On higher derivations of partially ordered sets

Abdelkarim Boua · Ahmed Y. Abdelwanis · Nadeem ur Rehman

Abstract The point of this paper is to present and study the idea of higher derivations on partially ordered sets that generalize the concept of derivations on partially ordered sets. Additionally, several characterization theorems on higher derivations are introduced. Moreover, the properties of the fixed points based on the higher derivations are examined. Finally, the properties of ideals and operations related to higher derivations are studied.

Keywords fixed points · ideals · partially ordered set

Mathematics Subject Classification (2010) 06E20 · 13N15 · 06Axx

1 Introduction

The notion of derivation introduced from the analytical theory is helpful in the research of structure and property in the algebraic system. In any case, the investigation of derivation in rings got a driving force not long after Herstein and Posner all the while acquired some astounding outcomes, especially for prime rings in 1957. Starting late many comprehended algebraists have studied derivations in rings, near rings, and various algebraic structures. The concept of derivation was extended to a higher derivation by Hasse and Schmidt [10]. Higher derivation of a ring has been studied by many authors on various algebraic structures, such as triangular rings, von Neumann algebras, for more details see ([1], [3], [6], [9] and [12]). In the year 2004 Jun and Xin [8] applied the notion of derivation in rings and near-ring theory to BCI-algebra. Recently, analytic and algebraic properties of lattices have been widely researched. The notion of lattice derivation was introduced and developed by Szasz [11], in which he established the main properties of derivations of lattices. In [13], Xin et al. introduced the notion of derivation for lattice and discussed some related properties also the definition of the derivation for lattice appeared in [2]. They gave some equivalent conditions under which the derivation is an isotone for lattices.
with the greatest element modular lattices, and distributive lattices, and characterized modular and distributive lattices by isotone derivation.

In 2017 Yılmaz Çeven [4], introduced the concept of higher derivations of lattices as follows: Let $L$ be a lattice, $J = \{0, 1, 2, \ldots, t\}$ or $J = N = \{0, 1, 2, \ldots\}$ (in this case $t \to \infty$) and $D = \{d_n\}_{n \in I}$ be a family of mappings of $P$ such that $d_0 = id_P$. Then $D$ is said to be a higher derivation of length $t$ on $L$ if for every $n \in J$, we have $d_n(x \wedge y) = \vee_{i+j=n} (d_i(x) \wedge d_j(y))$ for all $x, y \in L$.

Recently, Zhang and Li [14] introduced the notion of derivation on partially ordered sets and studied their basic properties. They also investigate the properties of the ideal and operations related to derivations.

Our research was mainly motivated by the studies in [4] and [14]. In the present article, we introduced the notion of higher derivation of a partially ordered set and some related properties are investigated for the higher derivation on a partially ordered set.

The paper is sorted out as takes after. In Section 3, we present the notion of higher derivations of partially ordered sets and concentrate their essential properties. Also, we examine the fixed sets considering the higher derivation. In Section 4, we examine the properties of ideals and the operations related to the higher derivation.

2 Preliminaries

To state our results precisely, we fix some notions. In this paper, $(P, \leq)$ always denotes a partially ordered set (poset). We additionally use the shorthand $P$ to indicate a poset. For $z \in P$, we write $z = \downarrow z = \{p \in P : p \leq z\}$ and $z = \uparrow z = \{p \in P : z \leq p\}$. For $B \subseteq P$, we define $l(B) = \{p \in P : p \leq b, \text{for all } b \in B\}$ the lower cone of $B$ and $u(B) = \{p \in P : b \leq p, \text{for all } b \in B\}$ the upper cone of $B$ dually. It is quickly clear that both are antitone and their compositions $l(u(\cdot))$ and $u(l(\cdot))$ are monotone. Also, we have $l(u(l(\cdot))) = l(\cdot), u(l(u(\cdot))) = u(\cdot)$ (see [5] for more details). If $B = \{b_1, b_2, \ldots, b_n\}$ is a finite subset, then we write $l(B) = l(b_1, b_2, \ldots, b_n)$ and $u(B) = u(b_1, b_2, \ldots, b_n)$ simply. Also, for $A \subseteq P$ and $B \subseteq P$, we will denote $l(A, B)$ for $l(A \cup B)$ and $u(A, B)$ for $u(A \cup B)$. For $B \subseteq P$, we write $\downarrow B = \{p \in P : p \leq a, \text{for some } a \in B\}$.

