Conformal quasi bi-slant submersions

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Abstract Maps have always been a fascinating topic for geometers that continually generates new ideas. We continue our study [20] on quasi bi-slant submersions by exploring the application of conformal maps between Riemannian manifolds. In present paper, we define conformal quasi bi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds. We study the geometry of leaves of distributions, work out integrability conditions of distributions on these submersions and obtain conditions for such submersions to be totally geodesic. Moreover, we mention some examples of conformal quasi bi-slant submersions.

Keywords Hermitian manifold · conformal submersions · conformal quasi bi-slant submersions

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1 Introduction

In Riemannian geometry there is always a shortage of suitable types of maps between Riemannian manifolds to compare their geometric properties. Isometric immersions and submersions are two fundamental maps. B. Fuglede [10] and T. Ishihara [15] introduced and studied about conformal maps between Riemannian manifolds independently. This notion is useful when we study harmonic morphism. These maps have several important applications in mathematics as well as in physics. Especially, within the Yang-Mills theory [25], Kaluza-Klein theory [13], supergravity and superstring theories [14], redundant robotic chains [6] e.t.c.

In the literature, the Riemannian submersions were independently introduced by O’Neill [21] and Gray [11]. Further, Watson [24] studied almost complex type of Riemannian submersions. In 1985, D. Chinea has extended it to different kinds of sub-classes of almost contact manifolds of almost Hermitian submersion [8]. Several good results can be found in [9] and [12] regarding Riemannian submersions. Different kinds of submersions between
Riemannian manifolds have been studied widely by many authors as semi-slant submersion [18] (see also [22]), conformal semi-slant submersions from Lorentzian para Sasakian manifolds [16], conformal semi-invariant submersions from almost contact metric manifolds [19], hemi-slant submersions [23], quasi bi-slant submersions [20] etc. Akyol and Sahin introduced conformal anti-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds [3], conformal semi-invariant submersions [5], conformal slant submersions [4] and Akyol studied conformal semi-slant submersions [1] in 2017.

In 2020, Kumar et al. introduced and studied conformal hemi-slant submersion from an almost Hermitian manifold onto a Riemannian manifold [17]. In 2021, Akyol [2] introduced and studied conformal generic submersions from almost Hermitian manifolds.

The present article is organized as follows: in section 2, some basic definitions about conformal submersions needed throughout this paper are given. In section 3, conformal quasi bi-slant submersions from an almost Hermitian manifolds onto a Riemannian manifold are defined. In section 4, some results on conformal quasi bi-slant submersions from a Kähler manifold onto a Riemannian manifold are obtained. In section 5, an example for this notion is provided.

2 Preliminaries

Definition 2.1 [7] Let $N_1$ and $N_2$ are two Riemannian manifolds with Riemannian metric $g_1$ and $g_2$, respectively, where $\dim N_1 = m$ and $\dim N_2 = n$. If $\pi$ is a smooth map from $(N_1, g_1)$ onto $(N_2, g_2)$, then $\pi$ is called semi-conformal or horizontally weakly conformal at $p \in N_1$ if either:

(i) $(\pi_*)_p = 0$, or
(ii) $(\pi_*)_p$ maps the horizontal space $\mathcal{H}_p = (\ker(\pi_*)_p)^\perp$ conformally onto $T_{\pi(p)}N_2$ i.e., $(\pi_*)_p$ is surjective and there exists a number $\Lambda(p) \neq 0$ such that

$$g_2(\pi_* V_1, \pi_* V_2) = \Lambda(p) g_1(V_1, V_2),$$

for all $V_1, V_2 \in \mathcal{H}_p$. \hfill (2.1)

A point $p$ of type (i) in Definition 2.1 is called a critical point of $\pi$, we shall call a point of type (ii) a regular point. At a critical point, $(\pi_*)_p$ has rank 0, at a regular point, $(\pi_*)_p$ has rank $n$ and $\pi$ is submersion. The number $\Lambda(p)$ is called the square dilation (of $\pi$ at $p$), it is necessarily non-negative, its square root $\sqrt{\Lambda(p)} = \lambda(p)$ is called the dilation (of $\pi$ at $p$). The map $\pi$ is called horizontally weakly conformal or semi-conformal (on $N_1$) if it is horizontally weakly conformal at every point of $N_1$. If $\pi$ has no critical points, then we call it a (horizontally) conformal submersion.

