b^{m+}b^{m-1} = b(b^{m-1}eb^{m-2})b \leq beb \leq b \), \( b \) is a \((m, m-1)\)-ideal element of \( S \). Since \( a \) is a minimal \((m, m-1)\)-ideal element of \( S \), \( b = a \). Hence \( a \) is a minimal bi-ideal element of \( S \).

Conversely assume that \( a \) is a minimal bi-ideal element of \( S \). As \( a^m ea^{m-1} = a(a^m ea^{m-2})a \leq aea \leq a \), \( a \) is an \((m, m-1)\)-ideal element of \( S \). To show that \( a \) is a minimal \((m, m-1)\)-ideal element of \( S \), take any \((m, m-1)\)-ideal element \( b \) of \( S \) such that \( b \leq a \). As \((b^{m+}b^{m-1})(b^{m+}b^{m-1}) = b^{m+}(b^{m-1}eb^{m-1})b^{m-1} \leq b^{m+}b^{m-1} \) and \((b^{m+}b^{m-1})e(b^{m+}b^{m-1}) = b^{m+}(b^{m-1}eb^{m-1})b^{m-1} \leq b^{m+}b^{m-1} \),
it follows that $b^m e b^{m-1}$ is a bi-ideal element of $S$. Since $a$ is a minimal bi-ideal element of $S$ and $b^m e b^{m-1} \leq a$, we have $b^m e b^{m-1} = a$. As $b^m e b^{m-1} \leq b$, $a \leq b$. Now, as $b \leq a$, we have $b = a$. Hence $a$ is a minimal $(m, m-1)$-ideal element of $S$.

Lemma 2.8 Let $S$ be a commutative poe-semigroup such that $e^2 = e$ and let $m$, $n$ be any non-negative integers. Then

1. $I_{<m,0>}$, the set of all $(m, 0)$-ideal elements of $S$, is a subsemigroup of $S$.
2. $I_{<0,n>}$, the set of all $(0, n)$-ideal elements of $S$, is a subsemigroup of $S$.
3. $I_{<m,n>}$, the set of all $(m, n)$-ideal elements of $S$, is a subsemigroup of $S$.

Proof. Straightforward.

3 $(m, n)$-regular le-semigroups

Definition 3.1 Let $S$ be a poe-semigroup and let $m$, $n$ be non-negative integers. An element $a \in S$ is said to be an $(m, n)$-regular element of $S$ if $a \leq a^m e a^n$. Further $S$ is said to be $(m, n)$-regular if every element of $S$ is $(m, n)$-regular. A $(1, 1)$-regular poe-semigroup is said to be regular.

It is clear from Definition 3.1 that, for each non-negative integers $m$ and $n$, every $(m, n)$-regular poe-semigroup is an $(r, s)$-regular poe-semigroup ($r \leq m, s \leq n$ are non-negative integers). In particular, for each $m, n \in \mathbb{Z}^+$, $(m, n)$-regular poe-semigroup is regular. On the other hand, for each $m \in \mathbb{Z}^+$, $(m, 0)$-regular poe-semigroup need not be a regular poe-semigroup.

Example 3.1 Let $S = \{x, y, z\}$. Define a binary operation $'$ and an order $' \leq'$ as follows:

$$
\begin{array}{c|ccc}
\cdot & x & y & z \\
x & z & x & z \\
y & x & y & z \\
z & z & z & z \\
\end{array}
$$

$$
\leq := \{(x, x), (y, y), (z, z), (z, x), (x, y), (z, y)\}.
$$

Clearly $S$ is a poe-semigroup with greatest element $e = y$. As $(x, x y x) \notin \leq$, $S$ is not a regular poe-semigroup, but $S$ is $(1, 0)$-regular as $a \leq ae$ for each $a \in S$.

Lemma 3.2 [5] Let $S$ be an le-semigroup, $a \in S$ and $m$, $n$ non-negative integers. Then the followings hold:

1. $(a \lor (a^m e \land e a^n))^m e \leq a^m e$.
2. $e (a \lor (a^m e \land e a^n))^n \leq e a^n$.
3. $(a)_{<m,n>}$ exists and $(a)_{<m,n>} = a \lor (a^m e \land e a^n)$.
Theorem 3.3 Let $S$ be an le-semigroup and $m$, $n$ be positive integers. Then $S$ is $(m, n)$-regular if and only if $a \wedge q = q^m a q^n$ for each $(m, n)$-quasi-ideal element $q$ and for each ideal-element $a$ of $S$.

Proof. Let $S$ be an $(m, n)$-regular le-semigroup, $q$ be an $(m, n)$-quasi-ideal element and $a$ be an ideal-element of $S$. Since $q^m a q^n \leq q^m e$ and $q^m a q^n \leq e q^n$, we have $q^m a q^n \leq q^m e \wedge e q^n$. As $q$ is an $(m, n)$-quasi-ideal element of $S$, we get $q^m e \wedge e q^n \leq q$. Thus $q^m a q^n \leq q$. Since $a$ is an ideal-element of $S$, we have $q^m a q^n \leq e a c \leq a c \leq a$. Therefore $q^m a q^n \leq q \wedge a$. As $S$ is $(m, n)$-regular, we have

\[
(a \wedge q) \leq (a \wedge q)^m e (a \wedge q)^n
\]

\[
\leq (a \wedge q)^m e ((a \wedge q)^n)^m e ((a \wedge q)^n)^n
\]

\[
= (a \wedge q)^m e ((a \wedge q)^n)^m (a \wedge q)^m \ldots (a \wedge q)^m e ((a \wedge q)^n)^n
\]

\[
\leq q^m e a \underbrace{a \ldots a}_{n \text{-times}} \underbrace{a \ldots a}_{n \text{-times}}
\]

\[
= q^m e a \underbrace{a \ldots a}_{n-1 \text{-times}} \underbrace{a \ldots a}_{n-1 \text{-times}} q^n
\]

\[
\leq q^m e a q^n
\]

Therefore $a \wedge q = q^m a q^n$.