Let $A \subseteq P$, $b$ is an upper bound if $a \leq b$ for all $a \in A$. A subset $D$ of $P$ is directed provided it is nonempty and every finite subset of $D$ has an upper bound in $D$. (Aside for nonemptiness, it is sufficient to assume every pair of element in $D$ has an upper bound in $D$). We say that $A$ is a lower bound if $A = \downarrow A$ (see [7]). A subset $J$ of $P$ is called an ideal if it is directed lower set.

The concept of derivation on partially ordered set introduced by Zhang and Li as follows:

Definition 2.1 [14, Definition 2.1] Let $P$ be a poset and $d : P \to P$ be a function. We call $d$ a derivation on $P$ if it satisfies the following conditions:
1. \( d((x,y)) = l(u(l(d(x),y),l(x,d(y)))) \) for all \( x,y \in P \);
2. \( l(d(u(x,y))) = l(u(d(x),d(y))) \) for all \( x,y \in P \).

### 3 Higher derivations of posets

Let \( P \) be a poset, \( J = \{0,1,\ldots,t\} \) or \( J = N = \{0,1,2,\ldots\} \) (in this case \( t \to \infty \)) and \( D = \{d_n\}_{n \in J} \) be a family of mappings of \( P \) such that \( d_0 = id_P \).

The following definition introduces the notion of higher derivation for a partially ordered sets as follows:

**Definition 3.1** Let \( P \) be a poset, \( D \) is said to be a higher derivation of length \( t \) on \( P \) if for every \( n \in J \) we have:

1. \( d_n((l(x),y)) = \bigcup_{i+j=n} l(u(l(d_{i}(x),d_{j}(y)), l(d_{j}(x),d_{i}(y)))) \) for all \( x,y \in P \);
2. \( l(d_n(u(x,y))) = l(u(d_n(x),d_n(y))) \) for all \( x,y \in P \).

If \( D \) is a higher derivation of length \( t \) on \( P \), then for all \( x,y \in P \), we have \( d_0((l(x),y)) = l(x,y) = l(u(l(d_0(x),d_0(y)))) \) and \( l(d_0(u(x,y))) = l(u(d_0(x),d_0(y))). \) Since \( d_0 = id_P \) so \( d_0 \) is a derivation on \( P \).

Also, \( d_1 \) is a derivation on \( P \) since

\[
\begin{align*}
d_1((l(x),y)) &= l(u(l(d_0(x),d_1(y)), l(d_1(x),d_0(y)))) \\
&= l(u(l(x,d_1(y)), l(d_1(x),y))) \text{ for all } x,y \in P.
\end{align*}
\]

\( d_2 \) is a mapping such that

\[
d_2((l(x),y)) = l(u(l(d_0(x),d_2(y))), l(d_2(x),d_0(y))) \cup l(u(l(d_1(x),d_1(y))))
\]

Furthermore, \( d_n \) is a mapping that satisfies

\[
d_n((l(x),y)) = l(u(l(d_0(x),d_n(y))), l(d_n(x),d_0(y))) \cup \cdots \cup l(u(l(d_{n-1}(x),d_{n-1}(y))), l(d_0(x),d_n(y))) \text{ for all } x,y \in P.
\]

**Remark 3.1** Suppose \( P = (P, \leq, \land, \lor) \) is a lattice, then we can prove that if \( D \) is a higher derivation on \( (P,\leq) \), then \( d \) is a higher derivation on lattice \( (P,\land,\lor) \).

Now we give an example of a higher derivation on a poset \( P \).

**Example 3.1** Let \( (P,\leq) = (\mathbb{R}, \leq) \). Define \( d_0, d_1, d_2, d_3 : \mathbb{R} \to \mathbb{R} \) by:

\[
d_0(x) = d_1(x) = d_2(x) = d_3(x) = x \text{ for all } x \in P.
\]

Then \( D = \{d_0, d_1, d_2, d_3\} \) is a higher derivation of length 3 on \( P \).

In this discussion, we begin with the following proposition, which is a generalization of Proposition 3.2 of [4].