In [21] the fundamental tensors of a submersion introduced by Watson. O’Neill’s tensors $\mathcal{A}$ and $\mathcal{T}$ for all $E_1, E_2 \in \Gamma(TN_1)$ are defined by

$$\mathcal{A}_{E_1 E_2} = V\nabla_{\mathcal{H}E_1} \mathcal{H}E_2 + \mathcal{H}\nabla_{\mathcal{H}E_1} V E_2,$$

(2.2)
where the vertical and horizontal projections are denoted by $\mathcal{V}$ and $\mathcal{H}$, and the Levi-Civita connection on $N_1$ is denoted by $\nabla$. In addition, from equations (2.2) and (2.3), we get

\begin{align}
\nabla_{V_1} V_2 &= \mathcal{T}_{V_1} V_2 + \mathcal{V} \nabla_{V_1} V_2, \\
\nabla_{V_1} Z_1 &= \mathcal{H} \nabla_{V_1} Z_1 + \mathcal{T}_{V_1} Z_1, \\
\nabla_{Z_1} V_1 &= A_{Z_1} V_1 + \mathcal{V} \nabla_{Z_1} V_1, \\
\nabla_{Z_1} Z_2 &= \mathcal{H} \nabla_{Z_1} Z_2 + A_{Z_1} Z_2,
\end{align}

for all $V_1, V_2 \in \Gamma(\ker \pi)$ and $Z_1, Z_2 \in \Gamma(\ker \pi)^\perp$, where $\mathcal{V} \nabla_{V_1} V_2 = \hat{\nabla}_{V_1} V_2$. If $Z_1$ is basic, then $A_{Z_1} V_1 = \mathcal{H} \nabla_{V_1} Z_1$.

It is seen that for $p \in N_1$, $V_1 \in V_p$, and $Z_1 \in \mathcal{H}_p$ the linear operators

$$A_{Z_1}, \quad \mathcal{T}_{V_1} : T_p N_1 \to T_p N_1,$$

are skew-symmetric, that is

$$g_1(A_{Z_1} W_1, W_2) = -g_1(W_1, A_{Z_1} W_2) \quad \text{and} \quad g_1(\mathcal{T}_{V_1} W_1, W_2) = -g_1(W_1, \mathcal{T}_{V_1} W_2),$$

for each $W_1, W_2 \in \Gamma(T_p N_1)$. When $\pi$ is horizontally conformal submersion we have the following:

**Proposition 2.2** [7] If $\pi$ be a horizontal conformal submersion between Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ with dilation $\lambda$ and $Z_1, Z_2 \in \mathcal{H}_p$, then

$$A_{Z_1} Z_2 = \frac{1}{2} \{ \mathcal{V}[Z_1, Z_2] - \lambda^2 g_1(Z_1, Z_2) \text{grad}(\frac{1}{\lambda^2}) \}.$$  

We know that if $\pi : (N_1, g_1) \to (N_2, g_2)$ is a smooth map then the differential $\pi_*$ of $\pi$ can be observed a section of the bundle $\text{Hom}(T N_1, \pi^{-1} T N_2) \to N_1$, where $\pi^{-1} T N_2$ is the bundle which has fibres $(\pi^{-1} T N_2)_p = T_{\pi(p)} N_2$, has a connection $\nabla$ induced from the Riemannian connection $\nabla^{N_1}$ and the pullback connection. The second fundamental form of $\pi$ is given by

$$\nabla_{\pi_*} (Z_1, Z_2) = \nabla^{N_1}_{Z_1} \pi_*(Z_2) - \pi_*(\nabla^{N_1}_{Z_1} Z_2),$$

for all $Z_1, Z_2 \in \Gamma(T N_1)$, where the pullback connection is denoted by $\nabla^\pi$. A differentiable map $\pi$ between two Riemannian manifolds is said to be totally geodesic if

$$\nabla_{\pi_*} (W_1, W_2) = 0, \quad \text{for all } W_1, W_2 \in \Gamma(T N_1).$$