Conversely assume that $a \wedge q = q^m a q^n$ for each $(m, n)$-quasi-ideal element $q$ and for each ideal-element $a$ of $S$. Take any $b \in S$. As $(b)_{<m,n>}$ and $(b)$ are $(m, n)$-quasi-ideal elements and ideal-elements of $S$ respectively, we have

\[
(b) \wedge (b)_{<m,n>} = (b)_{<m,n>} (b)_{<m,n>} \leq (b)_{<m,n>} e (b)_{<m,n>} \leq b^m e b^n.
\]

As $b \leq (b)_{<m,n>}$, we have $b \leq b^m e b^n$. Hence $S$ is $(m, n)$-regular.

\[\square\]

Theorem 3.4 [5] Let $S$ be an le-semigroup and $m$, $n$ be non-negative integers. Then, the following conditions are equivalent:

1. $S$ is $<m, n>$-regular.
2. $a^m e a^n = a$ for each $a \in I_{<m,n>}$.
3. $e q^n a q^n = a$ for each $q \in Q_{<m,n>}$.
4. $(a)_{<m,n>} = (a)_{<m,n>}$ for each $a \in S$.
5. $(a)_{<m,n>} = (a)_{<m,n>}$ for each $a \in S$.

Theorem 3.5 Let $S$ be an le-semigroup and let $m$, $n$ be non-negative integers. Then $S$ is $(m, n)$-regular if and only if $a \wedge b = a^m b^n$ for each $(m, 0)$-ideal element $a$ and for each $(0, n)$-ideal element $b$ of $S$.

Proof. The statement is trivially true for $m = 0 = n$. If $m = 0$ and $n \neq 0$ or $m \neq 0$ and $n = 0$, then the result follows by Theorem 3.4. So, let $m \neq 0$, $n \neq 0$, $a$ be any $(m, 0)$-ideal element and $b$ be any $(0, n)$-ideal element of $S$. Then $a^m b^n \leq a^m e \leq a$ and $a^m b^n \leq e b^n \leq b$. Therefore $a^m b^n \leq a \wedge b$. As
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S is \((m, n)-\)regular, we have
\[
(a \wedge b)^m e (a \wedge b)^n \leq a^m e b^n \leq a^m e b^{n-1} (b^m e b^n) e b^n \leq a^m e b^{n-1} b^{m-1} (b^m e b^n) e b^{n-1} (b^m e b^n) e b^n \leq \cdots \leq a^m e b^{n-1} b^{m-1} \cdots b^{m-1} (b^m e b^n) e b^n \leq a^m e b^n \leq a^m b^n.
\]
Therefore \(a \wedge b = a^m b^n\).

Conversely assume that \(a \wedge b = a^m b^n\) for each \((m, 0)\)-ideal element \(a\) and for each \((0, n)\)-ideal element \(b\) of \(S\). For any \(a \in S\), as \(< a >_{m, 0} >_{0, n} \leq e a^n\). As \(a^m e\) is an \((m, 0)\)-ideal element and \(e a^n\) is a \((0, n)\)-ideal element of \(S\), by hypothesis, we have
\[
a \leq < a >_{m, 0} \land < a >_{0, n} \leq a^m e \land e a^n = (a^m e)^n a^n \leq a^m e a^n.
\]
Hence \(S\) is \((m, n)\)-regular.

\[\square\]

**Theorem 3.6** Let \(S\) be an le-semigroup and let \(m, n\) be positive integers (either \(m \geq 2\) or \(n \geq 2\)). Then the following are equivalent:

1. Each \((m, n)\)-ideal element of \(S\) is idempotent.
2. For each \((m, n)\)-ideal elements \(a, b\) of \(S\), \(a \wedge b \leq a^m b^n\).
3. \(< a >_{m, n} \land < b >_{m, n} \leq (\langle a \rangle_{<m,n>} \land \langle b \rangle_{<m,n>})^n \forall a, b \in S\).
4. \(< a >_{m, n} \leq (\langle a \rangle_{<m,n>} \land \langle b \rangle_{<m,n>})^n \forall a \in S\).
5. \(S\) is \((m, n)\)-regular.

**Proof.** (1) \(\Rightarrow\) (2) Assume that each \((m, n)\)-ideal element of \(S\) is idempotent. Let \(a\) and \(b\) be any \((m, n)\)-ideal elements of \(S\). As \(a \wedge b\) is an \((m, n)\)-ideal element of \(S\), we have
\[
a \wedge b = (a \wedge b)^2 = (a \wedge b)^3 = \cdots = (a \wedge b)^m a \wedge b^n \leq a^m b^n.
\]
(2) \(\Rightarrow\) (3) and (3) \(\Rightarrow\) (4). Obvious.
(4) $\Rightarrow$ (5). Take any $a \in S$. Then, by (4), we have
\[
\begin{align*}
&< a ><_{m,n}> \\
&\leq (< a ><_{m,n}>)^m(< a ><_{m,n}>)^n \\
&\leq (< a ><_{m,n}>)^m(< a ><_{m,n}>)^{n-1}(< a ><_{m,n}>)^m(< a ><_{m,n}>)^n \\
&\leq (< a ><_{m,n}>)^m(e(< a ><_{m,n}>))^n \\
&= a^m e a^n \quad \text{(by Lemma 2.1).}
\end{align*}
\]
As $a \leq < a ><_{m,n}>$, we have $a \leq a^m e a^n$. Hence $S$ is $(m, n)$-regular.