**Proposition 3.2** Let \( P \) be a poset with the least element 0 and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). Then \( d_n(0) = 0 \) for all \( n \in J \).
Proof. Since \( d_0(x) = x \) for all \( x \in P \), then \( d_0(0) = 0 \), also we have \( d_1 \) is a derivation, so from [14], Proposition 2.1 (iv), we obtain \( d_1(0) = 0 \). Now for \( d_2 \) we get

\[
\begin{align*}
d_2(l(0)) &= d_2(l(0, 0)), \\
&= \bigcup_{i+j=2} l(u(l(d_i(0), d_j(0), l(d_j(0), d_i(0))))), \\
&= \bigcup_{i+j=2} l(u(l(d_i(0), d_j(0)))), \\
&= \bigcup_{i+j=2} l(d_i(0), d_j(0)), \\
&= l(d_0(0), d_2(0)) \cup l(d_1(0), d_1(0)) \cup l(d_2(0), d_0(0)), \\
&= l(0, d_2(0)) \cup l(0, 0) \cup l(d_2(0), 0), \\
&= l(0) \cup l(0) \cup l(0), \\
&= l(0).
\end{align*}
\]

But \( l(0) = \{0\} \), so we observe that \( d_2(0) = 0 \).

Now suppose that \( d_n(0) = 0 \) for \( n = 3, 4, \ldots, t - 1 \), we have

\[
\begin{align*}
d_t(l(0)) &= d_t(l(0, 0)), \\
&= \bigcup_{i+j=t} l(u(l(d_i(0), d_j(0), l(d_j(0), d_i(0))))), \\
&= \bigcup_{i+j=t} l(u(l(d_i(0), d_j(0)))), \\
&= \bigcup_{i+j=t} l(d_i(0), d_j(0)), \\
&= l(d_0(0), d_t(0)) \cup l(d_1(0), d_t-1(0)) \cup \ldots \cup l(d_t(0), d_0(0)), \\
&= l(0, d_t(0)) \cup l(0, 0) \cup l(0, 0) \cup \ldots \cup l(d_t(0), 0), \\
&= l(0) \cup l(0) \cup \ldots \cup l(0), \\
&= l(0).
\end{align*}
\]

Again by \( l(0) = \{0\} \), we obtain \( d_t(0) = 0 \). Hence we have \( d_n(0) = 0 \) for all \( n \in J \).

\[ \square \]

**Proposition 3.3** Let \( P \) be a poset and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). If \( x \leq y \), then \( d_n(x) \leq d_n(y) \) for all \( n \in J \).

Proof. Suppose that \( D \) is a higher derivation of length \( t \) on \( P \) and \( x \leq y \), then for all \( n \in J \) we get

\[
\begin{align*}
l(d_n(u(x, y))) &= l(d_n(u(y))) \\
&= l(u(d_n(x), d_n(y))).
\end{align*}
\]

But \( d_n(x) \in l(u(d_n(x), d_n(y))) \), so \( d_n(x) \in l(d_n(u(y))) \). Hence \( d_n(x) \leq d_n(y) \).

\[ \square \]

We have the next result which generalizes Proposition 3.3 in [4].
Proposition 3.4 Let $P$ be a poset and $D = \{d_n\}_{n \in J}$ be a higher derivation of length $t$ on $P$, then $d_1 \leq d_n$ for all $n \in J$.

Proof. Since $d_0 = id_P$, we get $d_1$ is a derivation. So from [[14], Proposition 2.1 (i)], we obtain $d_1(x) \leq x = d_0(x)$ for all $x \in P$.

Now for $d_2$, we have for all $x \in P$,

$$d_2(l(x)) = d_2(l(x,x)),$$
$$= \bigcup_{i+j=n} l(u(l(d_i(x),d_j(x)),l(d_j(x),d_i(x)))),$$
$$= \bigcup_{i+j=n} l(u(l(d_i(x),d_j(x))),$$
$$= \bigcup_{i+j=n} l(d_i(x),d_j(x)),$$
$$= l(d_0(x),d_2(x)) \cup l(d_1(x),d_1(x)) \cup l(d_2(x),d_0(x)),$$
$$= l(x,d_2(x)) \cup l(d_1(x),d_1(x)) \cup l(d_2(x),x),$$
$$= l(x,d_2(x)) \cup l(d_1(x)),$$
$$= l(d_1(x)) \cup l(d_2(x)).$$

So $l(d_1(x)) \subseteq d_2(l(x)) \subseteq l(d_2(x))$, for all $x \in P$. Since $d_1(x) \in l(d_1(x))$, we obtain $d_1(x) \in l(d_2(x))$, which forces that $d_1(x) \leq d_2(x)$ for all $x \in P$.