See [7] for details on totally geodesic maps. Now, we recollection the subsequent Lemma from [7]:
Lemma 2.3 Let \((N_1,g_1)\) and \((N_2,g_2)\) are two Riemannian manifolds. If \(\pi : (N_1,g_1) \to (N_2,g_2)\) horizontally conformal submersion between Riemannian manifolds, then for any horizontal vector fields \(Z_1, Z_2\) and vertical vector fields \(V_1, V_2\), we get

\[
(i)(\nabla_{\pi_*})(Z_1, Z_2) = Z_1(ln\lambda)\pi_*(Z_2) + Z_2(ln\lambda)\pi_*(Z_1) - g_1(Z_1, Z_2)\pi_*(\text{grad}\ln\lambda),
\]

\[
(ii)(\nabla_{\pi_*})(V_1, V_2) = -\pi_*(A^V_{V_1} V_2),
\]

\[
(iii)(\nabla_{\pi_*})(Z_1, V_1) = -\pi_*(\nabla^{N_1}_{Z_1} V_1).
\]

3 Conformal quasi bi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold

In this section, we introduce the notion of a conformal quasi bi-slant submersion from an almost Hermitian manifold onto a Riemannian manifold, investigate the effect of the existence of conformal quasi bi-slant submersion on the source manifold and target manifold. For the definition and detailed information of almost Hermitian, see the book of Yano and Kon [26]. Now, we first present the following notion.

Definition 3.1 Let \((N_1, g_1, J)\) be an almost Hermitian manifold and \((N_2, g_2)\) be a Riemannian manifold. A conformal submersion \(\pi : (N_1, g_1, J) \to (N_2, g_2)\) is said to be a conformal quasi bi-slant submersion or briefly (cqbss) if there exist three mutually orthogonal distributions \(D, D_1\) and \(D_2\) such that

\[
(i) \ker\pi_* = D \oplus D_1 \oplus D_2,
\]

\[
(ii) J(D) = D \text{ i.e., } D \text{ is invariant,}
\]

\[
(iii) J(D_1) \perp D_2 \text{ and } J(D_2) \perp D_1,
\]

\[
(iv) \text{for any non-zero vector field } Z_1 \in (D_1)_p, p \in N_1, \text{ the angle } \theta_1 \text{ between } JZ_1 \text{ and } (D_1)_p \text{ is constant and independent of the choice of point } p \text{ and } Z_1 \text{ in } (D_1)_p.
\]

\[
(v) \text{for any non-zero vector field } Z_2 \in (D_2)_q, q \in N_1, \text{ the angle } \theta_2 \text{ between } JZ_2 \text{ and } (D_2)_q \text{ is constant and independent of the choice of point } q \text{ and } Z_2 \text{ in } (D_2)_q.
\]

These angles \(\theta_1\) and \(\theta_2\) are said to be slant angles of the submersion.

In the section 3, we will denote a conformal quasi bi-slant submersion (cqbss) from an almost Hermitian manifold \((N_1, g_1, J)\) onto a Riemannian manifold \((N_2, g_2)\) by \(\pi\).

If \(\pi\) is a cqbss then, we get

\[
TN_1 = ker\pi_* \oplus (ker\pi_*)^\perp. \tag{3.1}
\]

For all \(Z_1 \in \Gamma(ker\pi_*)\), we have

\[
Z_1 = PZ_1 + QZ_1 + RZ_1, \tag{3.2}
\]

where projection morphisms of \(ker\pi_*\) onto \(D, D_1\) and \(D_2\) are denoted by \(P, Q\) and \(R\) respectively. For all \(V_1 \in \Gamma(ker\pi_*)\), we set
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\[ JV_1 = \phi V_1 + \omega V_1, \quad (3.3) \]

where \( \phi V_1 \in (\Gamma \ker \pi) \) and \( \omega V_1 \in (\Gamma \ker \pi) \perp \).

From equations (3.2) and (3.3), we get

\[ JV_1 = J(PV_1) + J(QV_1) + J(RV_1), \]
\[ = \phi(PV_1) + \omega(PV_1) + \phi(QV_1) + \omega(QV_1) + \phi(RV_1) + \omega(RV_1). \]

From \( JD = D \), we have \( \omega PV_1 = 0 \).