(5) $\Rightarrow$ (1). Take any $(m, n)$-ideal element $a$ of $S$. As $S$ is $(m, n)$-regular and $a$ is an $(m, n)$-ideal element, we have $a = a^m e a^n$. Now
\[
a^2 = (a^m e a^n)(a^m e a^n) \leq (a^m e a^n) = a
\]
and
\[
a = a^m e a^n = (a^m e a^n) e a^n \leq (a^m e a^n)(a^m e a^n) = aa = a^2.
\]
Therefore $a = a^2$. Hence each $(m, n)$-ideal element of $S$ is an idempotent.

\[\square\]

**Lemma 3.7** Let $S$ be an le-semigroup and let $m, n$ be non-negative integers. Then $S$ is $< m, 0 >$-regular ($(0, n)$-regular) if and only if $I_{< m, 0 >} (I_{< 0, n >})$, the set of all $(m, 0)$-ideal elements of $S$ ($(0, n)$-ideal elements of $S$), is $(m, 0)$-regular ($(0, n)$-regular).

**Proof.** When $m = 0$, the statement holds trivially because $e$ is the only $(0, 0)$-ideal element of $S$. So, let $m \neq 0$ and $a \in I_{< m, 0 >}$. Therefore $a^m e \leq a$. As $S$ is $(m, 0)$-regular, we have $a \leq a^m e$. Thus $a = a^m e$. Since $e \in I_{< m, 0 >}$, so $a$ is a $(m, 0)$-regular element of $I_{< m, 0 >}$. Hence $I_{< m, 0 >}$ is $(m, 0)$-regular.

Conversely assume that $I_{< m, 0 >}$ is $(m, 0)$-regular. Take any $a \in S$. As $< a ><_{m, 0}$ is in $I_{(m, 0)}$ and $I_{(m, 0)}$ is $(m, 0)$-regular, there exists $b \in I_{(m, 0)}$ such that $< a ><_{m, 0} = ( < a ><_{m, 0} >)^m b \leq ( < a ><_{m, 0} >)^m e$. By Lemma 2.1, $( < a ><_{m, 0} >)^m e = a^m e$. As $a \leq a <_{m, 0}$, we have $a \leq a^m e$. Hence $S$ is $(m, 0)$-regular.

\[\square\]

**Lemma 3.8** Let $S$ be an le-semigroup and let $m, n$ be non-negative integers. Then $S$ is $(m, n)$-regular if and only if $I_{< m, n >}$, the set of all $(m, n)$-ideal elements of $S$, is $(m, n)$-regular.

**Proof.** On the lines similar to the proof of Lemma 3.7.

\[\square\]

**Lemma 3.9** Let $S$ be an le-semigroup and let $m, n$ be non-negative integers. Then $S$ is $(m, n)$-regular if and only if $Q_{< m, n >}$, the set of all $(m, n)$-quasi-ideal elements of $S$, is $(m, n)$-regular.
Proof. If \( m = n = 0 \), then the result trivially holds as \( Q_{<0,0>} = \{e\} \). When \( m \neq 0 \) and \( n = 0 \) or \( m = 0 \) and \( n \neq 0 \), then the result follows by Lemma 3.7, as the set of all \((m, 0)\) \((0, n)\)-quasi-ideal elements \( Q_{<m,0>} \ (Q_{<0,n>}) \) coincides with the set of all \((m, 0)\) \((0, m)\)-ideal elements \( I_{<m,0>} \ (I_{<0,n>}) \). So, let \( m \neq 0 \) and \( n \neq 0 \) and \( a \in Q_{<m,n>} \). Therefore \( a^m e a^n \leq a^m e \land e a^n \leq a \). As \( S \) is \((m, n)\)-regular, we have \( a \leq a^m e a^n \). Since \( e \in Q_{<m,n>} \), it follows that \( a \) is an \((m, n)\)-regular element of \( Q_{<m,n>} \). Hence \( Q_{<m,n>} \) is \((m, n)\)-regular.

Conversely assume that \( Q_{<m,n>} \) is \((m, n)\)-regular and \( a \) be any element of \( S \). Then \( < a >_{m,n} \in Q_{<m,n>} \). Therefore there exists \( b \in Q_{<m,n>} \) such that

\[
(a)_{<m,n>} = ((a)_{<m,n>})^m b ((a)_{<m,n>})^n.
\]

Now \( ((a)_{<m,n>})^m b ((a)_{<m,n>})^n \leq ((a)_{<m,n>})^m e ((a)_{<m,n>})^n \) and, by Lemma 3.2, \( ((a)_{<m,n>})^m e ((a)_{<m,n>})^n \leq a^m e a^n \). As \( a \leq (a)_{<m,n>} \), we have \( a \leq a^m e a^n \). Therefore \( a \) is \((m, n)\)-regular and, hence, \( S \) is \((m, n)\)-regular.

\( \square \)

4 Characterization of relations \( \mathcal{I}_n, m\mathcal{I}, \mathcal{B}_m^n, Q_m^n \) and \( \mathcal{H}_m^n \)

Lemma 4.1 Let \( S \) be an le-semigroup, \( a \in S \) and let \( m, n \) be non-negative integers. Then

1. \( < a >_{m,n} <_{m,n} = < a >_{m,n} \).
2. \( ((a)_{<m,n>})_{<m,n>} = (a)_{<m,n>} \).

Proof. (1). By definition, we have

\[
<a>_{m,n} <_{m,n} = a \lor a^m e a^n <_{m,n} \\
= (a \lor a^m e a^n) \lor (a \lor a^m e a^n) e (a \lor a^m e a^n)^n \\
= (a \lor a^m e a^n) \lor (a^m e a^n) \text{ (by Lemma 2.1)} \\
= < a >_{m,n}.
\]

(2). Now

\[
((a)_{<m,n>})_{<m,n>} \\
= (a \lor (a^m e \land e a^n))_{<m,n>} \\
= (a \lor (a^m e \land e a^n)) \lor ((a \lor (a^m e \land e a^n))^m e \land e (a \lor (a^m e \land e a^n))^n). \\
\]