Now suppose that $d_1(x) \leq d_n(x)$ for $n = 2, 3, 4, \ldots, t-1$, then for all $x \in P$, we have

$$d_t(l(x)) = d_t(l(x,x)),$$
$$= \bigcup_{i+j=t} l(u(l(d_i(x),d_j(x)),l(d_j(x),d_i(x)))),$$
$$= \bigcup_{i+j=t} l(u(l(d_i(x),d_j(x))),$$
$$= \bigcup_{i+j=t} l(d_i(x),d_j(x)),$$
$$= l(d_0(x),d_t(x)) \cup l(d_1(x),d_{t-1}(x)) \cup \ldots \cup l(d_t(x),d_0(x)),$$
$$= l(x,d_t(x)) \cup l(d_1(x)).$$

Again $l(d_1(x)) \subseteq l(d_t(x))$, for all $x \in P$. Then we obtain $d_1(x) \in l(d_1(x)) \subseteq l(d_t(x))$. Which leads to $d_1(x) \leq d_t(x)$ for all $x \in P$. Hence, we conclude that $d_1(x) \leq d_n(x)$ for all $n \in J$, and $x \in P$. 

\[ \square \]

Corollary 3.5 Let $P$ be a poset and $D = \{d_n\}_{n \in J}$ be a higher derivation of length $t$ on $P$. Then

1. $l(d_i(x),d_j(y)) \subseteq d_n(l(x,y))$ for all $x, y \in P$ and $i + j = n$;
2. $l(d_i(x),d_j(x)) \subseteq d_n(l(x))$ for all $x, y \in P$ and $i + j = n$;
3. If $n$ is even then $d_n(x) \leq d_n(x)$ for all $x \in P$.

Proof. (1) For all $i, j \in J$, we have

$$l(d_i(x),d_j(y)) = l(u(l(d_i(x),d_j(y))))$$
$$\subseteq l(u(l(d_i(x),d_j(y)),l(d_j(x),d_i(y)))).$$
So for all $i, j \in J$ such that $i + j = n$ we get
\[
l(d_i(x), d_j(y)) \subseteq \bigcup_{i+j=n} (l(\bigcup(l(d_i(x), d_j(y))), l(d_j(x), d_i(y)))) = d_n(l(x, y)).
\]

(2) Similarly as (1).

(3) If $n = 0$ then $d_{\frac{n}{2}}(x) = d_0(x) = x$ for all $x \in P$.

Also if $n = 2$, then $d_{\frac{n}{2}}(x) = d_1(x) \leq d_2(x)$, from Proposition 3.4.

Now let $n \in J$ be an even number and $d_{\frac{n}{2}}(x) \leq d_n(x)$ for all $x \in P$ and

\[s = 0, 2, 4, ..., n - 2,\]

then we can see that
\[
d_n(l(x)) = d_n(l(x, x)),
\]
\[
= \bigcup_{i+j=n} l(\bigcup(l(d_i(x), d_j(x), l(d_j(x), d_i(x))))),
\]
\[
= \bigcup_{i+j=n} l(\bigcup(l(d_i(x), d_j(x))))
\]
\[
= \bigcup_{i+j=n} l(d_i(x), d_j(x)),
\]
\[
= l(d_0(x), d_n(x)) \cup l(d_1(x), d_{n-1}(x)) \cup ... \cup l(d_n(x), d_0(x)),
\]
\[
= l(x, d_n(x)) \cup l(d_1(x)) \cup l(d_2(x), d_{n-2}(x)) \cup ... \cup l(d_{\frac{n}{2}}(x)),
\]

So $l(d_{\frac{n}{2}}(x)) \subseteq l(d_n(x)) \subseteq l(d_1(x))$ for all $x \in P$. Thus $d_{\frac{n}{2}}(x) \leq d_n(x)$ for all $x \in P$. Hence $d_{\frac{n}{2}}(x) \leq d_n(x)$ for all $x \in P$, and for every even number $n \in J$.

\[\Box\]

**Proposition 3.6** Let $P$ be a poset and $D = \{d_n\}_{n \in J}$ be a higher derivation of length $t$ on $P$. Then

1. $d_n(x) \leq x$ for all $x \in P$; and $n \in J$;
2. $l(d_n(x), d_n(y)) \subseteq d_n(l(x, y))$ for all $x, y \in P$.