Therefore, we obtain

\[ JV_1 = \phi(PV_1) + \phi QV_1 + \omega QV_1 + \phi RV_1 + \omega RV_1. \quad (3.4) \]

So, we have

\[ J(\ker \pi) = D \oplus (\phi D_1 \oplus \phi D_2) \oplus (\omega D_1 \oplus \omega D_2), \quad (3.5) \]

where orthogonal direct sum is denoted by \( \oplus \).

If \( Z_1 \in \Gamma(D_1) \) and \( Z_2 \in \Gamma(D_2) \) then we have

\[ g_1(Z_1, Z_2) = 0, g_1(JZ_1, Z_2) = g_1(Z_1, JZ_2) = 0, \]
\[ g_1(\phi Z_1, Z_2) = 0, g_1(Z_1, \phi Z_2) = 0. \]

If \( V_1 \in \Gamma(D) \) and \( V_2 \in \Gamma(D_1) \) then, we get \( g_1(\phi \ V_1, V_2) = 0 \), as \( D \) is invariant i.e., \( JV_1 \in \Gamma(D) \).

Similarly, for all \( W_1 \in \Gamma(D) \) and \( W_2 \in \Gamma(D_2) \), we obtain \( g_1(\phi W_1, W_2) = 0 \).

Hence, we get \( g_1(\phi W_1, \phi W_2) = 0, g_1(\omega W_1, \omega W_2) = 0 \), for all \( W_1 \in \Gamma(D_1) \) and \( W_2 \in \Gamma(D_2) \). So, we have \( \phi D_1 \cap \phi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\} \). If \( \theta_2 = \frac{\pi}{2} \), then \( \phi R = 0 \) and \( D_2 \) is anti-invariant, i.e., \( J(D_2) \subseteq (\ker \pi) \perp \). In this case, \( D_2 \) denoted by \( D \perp \).

We also get

\[ J(\ker \pi) = D \oplus \phi D_1 \oplus \omega D_1 \oplus JD \perp. \quad (3.6) \]

Since \( \omega D_1 \subseteq (\ker \pi) \perp, \omega D_2 \subseteq (\ker \pi) \perp \). So we can write

\[ (\ker \pi) \perp = \omega D_1 \oplus \omega D_2 \oplus \mu, \]

where \( \mu \) is orthogonal complement of \( (\omega D_1 \oplus \omega D_2) \) in \( (\ker \pi) \perp \).

In addition, for any \( V_1 \in \Gamma(ker \pi) \perp \), we get

\[ JV_1 = BV_1 + CV_1, \quad (3.7) \]

where \( BV_1 \in \Gamma(ker \pi) \) and \( CV_1 \in \Gamma(\mu) \)

\textbf{Lemma 3.2} If \( \pi \) is a cqbss then, we have
ϕ^2 V_1 + B \omega V_1 = -V_1, \omega \phi V_1 + C \omega V_1 = 0,
\omega B V_2 + C^2 V_2 = -V_2, \phi B V_2 + BC V_2 = 0,
for all \( V_1 \in \Gamma(\ker \pi) \) and \( V_2 \in \Gamma(\ker \pi) \).

Proof. From equations (3.3) and (3.7), we get the Lemma 3.2. □

Lemma 3.3 If \( \pi \) is a cqbss then, we have
(i) \( \phi^2 W_1 = - (\cos^2 \theta_1) W_1 \),
(ii) \( g_1(\phi W_1, \phi W_2) = \cos^2 \theta_1 g_1(W_1, W_2) \),
(iii) \( g_1(\omega W_1, \omega W_2) = \sin^2 \theta_1 g_1(W_1, W_2) \),
for all \( W_1, W_2 \in \Gamma(D_1) \).

Proof. By Lemma 3.2 in [20], the above Lemma holds. □

Lemma 3.4 If \( \pi \) is a cqbss then, we have
(i) \( \phi^2 V_1 = - (\cos^2 \theta_2) V_1 \),
(ii) \( g_1(\phi V_1, \phi V_2) = \cos^2 \theta_2 g_1(V_1, V_2) \),
(iii) \( g_1(\omega V_1, \omega V_2) = \sin^2 \theta_2 g_1(V_1, V_2) \),
for all \( V_1, V_2 \in \Gamma(D_2) \).

4 Conformal quasi bi-slant submersions from a Kähler manifold onto a Riemannian manifold

In this section, we will study conformal quasi bi-slant submersions (cqbss) from a Kähler manifold onto a Riemannian manifold. For the definition and detailed information of Kähler manifolds, see the book of Yano and Kon [26].