Now, by Lemma 3.2, \( (a \lor (a^m e \land e a^n))^m e \leq a^m e \) and \( e ((a \lor (a^m e \land e a^n))^n \leq e a^n \). So \( (a \lor (a^m e \land e a^n))^m e \land e (a \lor (a^m e \land e a^n))^n \leq a^m e \land e a^n \). Therefore

\[
((a)_{<m,n>})_{<m,n>} \\
= (a \lor (a^m e \land e a^n)) \lor ((a \lor (a^m e \land e a^n))^m e \land e (a \lor (a^m e \land e a^n))^n) \\
\leq a \lor (a^m e \land e a^n) \\
= (a)_{<m,n>}.
\]

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Also
\[
(a)_{m,n} = a \lor (a^m e \land e a^n) \\
\leq (a \lor (a^m e \land e a^n)) \lor ((a \lor (a^m e \land e a^n)) e (a \lor (a^m e \land e a^n)))^n \\
= ((a)_{m,n})_{m,n}.
\]

Hence \((a)_{m,n} = (a)_{m,n}\), as required. \(\Box\)

**Corollary 4.2** Let \(S\) be an le-semigroup, \(a \in S\) and let \(m, n\) be non-negative integers. Then
1. \(<< a >_{m,0} >_{m,0} = << a >_{m,0}\); \(<< a >_{0,n} >_{0,n} = << a >_{0,n}\).

**Remark 4.1** Let \(S\) be an le-semigroup and let \(m, n\) be non-negative integers. If \(a, b \in S\) such that \(a \leq b\), then
1. \(<< a >_{m,0} \leq b >_{m,0}\); \(<< a >_{0,n} \leq b >_{0,n}\); \(<< a >_{m,n} \leq b >_{m,n}\); \(<< a >_{m,n} \leq (b)_{m,n}\).

**Lemma 4.3** Let \(S\) be an le-semigroup, \(a \in S\) and let \(m, n\) be non-negative integers. Then
1. \(<< (a)_{m,n} >_{m,0} \leq << a >_{m,0}\); \(<< (a)_{m,n} >_{0,n} = < a >_{0,n}\).

**Proof.** Straightforward. \(\Box\)

**Lemma 4.4** Let \(S\) be an le-semigroup, \(a \in S\) and let \(m, n\) be non-negative integers. Then
1. \(< (a)_{m,n} >_{m,0} = < a >_{m,0}\); \(< (a)_{m,n} >_{0,n} = < a >_{0,n}\).

**Proof.** On the lines similar to the proof of Lemma 4.1. \(\Box\)

**Definition 4.5** Let \(S\) be an le-semigroup and let \(m, n\) be positive integers. We define the relations \(I_n\), \(m, I\), \(H^n_m\), \(B^n_m\), and \(Q^n_m\) as follows:
\[
I_n = \{(a, b) \in S \times S \mid < a >_{m,0} = b >_{n,0}\}; \\
m, I = \{(a, b) \in S \times S \mid < a >_{m,0} = b >_{m,0}\}; \\
H^n_m = m, I \cap I_n; \\
B^n_m = \{(a, b) \in S \times S \mid < a >_{m,n} = b >_{m,n}\}; \\
Q^n_m = \{(a, b) \in S \times S \mid (a)_{m,n} = (b)_{m,n}\}.
\]

Clearly all the relations defined above are equivalence relations on \(S\).
Lemma 4.6 Let $S$ be an le-semigroup. If $a, b \in S$ are $mI$-related (resp. $H_1$-related), then $a^m e = b^m e$ (resp. $ea^n = eb^n$).

Proof. Suppose $(a, b) \in mI$. Then, by definition, $<a >_{< m,0> } = <b >_{< m,0> }$ i.e. $a \vee a^m e = b \vee b^m e$. Therefore $a \leq b \vee b^m e$ and $b \leq a \vee a^m e$. Thus $a^m e \leq (b \vee b^m e)^m e \leq b^m e$ (by Lemma 2.1). Similarly, from $b \leq a \vee a^m e$, we have $b^m e \leq a^m e$. Hence $a^m e = b^m e$.

Dually it may be shown that if $(a, b) \in I_n$, then $ea^n = eb^n$.

\[\square\]

Lemma 4.7 Let $S$ be an le-semigroup. If $a, b \in S$ are $H_m$-related, then $a^m e = b^m e$, $ea^n = eb^n$ and $a^m ea^n = b^m eb^n$.

Proof. Suppose $(a, b) \in H_m$. Then, by definition, $(a, b) \in mI$ and $(a, b) \in I_n$. Now, by Lemma 4.6, $a^m e = b^m e$ and $ea^n = eb^n$. Also, as $a^m e = b^m e$ and $ea^n = eb^n$, we have $a^m ea^n = b^m ea^n = b^m eb^n$, as required.

\[\square\]

Lemma 4.8 In any le-semigroup $S$, $B_m \subseteq H_m$.

Proof. Let $(a, b) \in B_m$. Then $<a >_{< m,0> } = <b >_{< m,0> }$ i.e., $a \vee a^m e = b \vee b^m e$. So $a \leq b \vee b^m e$ and $b \leq a \vee a^m e$. Now $a^m e \leq (b \vee b^m e)^m e$ and $b^m e \leq (a \vee a^m e)^m e$. Thus, by Lemma 2.1, $a^m e \leq b^m e$ and $b^m e \leq a^m e$.

Now

$<a >_{< m,0> } = a \vee a^m e \leq b \vee b^m e \vee a^m e \leq b \vee b^m e \vee b^m e = <b >_{< m,0> }$ and

$<b >_{< m,0> } = b \vee b^m e \leq a \vee a^m e \vee b^m e \leq a \vee a^m e \vee a^m e = <a >_{< m,0> }$.