**Proof.** (1) It is obvious that $d_0(x) \leq x$ and $d_1(x) \leq x$ for all $x \in P$.

Since $d_1(x) \leq d_2(x)$ and
\[
d_2(l(x)) = d_2(l(x, x))
\]
\[
= l(x, d_2(x)) \cup l(d_1(x)) \cup l(x, d_2(x)) \quad \text{for all} \quad x \in P,
\]
so $l(x, d_2(x)) \subseteq l(d_2(x))$ for all $x \in P$. This means that $d_2(x) \leq x$ for all $x \in P$.

Now assume that $d_n(x) \leq x$ for all $x \in P$ and $n = 3, 4, ..., t - 1$, then by Propositions 3.5, 3.4, we get
\[
d_t(l(x)) = l(d_0(x), d_t(x)) \cup l(d_1(x), d_{t-1}(x)) \cup ... \cup l(d_t(x), d_0(x))
\]
\[
= l(x, d_t(x)) \cup l(d_1(x)) \cup l(d_2(x), d_{t-2}(x)) \cup ... \cup l(x, d_t(x))
\]
\[
= l(x, d_t(x)) \quad \text{for all} \quad x \in P.
\]

Thus $d_t(x) \in d_t(l(x)) = l(x, d_t(x))$ for all $x \in P$. Hence $d_t(x) \leq x$ for all $x \in P$.

(2) By Proposition 3.5 (1), we get $l(d_0(x), d_n(y)) \subseteq d_n(l(x, y))$, for all $x, y \in P$. 

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P. So \( l(x, d_n(y)) \subseteq d_n(l(x, y)) \), but by (1), we have \( d_n(x) \leq x \) for all \( x \in P \). Thus \( l(d_n(x), d_n(y)) \subseteq l(x, d_n(y)) \subseteq d_n(l(x, y)) \) for all \( x, y \in P \).

\[ \square \]

Now, we are in position to prove our first Theorem which generalize Theorem 3.7 of [4].

**Theorem 3.7** Let \( P \) be a poset and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). Then \( d_n^2 = d_n \) for all \( n \in J \).

**Proof.** Assume that \( d_n(1) = 1 \) for all \( n \in J \), and \( x \in P \). Then

\[
d_n(l(x)) = d_n(l(x, 1)), \]

\[
= l(d_0(x), d_n(1)) \cup l(d_1(x), d_{n-1}(1)) \cup \ldots \cup l(d_n(x), d_0(1)), \]

\[
= l(x, 1) \cup l(d_1(x), 1) \cup l(d_2(x), 1) \cup \ldots \cup l(d_n(x), 1), \]

\[
= l(x). \]

Then \( d_n(x) = x \) for all \( x \in P \). The converse is clear.

\[ \square \]

**Theorem 3.8** Let \( P \) be a poset with a greatest element \( 1 \) and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). Then \( d_n(1) = 1 \) for all \( n \in J \), if and only if \( d_n(x) = x \) for all \( x \in P \) and \( n \in J \).

**Proof.** Assume that \( d_n(1) = 1 \) for all \( n \in J \), and \( x \in P \). Then

\[
d_n(l(x)) = d_n(l(x, 1)), \]

\[
= l(d_0(x), d_n(1)) \cup l(d_1(x), d_{n-1}(1)) \cup \ldots \cup l(d_n(x), d_0(1)), \]

\[
= l(x, 0) \cup l(d_1(x), 0) \cup l(d_2(x), 0) \cup \ldots \cup l(d_n(x), 0), \]

\[
= l(0) = \{0\}. \]

Then \( d_n(x) = x \) for all \( x \in P \). The converse is clear.

\[ \square \]

**Theorem 3.9** Let \( P \) be a poset with a least element \( 0 \) and a greatest element \( 1 \) and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). Then \( d_n(1) = 0 \) for all \( n \in J \), if and only if \( d_n(x) = 0 \) for all \( x \in P \) and \( n \in J \).