From now on we will denote a conformal quasi bi-slant submersion (cqbss) from a Kähler manifold \( (N_1, g_1, J) \) onto a Riemannian manifold \( (N_2, g_2) \) by \( \pi \).

Lemma 4.1 If \( \pi \) is a cqbss then, we have

\[
\nabla_{W_1} \phi W_2 + T_{W_1} \omega W_2 = \phi \nabla_{W_1} W_2 + B T_{W_1} W_2,
\]

\[
T_{W_1} \phi W_2 + H \nabla_{W_1} \omega W_2 = \omega \nabla_{W_1} W_2 + C T_{W_1} W_2,
\]

\[
\nabla_{Z_1} B Z_2 + A_{Z_1} C Z_2 = \phi A_{Z_1} Z_2 + B H \nabla_{Z_1} Z_2,
\]

\[
A_{Z_1} B Z_2 + H \nabla_{Z_1} C Z_2 = \omega A_{Z_1} Z_2 + C H \nabla_{Z_1} Z_2,
\]

\[
\nabla_{W_1} B Z_1 + T_{W_1} C Z_1 = \phi T_{W_1} Z_1 + B H \nabla_{W_1} Z_1,
\]

\[
T_{W_1} B Z_1 + H \nabla_{W_1} C Z_1 = \omega T_{W_1} Z_1 + C H \nabla_{W_1} Z_1,
\]

\[
\nabla_{Z_1} \phi W_1 + A_{Z_1} \omega W_1 = B A_{Z_1} W_1 + \phi \nabla_{Z_1} W_1,
\]

\[
A_{Z_1} \phi W_1 + H \nabla_{Z_1} \omega W_1 = C A_{Z_1} W_1 + \omega \nabla_{Z_1} W_1,
\]

for any \( W_1, W_2 \in \Gamma(\ker \pi) \) and \( Z_1, Z_2 \in \Gamma(\ker \pi) \).
Proof. From equations (2.4), (2.5), (2.6), (2.7), (3.3) and (3.7), we get equations (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8).

\[ (\nabla W, \phi)W_2 = \mathcal{V}\nabla W, \phi W_2 - \phi \mathcal{V}\nabla W, W_2, \]  \( (\nabla W, \omega)W_2 = \mathcal{H}\nabla W, \omega W_2 - \omega \mathcal{V}\nabla W, W_2, \]  \( (\nabla Z_1, C)Z_2 = \mathcal{H}\nabla Z_1, CZ_2 - \mathcal{C}\nabla Z_1, Z_2, \]  \( (\nabla Z_1, B)Z_2 = \mathcal{V}\nabla Z_1, BZ_2 - B\mathcal{H}\nabla Z_1, Z_2, \)

for any \( W \), \( V \), \( Z \) and \( \Pi \), respectively.

The slant distribution \( D_2 \) is integrable if and only if
\[ \pi \ast \phi_1 + \phi_2 = 0, \]
for all \( \Pi \), \( V \), \( Z \) and \( \phi \)

Theorem 4.2 The invariant distribution \( D \) is integrable if and only if
\[ g_1(\mathcal{T}_Z, \omega V_1, JPZ_2) - g_1(\mathcal{T}_Z, \omega V_1, JPZ_1) = g_1(\mathcal{V}\nabla Z_1, JPZ_2 - \mathcal{V}\nabla Z_1, JPZ_1, \phi QV_1 + \phi RV_1), \]
for all \( Z_1, Z_2 \in \Gamma(D) \) and \( V_1 \in \Gamma(D_1 \oplus D_2) \).

Proof. For all \( Z_1, Z_2 \in \Gamma(D) \) and \( V_1 \in \Gamma(D_1 \oplus D_2) \), using equations (2.4), (3.2) and (3.3), we get
\[ g_1([Z_1, Z_2], V_1) = g_1(\nabla Z_1, JZ_2, JV_1) - g_1(\nabla Z_2, JZ_1, JV_1), \]
\[ = g_1(\mathcal{V}\nabla Z_1, JPZ_2 - \mathcal{V}\nabla Z_1, JPZ_1, \phi QV_1 + \phi RV_1), \]

for all \( Z_1, Z_2 \in \Gamma(D) \) and \( V_1 \in \Gamma(D_1 \oplus D_2) \).