Therefore $<a >_{< m,0> } = <b >_{< m,0> }$. Similarly we may show that $<a >_{< n,0> } = <b >_{< n,0> }$. Thus $(a, b) \in H_m$. Hence $B_m \subseteq H_m$.

\[\square\]

Remark 4.2 The converse of the Lemma 4.8 is not true in general as shown by the following example:

Example 4.1 [15] Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the multiplicative semigroup under multiplication mod 8. Then the power semigroup $P(Z_8) = (P(Z_8), \cup, \cap, 1)$, where $1 \cdot$ is the extension of the multiplication of $Z_8$ to the power set $P(Z_8)$ of $Z_8$, is an le-semigroup with the largest element $e = Z_8$.

Observe that

$\{2\} \cup \{2\}e\{2\} = \{2\} \cup \{2\}Z_8\{2\} = \{0, 2, 4\},$

$\{6\} \cup \{6\}e\{6\} = \{6\} \cup \{6\}Z_8\{6\} = \{0, 4, 6\},$

which shows that $\{2\}$ is not $B(= B_1)$-related to $\{6\}$. On the other hand, as

$\{2\} \cup \{2\}e = \{2\} \cup \{2\}Z_8 = \{0, 2, 4, 6\},$

$\{6\} \cup \{6\}e = \{6\} \cup \{6\}Z_8 = \{0, 2, 4, 6\},$

we have $\{2\}R\{6\}$. Since $P(Z_8)$ is commutative, the relations $R(= I)$ and $H(= H_1)$ coincide. Thus $\{2\}H_1\{6\}$ which implies that $H_1 \subseteq B_1$.
Theorem 4.9 In any \((m,n)\)-regular le-semigroup \(S\), \(B^n_m = H^n_m\).

Proof. Let \((a,b) \in H^n_m\). Then, by Lemma 4.7, \(a^mea^n = b^meb^n\). As \(S\) is \((m,n)\)-regular, we have \(a \leq a^mea^n \text{ and } b \leq b^meb^n\). So \(< a >_{<m,n>} = a \lor a^mea^n \text{ and } < b >_{<m,n>} = b \lor b^meb^n\). Thus \(< a >_{<m,n>} = < b >_{<m,n>}\) i.e., \((a,b) \in B^n_m\). So \(H^n_m \subseteq B^n_m\). Hence, by Lemma 4.8, \(B^n_m = H^n_m\).

Lemma 4.10 In any le-semigroup \(S\), \(B^n_m \subseteq Q^n_m\).

Proof. Let \((a,b) \in B^n_m\). Then \(< a >_{<m,n>} = < b >_{<m,n>}\) i.e., \(a \lor a^mea^n = b \lor b^meb^n\). So \(a \leq b \lor b^meb^n\) and \(b \leq a \lor a^mea^n\). Therefore \(a^me \leq (b \lor b^meb^n)^m e \leq b^m e\) and \(ea^n \leq e(b \lor b^meb^n)^n \leq eb^n\). Thus \(a^me \land ea^n \leq b^m e \land eb^n\). Now

\[(a)_{<m,n>} = a \lor (a^me \land ea^n) \leq b \lor b^meb^n \lor (a^me \land ea^n) \leq b \lor b^meb^n \lor (b^me \land eb^n) = (b)_{<m,n>}.
\]

Similarly, as \(b \leq a \land a^mea^n\), we may show that \((b)_{<m,n>} \leq (a)_{<m,n>}\) i.e., \((a,b) \in Q^n_m\). Hence \(B^n_m \subseteq Q^n_m\), as required.

Remark 4.3 The converse of the Lemma 4.10 is not true in general. It is easy to verify from Example 4.1 that \(\{2\} \neq Q^1_1 = Q\{6\}\) while \(\{2\} \neq B^1_1 = B\{6\}\).

Theorem 4.11 Let \(S\) be an \((m,n)\)-regular le-semigroup. Then \(Q^n_m = B^n_m\).

Proof. Let \((a,b) \in Q^n_m\). Then \((a)_{<m,n>} = (b)_{<m,n>}\) i.e., \(a \lor (a^me \land ea^n) = b \lor (b^me \land eb^n)\). So \(a \leq b \lor (b^me \land eb^n)\) and \(b \leq a \lor (a^me \land ea^n)\). As \(Q^n_m \subseteq H^n_m\), we have \((a,b) \in H^n_m\). So, by Lemma 4.7, \(a^mea^n = b^meb^n\). Now

\[< a >_{<m,n>} = a \lor a^mea^n \leq b \lor (b^me \land eb^n) \lor a^mea^n \quad \text{(because } a \leq b \lor (b^me \land eb^n)\)]

\[\leq b \lor (b^me \land eb^n) \lor b^meb^n \quad \text{(as } a^mea^n = b^meb^n\)]

\[= b \lor b^meb^n \quad \text{(because } b^meb^n \leq b^m e \land eb^n)\]

\[= b \lor (b^m e)^n (eb^n) \quad \text{(by Theorem 3.5)}\]

\[\leq < b >_{<m,n>} .
\]

Similarly, as \(b \leq a \lor (a^me \land ea^n)\), we have \(< b >_{<m,n>} \leq < a >_{<m,n>}\).

Therefore \(< a >_{<m,n>} = < b >_{<m,n>}\). Thus \((a,b) \in B^n_m\). This implies that \(Q^n_m \subseteq B^n_m\). Hence, by Lemma 4.10, \(Q^n_m = B^n_m\).

Lemma 4.12 In any le-semigroup, \(Q^n_m \subseteq H^n_m\).