**Proof.** Assume that \( d_n(1) = 0 \) for all \( n \in J \), and \( x \in P \). Then

\[
d_n(l(x)) = d_n(l(x, 1)), \]

\[
= l(d_0(x), d_n(1)) \cup l(d_1(x), d_{n-1}(1)) \cup \ldots \cup l(d_n(x), d_0(1)), \]

\[
= l(x, 0) \cup l(d_1(x), 0) \cup l(d_2(x), 0) \cup \ldots \cup l(d_n(x), 0), \]

\[
= l(0) = \{0\}. \]

Then \( d_n(x) = 0 \) for all \( x \in P \). The converse is clear.

\[ \square \]

**Theorem 3.10** Let \( P \) be a poset and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). Then \( d_n^2 = d_n \) for all \( n \in J \).
Proof. Since $d^2_n(x) = d_n(d_n(x)) \leq d_n(x) \leq x$ for all $x \in P$, from Proposition 3.6 (1). Also, since $l(d_i(x), d_j(d_n(x))) \leq l(d_i(x), d_n(x)) \leq l(d_n(x))$ for all $x \in P$ and $i + j = n$. Then
\begin{align*}
l(d^2_n(x)) &= l(x, d_n(d_n(x))), \\
&= l(d_0(x), d_n(x)), \\
&= l(d_n(x))
\end{align*}
for all $x \in P$. Hence $d^2_n(x) = d(x)$ for all $x \in P$. 
\hfill \Box

**Theorem 3.11** Let $P$ be a poset with a greatest element $1$ and $D = \{d_n\}_{n \in J}$ be a higher derivation of length $t$ on $P$. If $x \leq d_n(1)$, then $d_n(x) = x$ for all $x \in P$ and $n \in J$.

Proof. Since $d_i(x) \leq x \leq d_j(1)$ for all $x \in P$ and $i + j = n$. Then
\begin{align*}
d_n(l(x)) &= d_n(l(x, 1)), \\
&= l(d_0(x), d_n(1)) \cup l(d_1(x), d_{n-1}(1)) \cup \ldots \cup l(d_n(x), d_0(1)), \\
&= l(x) \cup l(d_1(x)) \cup l(d_2(x)) \cup \ldots \cup l(d_n(x)), \\
&= l(x).
\end{align*}
Hence $d_n(x) = x$ for all $x \in P$.
Let $P$ be a poset and $D = \{d_n\}_{n \in J}$ be a higher derivation of length $t$ on $P$. Let $\text{Fix}_D(P) = \{x \in P : d_n(x) = x, \forall n \in J\}$. If $P$ has a least element $0$, then $0 \in \text{Fix}_D(P)$ by Proposition 3.2. Hence $\text{Fix}_D(P) \neq \emptyset$. 
\hfill \Box

**Proposition 3.12** Let $D = \{d_n\}_{n \in J}, T = \{t_n\}_{n \in J}$ are two higher derivations of length $t$ on $P$. Then $d_n = t_n$ for all $n \in J$ if and only if $\text{Fix}_D(P) = \text{Fix}_T(P)$.

Proof. It is clear that if $d_n = t_n$, for all $n \in J$ then $\text{Fix}_D(P) = \text{Fix}_T(P)$. Conversely, let $\text{Fix}_D(P) = \text{Fix}_T(P)$, and $x \in P$. But by Theorem 3.10 we get $d_n(x) \in \text{Fix}_D(P) = \text{Fix}_T(P)$, so $t_n(d_n(x)) = d_n(x)$. Similarly, we also have $d_n(t_n(x)) = t_n(x)$. So by Propositions 3.3, 3.6, we obtain $d_n(x) = t_n(x)$, for all $n \in J$ and $x \in P$. 
\hfill \Box

**Proposition 3.13** Let $P$ be a poset with a least element $0$ and $D = \{d_n\}_{n \in J}$ be a higher derivation of length $t$ on $P$. Then the following hold.
1. $\text{Fix}_D(P) \neq \emptyset$,
2. If $x \in \text{Fix}_D(P)$, and $y \leq x$ then $y \in \text{Fix}_D(P)$,
3. If $P$ is directed, then, for any $x, y \in \text{Fix}_D(P)$, there exists $z \in \text{Fix}_D(P)$ satisfying $x, y \leq z \leq z$. 

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Proof. (1) Since \( d_n(0) = 0 \) for all \( n \in J \), then \( 0 \in \text{Fix}_D(P) \).

(2) Assume that \( x \in \text{Fix}_D(P) \) and \( y \leq x \), then \( d_n(x) = x \) for all \( n \in J \).