Theorem 4.3 The slant distribution \( D_1 \) is integrable if and only if
\[ \frac{1}{\lambda^2}g_2((\nabla \pi_v)(V_1, \omega V_2) - (\nabla \pi_v)(V_2, \omega V_1), \pi_v(\omega RZ_1)) \]
\[ = g_1(\mathcal{T}_{v_1}, \omega V_2 - \mathcal{T}_{v_2}, \omega V_1, JPZ_1 + \phi RZ_1) - g_1(\mathcal{T}_{v_1}, \omega \phi V_2 - \mathcal{T}_{v_2}, \omega \phi V_1, Z_1), \]
for all \( V_1, V_2 \in \Gamma(D_1) \) and \( Z_1 \in \Gamma(D \oplus D_2) \).
Proof. For all \( V_1, V_2 \in \Gamma(D_1) \) and \( Z_1 \in \Gamma(D \oplus D_2) \), we have 
\[
g_1([V_1, V_2], Z_1) = g_1(\nabla_{V_1} V_2, Z_1) - g_1(\nabla_{V_2} V_1, Z_1).
\]
Using equations (2.4), (2.5), (3.2), (3.3) and the Lemma 3.3(i), we have 
\[
g_1([V_1, V_2], Z_1)
= g_1(\nabla_{V_1} J V_2, J Z_1) - g_1(\nabla_{V_2} J V_1, J Z_1),
= g_1(\nabla_{V_1} \phi V_2, J Z_1) + g_1(\nabla_{V_2} \omega V_2, J Z_1) - g_1(\nabla_{V_2} \omega V_1, J Z_1) - g_1(\nabla_{V_1} \phi V_1, J Z_1),
= \cos^2 \theta_1 g_1(\nabla_{V_1} V_2, Z_1) - \cos^2 \theta_1 g_1(\nabla_{V_2} V_1, Z_1) - g_1(T_{V_1} \omega \phi V_2 - T_{V_2} \omega \phi V_1, Z_1) \nonumber
+ g_1(H \nabla_{V_1} \omega V_2 + T_{V_1} \omega V_2, J P Z_1 + \phi R Z_1 + \omega R Z_1)
- g_1(H \nabla_{V_2} \omega V_1 + T_{V_2} \omega V_1, J P Z_1 + \phi R Z_1 + \omega R Z_1).
\]

Now, we have
\[
sin^2 \theta_1 g_1([V_1, V_2], Z_1)
= g_1(T_{V_1} \omega V_2 - T_{V_2} \omega V_1, J P Z_1 + \phi R Z_1) + g_1(H \nabla_{V_1} \omega V_2 - H \nabla_{V_2} \omega V_1, \omega R Z_1) - g_1(T_{V_1} \omega \phi V_2 - T_{V_2} \omega \phi V_1, Z_1).
\]

From \( \pi \) is conformal, the equation (2.10) and the Lemma 2.3, we get
\[
sin^2 \theta_1 g_1([V_1, V_2], Z_1)
= g_1(T_{V_1} \omega V_2 - T_{V_2} \omega V_1, J P Z_1 + \phi R Z_1) - \frac{1}{\lambda^2} g_2((\nabla_{\pi_\tau})(V_1, \omega V_2) - (\nabla_{\pi_\tau})(Z_1, \omega Z_2) - (\nabla_{\pi_\tau})(Z_2, \omega Z_1), \pi_\tau(\omega Q W_1))
= g_1(T_{Z_1} \omega Z_2 - T_{Z_2} \omega Z_1, J P W_1 + \phi Q W_1) - g_1(T_{Z_1} \omega \phi Z_2 - T_{Z_2} \omega \phi Z_1, W_1),
\]
for all \( Z_1, Z_2 \in \Gamma(D_2) \) and \( W_1 \in \Gamma(D \oplus D_2) \).

\[\square\]

In a similar way, we obtain the following Theorem:

**Theorem 4.4** The slant distribution \( D_2 \) is integrable if and only if
\[
\frac{1}{\lambda^2} g_2((\nabla_{\pi_\tau})(Z_1, \omega Z_2) - (\nabla_{\pi_\tau})(Z_2, \omega Z_1), \pi_\tau(\omega Q W_1)) = g_1(T_{Z_1} \omega Z_2 - T_{Z_2} \omega Z_1, J P W_1 + \phi Q W_1) - g_1(T_{Z_1} \omega \phi Z_2 - T_{Z_2} \omega \phi Z_1, W_1),
\]
for all \( Z_1, Z_2 \in \Gamma(D_2) \) and \( W_1 \in \Gamma(D \oplus D_2) \).