Proof. Let \(S\) be any le-semigroup and let \((a,b) \in Q^n_m\). Then \((a)_{<m,n>} = (b)_{<m,n>}\) i.e. \(a \lor (a^me \land ea^n) = b \lor (b^me \land eb^n)\). So \(a \leq b \lor (b^me \land eb^n)\) and \(b \leq a \lor (a^me \land ea^n)\). Now
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\[ <a>_{<m,0>} = a \lor a^m e \leq (b \lor (b^m e \land eb^n)) \lor (b \lor (b^m e \land eb^n))^n e \leq <b>_{<m,0>} \]

and

\[ <a>_{<0,n>} = a \lor ea^n \leq (b \lor (b^m e \land eb^n)) \lor e(b \lor (b^m e \land eb^n))^n e \leq <b>_{<0,n>} . \]

Similarly, as \( b \leq a \lor (a^m e \land ea^n) \), we have \( <b>_{<m,0>} \leq <a>_{<m,0>} \) and \( <b>_{<0,n>} \leq <a>_{<0,n>} \). Therefore \( <a>_{<m,0>} = <b>_{<m,0>} \) and \( <a>_{<0,n>} = <b>_{<0,n>} \) implying that \((a,b) \in H^m_m\). Hence \( Q^m_n \leq H^m_m \), as required.

\[ \square \]

**Corollary 4.13** Let \( S \) be an \((m, n)\)-regular le-semigroup. Then \( B^m_n = Q^m_n = H^m_m \).

**Lemma 4.14** If \( B_x \) and \( B_y \) are two \((m, n)\)-regular \( B^m_n \)-classes contained in the same \( H^m_m \)-class of an le-semigroup \( S \), then \( B_x = B_y \).

**Proof.** As \( x \) and \( y \) are \((m, n)\)-regular elements of \( S \), \( x \leq x^m e x^n \) and \( y \leq y^m e y^n \). Therefore \( x_{<m,n>} = x \lor (x^m e x^n) = x^m e x^n \) and \( y_{<m,n>} = y \lor (y^m e y^n) = y^m e y^n \). Since \( x \) and \( y \) are contained in the same \( H^m_m \)-class, by Lemma 4.7, \( x^m e x^n = y^m e y^n \). So \( x_{<m,n>} = y_{<m,n>} \). Therefore \( x B^m_n y \). Hence \( B_x = B_y \).

\[ \square \]

**Lemma 4.15** If \( Q_x \) and \( Q_y \) are two \((m, n)\)-regular \( Q^m_n \)-classes contained in the same \( H^m_m \)-class of an le-semigroup \( S \), then they coincide.

**Proof.** As \( x \) and \( y \) are \((m, n)\)-regular elements of \( S \), \( x \leq x^m e x^n \leq x^m e \land ex^n \) and \( y \leq y^m e y^n \leq y^m e \land ey^n \). Therefore \( x_{<m,n>} = x \lor (x^m e \land ex^n) = x^m e \land ex^n \) and \( y_{<m,n>} = y \lor (y^m e \land ey^n) = y^m e \land ey^n \). Since \( x \) and \( y \) are contained in the same \( H^m_m \)-class, by Lemma 4.7, \( x^m e = y^m e \) and \( ex^n = ey^n \). Therefore \( x^m e \land ex^n = y^m e \land ex^n \). This implies that \( (x)_{<m,n>} = (y)_{<m,n>} \). So \( x Q^m_n y \) implies \( Q_x = Q_y \), as required.

\[ \square \]

5 \((m, 0)\)-regularity \([(0, n)\)-regularity\] and \((m, n)\)-right weakly regularity of \( B^m_n \)-classes, \( Q^m_n \)-classes and \( H^m_m \)-classes

**Proposition 5.1** A \( Q^m_n \)-class \( Q \) of an le-semigroup \( S \) is \((m, 0)\)-regular \([(0, n)\)-regular\] if it contains an \((m, 0)\)-regular \([(0, n)\)-regular\] element.

**Proof.** Let \( a \in Q \) be an \((m, 0)\)-regular element and \( b \in Q \). Then \( (b)_{<m,n>} = (a)_{<m,n>} = a \lor (a^m e \land a^n) \leq a \lor a^m e = a^m e \) (since \( a \) is \((m, 0)\)-regular, \( a \leq a^m e \)). By Lemma 4.10 and Lemma 4.7, \( a^m e = b^m e \). So \( (b)_{<m,n>} \leq b^m e \). Thus \( b \leq b^m e \). Therefore \( b \) is a \((m, 0)\)-regular element and, hence, \( Q \) is \((m, 0)\)-regular. The dual statement may be proved on similar lines.

\[ \square \]
**Proposition 5.2** An \( \mathcal{H}_{m,n} \)-class \( H \) of a \( \vee \)-semigroup \( S \) is \((m,0)\)-regular \([(0,n)\)-regular] if it contains an \((m,0)\)-regular \([(0,n)\)-regular] element.

**Proof.** Let \( a \in H \) be an \((m,0)\)-regular element and \( c \in H \). Then \(< a >_{m,0}^c = a \vee a^m e = a^m e \) (since \( a \) is \((m,0)\)-regular, \( a \leq a^m e \)). By Lemma 4.7, \( a^m e = b^m e \). This implies that \(< b >_{m,0}^c \leq b^m e \). Hence \( b \leq b^m e \). So \( b \) is \((m,0)\)-regular element. Hence \( H \) is \((m,0)\)-regular. The dual statement follows on the similar lines.

\[\square\]

**Corollary 5.3** An \( \vee \)-semigroup \( S \) is \((m,0)\)-regular \([(0,n)\)-regular] if and only if each \( \mathcal{Q}_{m,n} \)-class \((\mathcal{H}_{m,n} \)-class) of \( S \) contains an \((m,0)\)-regular \([(0,n)\)-regular] element.

**Definition 5.4** Let \( S \) be a \( \text{po}e \)-semigroup and \( m, n \) be positive integers. An element \( a \) of \( S \) is said to be an \((m,n)\)-right regular element if \( a \leq a^m e a^n e \). Further \( S \) is said to be \((m,n)\)-right weakly regular if each element of \( S \) is \((m,n)\)-right weakly regular.

**Proposition 5.5** A \( \mathcal{B}_{m,n} \)-class \( B \) of a \( \vee \)-semigroup \( S \) is \((m,n)\)-right weakly regular if it contains an \((m,n)\)-right weakly regular element.