Then by Proposition 3.6 (1), we get

\[
\begin{align*}
\quad d_n(l(y)) &= d_n(l(x, y)) \\
&= l(x, d_n(y)) \cup l(d_1(x), d_{n-1}(y)) \cup \ldots \cup l(d_n(x), y).
\end{align*}
\]

So \( d_n(l(y)) = l(x, y) = l(y) \) for all \( x, y \in P \). Thus \( d_n(y) = y \) for all \( n \in J \), i.e. \( y \in \text{Fix}_D(P) \).

(3) Assume that \( P \) is directed then for any \( x, y \in P \) there exists \( v \in P \) such that \( x \leq v \) and \( y \leq v \). Since \( x, y \in \text{Fix}_D \), then \( d_n(x) = x, d_n(y) = y \). But \( d_n(x) = x \leq d_n(v) \) and \( d_n(y) = y \leq d_n(v) \), for all \( n \in J \). Put \( z = d_n(v) \), hence by Theorem 3.10, we get \( z \in \text{Fix}_D(P) \).

\[ \square \]

**Corollary 3.14** If \( P \) is a directed poset with the least element 0, then \( \text{Fix}_D(P) \) is an ideal of \( P \).

4 The ideals and operations related with higher derivations

Throughout, this section \( P \) denotes a poset with the least element 0.

**Theorem 4.1** Let \( P \) be a poset with a least element 0 and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). Then \( \ker D = \{x \in P : d_n(x) = 0, \forall n \in J\} \) is a nonempty lower set of \( P \).

**Proof.** From Proposition 3.2 we have \( d_n(0) = 0 \), for all \( n \in J \). So \( 0 \in \ker D \), and \( \ker D \neq \phi \). Assume that \( x \in \ker D \) and \( y \leq x \), then \( d_n(x) = 0 \) for all \( n \in J \). By Proposition 3.3 we get \( d_n(y) \leq d_n(x) = 0 \), for all \( n \in J \). Thus \( d_n(y) = 0 \), for all \( n \in J \), and this shows that \( y \in \ker D \).

\[ \square \]

**Proposition 4.2** Let \( P \) be a poset with a least element 0, \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \) and \( I \) be an ideal of \( P \). Then \( d_n^{-1}(I) \) is a nonempty lower set of \( P \) such that \( \ker D \subseteq d_n^{-1}(I) \), for all \( n \in J \).

**Proof.** Since \( d_n(0) = 0 \), for all \( n \in J \) we have \( 0 \in d_n^{-1}(I) \), and then \( d_n^{-1}(I) \neq \phi \), for all \( n \in J \). Suppose \( x \in d_n^{-1}(I) \) and \( y \leq x \), then \( d_n(x) \in I \), for all \( n \in J \).

By Proposition 3.3 \( d_n(y) \leq d_n(x) \in I \), for all \( n \in J \). Hence \( d_n(y) \in I \), and this shows that \( y \in d_n^{-1}(I) \). Hence \( d_n^{-1}(I) \) is a nonempty lower set of \( P \). On the other hand, note that \( \ker D = d_n^{-1}(\{0\}) \subseteq d_n^{-1}(I) \), for all \( n \in J \).

\[ \square \]

**Proposition 4.3** Let \( P \) be a poset and \( D = \{d_n\}_{n \in J} \) be a higher derivation of length \( t \) on \( P \). If \( I, J \) are two ideals of \( P \), then we have:

1. if \( I \subseteq J \), then \( d_n(I) \subseteq d_n(J) \), for all \( n \in J \);
2. \( d_n(I \cap J) = d_n(I) \cap d_n(J) \), for all \( n \in J \).
Proof. (1) Assume that \( n \in J \) and \( x \in d_n(I) \), then there exist \( y \in I \subseteq J \) such that \( x = d_n(y) \). Hence \( x \in d_n(J) \), and this shows that \( d_n(I) \subseteq d_n(J) \).

(2) It is clear that \( d_n(I \cap J) \subseteq d_n(I) \cap d_n(J) \).