**Theorem 4.5** The horizontal distribution \( \pi \) defines a totally geodesic foliation on \( N_1 \) if and only if
\[
\frac{1}{\lambda^2} \{ g_2(\nabla_{V_1} \pi_\tau(V_2), \pi_\tau(\omega \phi PU_1 + \omega \phi QU_1 + \omega R U_1)) - g_2(\nabla_{V_1} \pi_\tau(V_2), \pi_\tau(\omega U_1)) \}
= g_1(A_{V_1} V_2, P U_1 + \cos^2 \theta_2 Q U_1 + \cos^2 \theta_2 R U_1) + g_1(A_{V_1} B V_2, \omega U_1) + g_1(V_1, \text{grad} \lambda) g_1(V_2, \omega \phi PU_1 + \omega \phi QU_1 + \omega R U_1) + g_1(V_2, \text{grad} \lambda) g_1(V_1, \omega \phi PU_1 + \omega \phi QU_1 + \omega R U_1) - g_1(V_1, V_2) g_1(\text{grad} \lambda, \omega \phi PU_1 + \omega \phi QU_1 + \omega R U_1) - g_1(C V_2, \text{grad} \lambda) g_1(V_1, \omega U_1) + g_1(V_1, C V_2) g_1(\text{grad} \lambda, \omega U_1),
\]
for all \( V_1, V_2 \in \Gamma(\ker \pi_\tau) \) and \( U_1 \in \Gamma(\ker \pi_\tau) \).
Proof. For all \( V_1, V_2 \in \Gamma(\ker\pi^*) \) and \( U_1 \in \Gamma(\ker\pi^*) \), we have
\[
g_1(\nabla V_1 V_2, U_1) = g_1(\nabla V_1 V_2, P U_1 + Q U_1 + R U_1).
\]

Using equations (2.6), (2.7), (3.2), (3.3) and the Lemmas 3.3(i) and 3.4(i), we have
\[
g_1(\nabla V_1 V_2, U_1) = g_1(\nabla V_1 J V_2, J P U_1) + g_1(\nabla V_1 J V_2, J Q U_1) + g_1(\nabla V_1 J V_2, J R U_1),
\]
\[
= g_1(\mathcal{A}_V V_2, P U_1 + \cos^2 \theta_1 Q U_1 + \cos^2 \theta_2 R U_1)
\]
\[
- g_1(H \nabla V_1 V_2, \omega \phi P U_1 + \omega \phi Q U_1 + \omega \phi R U_1)
\]
\[
+ g_1(\mathcal{A}_V V_2, B V_1 + H \nabla V_1 CV_2, \omega PU_1 + \omega QU_1 + \omega RU_1).
\]

Now, since \( \omega PU_1 + \omega QU_1 + \omega RU_1 = U_1 \) and \( \omega PU_1 = 0 \), one obtains
\[
g_1(\nabla V_1 V_2, U_1) = g_1(\mathcal{A}_V V_2, P U_1 + \cos^2 \theta_1 Q U_1 + \cos^2 \theta_2 R U_1)
\]
\[
- g_1(H \nabla V_1 V_2, \omega \phi P U_1 + \omega \phi Q U_1 + \omega \phi R U_1)
\]
\[
+ g_1(\mathcal{A}_V V_2, B V_1 + H \nabla V_1 CV_2, U_1).
\]