**Proof.** Let \( a \in B \) be an \((m,n)\)-right weakly regular element and \( b \in B \) be any element of \( B \). Then \( a \leq a^m e a^n e \). So, it follows that \( a^m e a^n \leq (a^m e a^n e)^n e (a^m e a^n e)^n \leq a^m e a^n e \). Since \( a, b \in B \), we have \(< a >_{m,n}^c = a \vee a^m e a^n e \leq a \vee a^m e a^n e = a^m e a^n e \). By Lemmas 4.8 and 4.7, we have \( a^m e a^n = b^m e b^n \). This implies that \(< b >_{m,n}^c \leq b^m e b^n \). So \( b \leq b^m e b^n \). Thus \( b \) is an \((m,n)\)-right weakly regular element of \( B \). Hence \( B \) is \((m,n)\)-right weakly regular.

\[\square\]

**Proposition 5.6** A \( \mathcal{Q}_{m,n} \)-class \( Q \) of an \( \text{po}e \)-semigroup \( S \) is \((m,n)\)-right weakly regular if it contains an \((m,n)\)-right weakly regular element.

**Proof.** Let \( a \in Q \) be an \((m,n)\)-right weakly regular element and \( b \in Q \) be any element of \( Q \). Then \( a \leq a^m e a^n e \). Thus \( a^m e \leq (a^m e a^n e)^m e \leq a^m e a^n e \). Since \( a, b \in Q \), we have \( (b)_{m,n}^c = (a)_{m,n}^c = a \vee (a^m e a^n e) \leq a \vee a^m e a^n e \leq a \vee a^m e a^n e = a^m e a^n e \). As, by lemmas 4.10 and 4.7, \( a^m e a^n = b^m e b^n \), we have \( (b)_{m,n}^c \leq b^m e b^n \). Thus \( b \leq b^m e b^n \). This implies that \( b \) is an \((m,n)\)-right weakly regular element of \( Q \) and, hence, \( Q \) is \((m,n)\)-right weakly regular.

\[\square\]

**Proposition 5.7** An \( \mathcal{H}_{m,n} \)-class \( H \) of a \( \vee \)-semigroup \( S \) is \((m,n)\)-right weakly regular if it contains an \((m,n)\)-right weakly regular element.

**Proof.** On the lines similar to the proof of Proposition 5.6.

\[\square\]

**Corollary 5.8** An \( \text{po}e \)-semigroup \( S \) is \((m,n)\)-right weakly regular if and only if each \( \mathcal{B}_{m,n} \)-class \((\mathcal{Q}_{m,n} \)-class, \( \mathcal{H}_{m,n} \)-class) of \( S \) contains an \((m,n)\)-right weakly regular element.
6 Strong \((m, n)\)-quasi-ideal element

In a poe-semigroup \(S\), meet of an \((m, 0)\)-ideal element \(a\) and a \((0, n)\)-ideal element \(b\) is an \((m, n)\)-quasi-ideal if \((a \land b)^m \land c(a \land b)^n\) exists. The converse statement does not hold in general. As an example [5], let \(S = \{a, b, c, d\}\). Define a binary operation \(\cdot\) as follows:

\[
\begin{array}{cccc}
  a & b & c & d \\
  a & a & a & a \\
  b & a & a & d \\
  c & a & c & c \\
  d & a & c & b \\
\end{array}
\]

Then the set \(\mathcal{P}(S)\) of all non-empty subsets of \(S\) is a poe-semigroup under the binary operation of set products induced by the binary operation defined by the above table and partially ordered by set inclusion. The set \(\{a, b\}\) is an \((1, 1)\)-quasi-ideal element of \(S\). The sets \(\{a\}\), \(\{a, b, c, d\}\), \(\{a, c\}\) and \(\{a\}\), \(\{a, b, c, d\}\), \(\{a, d\}\) are the \((1, 0)\) and \((0, 1)\)-ideal element of \(S\) respectively, but \(\{a, b\}\) is not the intersection of any \((1, 0)\) and \((0, 1)\)-ideal element.

**Definition 6.1** Let \(S\) be a \(\vee\)-e-semigroup and \(q\) be an \((m, n)\)-quasi-ideal element of \(S\). Then \(q\) is said to be a strong \((m, n)\)-quasi-ideal element of \(S\) if \(q\) can be written as the meet of an \((m, 0)\)-ideal element and a \((0, n)\)-ideal element.

**Theorem 6.2** Let \(S\) be a \(\vee\)-e-semigroup and \(q\) be an \((m, n)\)-quasi-ideal element of \(S\). Then the following are equivalent:
1. \(q\) is a strong \((m, n)\)-quasi-ideal element of \(S\);
2. \(< q >_{<m,0>} \land < q >_{<0,n>} = q\);
3. \(< a >_{<m,0>} \land < b >_{<0,n>} \leq q \quad (\forall a, b \leq q)\).

**Proof.** (1) \(\Rightarrow\) (2). Let \(q\) be a strong \((m, n)\)-quasi-ideal element of \(S\). Then \(q = a \land b\) for some \((m, 0)\)-ideal element \(a\) and \((0, n)\)-ideal element \(b\) of \(S\). Therefore \(< a >_{<m,0>} = a\) and \(< b >_{<0,n>} = b\). Now, as \(< q >_{<m,0>} = < a \land b >_{<m,0>} = < a >_{<m,0>} \land < a >_{<0,n>} = < a \land b >_{<m,0>} \leq < b >_{<0,n>} = b\), it follows that \(< q >_{<m,0>} \land < q >_{<0,n>} \leq a \land b = q\). Again, as \(q^m e \land eq^n \leq q, \quad q = q \lor (q^m e \land eq^n)\). Therefore \(q \leq q \lor q^n = < q >_{<m,0>}\) and \(q \leq q \lor q^n = < q >_{<0,n>}\). Thus \(q \leq < q >_{<m,0>} \land < q >_{<0,n>} = q\), as required.

(2) \(\Rightarrow\) (3). Obvious.