Conversely, let \( x \in d_n(I \cap J) \), then there exist \( a \in I, b \in J \) such that \( d_n(a) = x \) and \( d_n(b) = x \). Then by proposition 3.6 (1) and Theorem 3.10, we get

\[
d_n(l(a, d_n(b))) = l(a, d_n(b)) \cup l(d_1(a), d_n(b)) \cup \ldots \cup l(d_n(a), b) = l(x)
\]

But \( x \in l(x) \), so \( x \in d_n(l(a, d_n(b))) \). Thus there exist \( z \in l(a, d_n(b)) \) such that \( d_n(z) = x \). By \( z \leq a \) and \( z \leq d_n(b) \) \( \leq b \) so we have \( z \in I \cap J \). Hence \( x \in d_n(I \cap J) \), and \( d_n(I) \cap d_n(J) \subseteq d_n(I \cap J) \), for all \( n \in J \).

\( \square \)

Theorem 4.4 Let \( P \) be a poset and \( D = \{d_n\}_{n \in J}, T = \{t_n\}_{n \in J} \) be two higher derivations of length \( t \) on \( P \). Then \( d_n \) and \( t_n \) commute for all \( n \in J \).

Proof. Assume that \( D = \{d_n\}_{n \in J}, T = \{t_n\}_{n \in J} \) be two higher derivations of length \( t \) on \( P \). So, for any \( x \in P \), and \( n \in J \)

\[
d_n(l(t_n(x))) = d_n(l(x, t_n(x))),
\]

\[
= l(x, d_n(t_n(x))) \cup l(d_1(x), d_n(t_n(x))) \cup \ldots \cup l(d_n(x), t_n(x)),
\]

\[
= l(x).
\]

And

\[
t_n(l(d_n(x))) = t_n(l(x, d_n(x))),
\]

\[
= l(x, t_n(d_n(x))) \cup l(t_1(x), t_n-1(d_n(x))) \cup \ldots \cup l(t_n(x), d_n(x)),
\]

\[
= l(x).
\]

But \( d_n t_n(x) = d_n(t_n(x)) \in d_n(l(t_n(x))) \). Hence \( d_n t_n(x) \in t_n(l(d_n(x))) \). Then there exists \( y \in l(l(x)) \) such that \( d_n t_n(x) = t_n(y) \). By Proposition 3.3 \( t_n(y) \leq t_n(d_n(x)) \), and therefore \( d_n t_n(x) \leq t_n d_n(x) \). Similarly, we can prove that \( t_n d_n(x) \leq d_n t_n(x) \). This means \( d_n t_n(x) = t_n d_n(x) \).

\( \square \)

Theorem 4.5 Let \( P \) be a poset and \( D = \{d_n\}_{n \in J}, T = \{t_n\}_{n \in J} \) be two higher derivations of length \( t \) on \( P \). Then \( d_n \leq t_n \) for all \( n \in J \) if and only if \( t_n d_n = d_n \), for all \( n \in J \).

Proof. Let \( D = \{d_n\}_{n \in J}, T = \{t_n\}_{n \in J} \) be two higher derivations of length \( t \) on \( P \), with \( d_n \leq t_n \), for all \( n \in J \). So, for any \( x \in P \), we have \( d_n(x) \in Fix_D(P) \) i.e. \( d_n(x) = d_n(d_n(x)) \leq t_n(d_n(x)) \). Also by Proposition 3.7, \( t_n d_n(x) \leq d_n(x) \). Thus \( t_n d_n(x) = d_n(x) \). This shows that \( t_n d_n = d_n \). On the other hand, for any \( x \in P \), and \( n \in J \) we find that \( d_n(x) = t_n d_n(x) \leq t_n(x) \). From Proposition 3.3, we conclude that \( d_n \leq t_n \), for all \( n \in J \).

\( \square \)
Acknowledgements For the third author, this research is supported by the National Board of Higher Mathematics (NBHM), India, Grant No. 02011/16/2020 NBHM (R. P.)

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Received: 02.X.2020 / Revised: 18.XI.2020 / Accepted: 25.XI.2020

AUTHORS

Abdelkarim Boua (Corresponding author),
Polydisciplinary Faculty, LSI, Taza,
Sidi Mohammed Ben Abdellah University,
Fez, Morocco,
E-mail: abdelkarimboua@yahoo.fr

Ahmed Y. Abdelwanis,
Department of Mathematics, Faculty of Science,
Cairo University,
Giza, Egypt,
E-mail: ayunis@sci.cu.edu.eg

Nadeem ur Rehman,
Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002 India,
E-mail: rehman100@gmail.com
and nu.rehman.mm@amu.ac.in