(3) \(\Rightarrow\) (1). As \(q \leq q\), by (3), we have \(< q >_{<m,0>} \land < q >_{<0,n>} \leq q\). Also, as \(q \leq < q >_{<m,0>} \land < q >_{<0,n>}\), we have \(< q >_{<m,0>} \land < q >_{<0,n>} = q\).

Since \(< q >_{<m,0>}\) and \(< q >_{<0,n>}\) are \((m, 0)\) and \((0, n)\)-ideal elements of \(S\) respectively, by definition, \(q\) is a strong \((m, n)\)-quasi-ideal element of \(S\).

**Lemma 6.3** Let \(S\) be an le-semigroup and let \(a \in S\). Then the strong \((m, n)\)-quasi-ideal element of \(S\) generated by the element \(a\), denoted by \(q_s(a)\), is equal to

\(< a >_{<m,0>} \land < a >_{<0,n>}\).
Proof. Let a ∈ S. Then <a><m,0> ∧ <a><0,n> is a strong (m, n)-quasi-ideal element of S. Let q be any strong (m, n)-quasi-ideal element of S such that a ≤ q. Then, by Theorem 6.2, we have <a><m,0> ∧ <a><0,n> ≤ q. Therefore qa(a) = <a><m,0> ∧ <a><0,n>.

Theorem 6.4 Let S be an (m, n)-regular le-semigroup. Then each (m, n)-quasi-ideal element is a strong (m, n)-quasi-ideal element.

Proof. Let S be an (m, n)-regular le-semigroup and q be an (m, n)-quasi-ideal element of S. As q ≤ qmqn, we have q ≤ qmqn and q ≤ eqn. Therefore qmqn = q ∨ qmqn = <q><m,0> and eqn = q ∨ eqn = <q><0,n>. As q ≤ <q><m,0> ∧ <q><0,n>, we have <q><m,0> ∧ <q><0,n> ≤ q. Hence, by Theorem 6.7, q is a strong (m, n)-quasi-ideal element of S.

Lemma 6.5 Each (m, n)-quasi-ideal element in a distributive le-semigroup is a strong (m, n)-quasi-ideal element.

Proof. Straightforward.

Lemma 6.6 Each Hm-class H of S has a strong (m, n)-quasi-ideal element which is the greatest element of H and is equal to <a><m,0> ∧ <a><0,n> for a ∈ H.

Proof. Let a ∈ H. Then, by Lemma 4.3, <a><m,0> ∧ <a><0,n> = <a><m,0> and <a><m,0> ∧ <a><0,n> = <a><m,0>. So, we have <a><m,0> ∧ <a><m,0>, a) ∈ mI and <a><m,0> ∧ <a><m,0>, a) ∈ mI. Now take any element b ∈ H. As b ≤ <b><m,0>=<b><m,0> and b ≤ <b><n,0>=<b><n,0>, we have b ≤ <a><m,0> ∧ <a><n,0>. Therefore <a><m,0> ∧ <a><n,0> is the greatest strong (m, n)-quasi-ideal element of H.

Theorem 6.7 Let S be an le-semigroup. Then Hm = Qm if and only if each (m, n)-quasi-ideal element is a strong (m, n)-quasi-ideal element.

Proof. Let (a, b) ∈ Hm. Therefore <a><m,0>=<b><m,0> and <a><0,n>=<b><0,n>. As each (m, n) quasi-ideal element is a strong (m, n) quasi-ideal element, therefore, by Theorem 6.2, we have

(a)<m,n> = <a><m,0> ∧ <a><0,n> = <a><m,0> ∧ <a><0,n> = <b><m,0> ∧ <b><0,n> = <b><m,0> ∧ <b><0,n> = <b><m,0> ∧ <b><0,n> = (b)<m,n>.
Thus \((a, b) \in Q^n_m\). Hence, by Lemma 4.10, \(Q^n_m = H^n_m\).

Conversely, assume that \(q\) be any \((m, n)\)-quasi-ideal element of \(S\). Therefore \(q \in Q^n_m(a)\) for some \(a \in S\). Let \(x \in Q^n_m(a)\). Then \((x)_{<m,n>} = (q)_{<m,n>} = q\) implies \(x \leq q\). Thus \(x\) is the greatest element of \(Q^n_m(a)\). By assumption \(Q^n_m(a) = H^n_m(a)\). Therefore, by Lemma 6.3, we have \(q = <a >_{<m,n>} \wedge <a >_{<0,n>}\). Hence \(q\) is a strong \((m, n)\)-quasi-ideal element of \(S\).

\[ \square \]

**Corollary 6.8** In any distributive \(le\)-semigroup, \(H^n_m = Q^n_m\).

**Proof.** The proof follows by Lemma 4.15 and Theorem 6.7.

\[ \square \]

**Motivation and Application:** The main motivation of the present paper is to introduce the equivalence relations \(mI, I_n, B^m, Q^n_m\) and \(H^n_m\) on an \(le\)-semigroup and enhance the understanding of different classes of \(le\)-semigroups ((\(m, n\))-regular, \((0, n)\)-regular, \((m, n)\)-right weakly regular) by considering the structural influence of the equivalence relations \(mI, I_n, B^m, Q^n_m\) and \(H^n_m\). In particular, if we take \(m = 1 = n\) in the equivalence relations \(mI, I_n, B^m, Q^n_m\) and \(H^n_m\), then we get the equivalence relations \(L, R, B, Q\) and \(H\). Again, if we take \(m = 1 = n\) in Theorem 3.5, Lemma 4.6, Lemma 4.7, Lemma 4.8, Theorem 4.9, Lemma 4.10, Theorem 4.11, Lemma 4.12, Lemma 4.14, Lemma 4.15, Corollary 6.8, then all results of the papers [15–17] are deduced as corollaries which is the main motivation of the paper and a testimony of the genuineness of the notions introduced in the paper.

**References**


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