Since \( \pi \) is conformal, the equation (2.10) and the Lemma 2.3, we get
\[
g_1(\nabla V_1 V_2, U_1)
\]
\[
= g_1(\mathcal{A}_V V_2, P U_1 + \cos^2 \theta_1 Q U_1 + \cos^2 \theta_2 R U_1) + g_1(\mathcal{A}_V V_2, B U_1) +
\]
\[
\frac{1}{\lambda^2} g_2(\mathcal{V}_1, \mathcal{V}_2, \pi_*(\omega U_1)) -
\]
\[
\frac{1}{\lambda^2} g_2(\mathcal{V}_1, \pi_*(\omega \phi P U_1 + \omega \phi Q U_1 + \omega \phi R U_1)) +
\]
\[
g_1(\mathcal{V}_1, \mathcal{V}_1) g_1(\mathcal{V}_1, \mathcal{V}_1) + g_1(\mathcal{V}_1, \mathcal{V}_1) g_1(\mathcal{V}_1, \mathcal{V}_1) -
\]
\[
g_1(\mathcal{V}_1, \mathcal{V}_1) g_1(\mathcal{V}_1, \mathcal{V}_1) + g_1(\mathcal{V}_1, \mathcal{V}_1) g_1(\mathcal{V}_1, \mathcal{V}_1),
\]
which completes the proof.

\( \square \)

**Theorem 4.6** The vertical distribution \((\ker\pi_*)\) defines a totally geodesic foliation on \( N_1 \) if and only if
\[
\frac{1}{\lambda^2} \{ g_2(\nabla \omega Z_2, \pi_*(\phi Z_1), \pi_*(J CV_1)) - g_2((\nabla \pi_*)(\omega Z_2, \phi Z_1), \pi_*(J CV_1)) +
\]
\[
g_2((\nabla \pi_*)(Z_1, \omega \phi P Z_2) + (\nabla \pi_*)(Z_1, \omega \phi Q Z_2) + (\nabla \pi_*)(Z_1, \omega \phi R Z_2), \pi_*(V_1)) \}
\]
\[
= g_1(T_2, P Z_2, V_1) + \cos^2 \theta_1 g_1(T_2, Q Z_2, V_1) + \cos^2 \theta_2 g_1(T_2, R Z_2, V_1) +
\]
\[
g_1(T_2, \omega Z_2, B V_1) - g_1(\omega Z_1, \omega Z_2) g_2(\mathcal{V}_1, \mathcal{V}_1),
\]
for all \( Z_1, Z_2 \in \Gamma(\ker\pi_*) \) and \( V_1 \in \Gamma(\ker\pi_*)^\perp \).
Proof. For all $Z_1, Z_2 \in \Gamma(\ker \pi)$ and $V_1 \in \Gamma(\ker \pi^\perp)$, using equation (3.2), we get

$$g_1(\nabla Z_1 Z_2, V_1) = g_1(\nabla Z_1 JPZ_2, JV_1) + g_1(\nabla Z_1 JQZ_2, JV_1) + g_1(\nabla Z_1 JRZ_2, JV_1).$$

Now, using equations (2.4), (2.5), (3.3) and Lemmas 3.3(i) and 3.4(i), we have

$$g_1(\nabla Z_1 Z_2, V_1) = g_1(T_{Z_1} PZ_2, V_1) + \cos^2 \theta_1 g_1(T_{Z_1} QZ_2, V_1) + \cos^2 \theta_2 g_1(T_{Z_1} RZ_2, V_1) - g_1(H\nabla Z_1, \omega PZ_2 + H\nabla Z_1, \phi QZ_2 + H\nabla Z_1, \phi RZ_2, V_1) + g_1(\nabla Z_1, \omega PZ_2 + \nabla Z_1, \phi QZ_2 + \nabla Z_1, \phi RZ_2, JV_1).$$

Now, since $\omega PZ_2 + \omega QZ_2 + \omega RZ_2 = \omega Z_2$ and $\omega PZ_2 = 0$, we have

$$g_1(\nabla Z_1 Z_2, V_1) = g_1(T_{Z_1} PZ_2, V_1) + \cos^2 \theta_1 g_1(T_{Z_1} QZ_2, V_1) + \cos^2 \theta_2 g_1(T_{Z_1} RZ_2, V_1) - g_1(H\nabla Z_1, \omega PZ_2 + H\nabla Z_1, \omega QZ_2 + \nabla Z_1, \omega \phi RZ_2, V_1) + g_1(\nabla Z_1, \omega Z_2, BV_1) + g_1(\nabla Z_1, \omega Z_2, CV_1).$$

From $\pi$ is conformal, the equation (2.10) and the Lemma 2.3, we get

$$g_1(\nabla Z_1 Z_2, V_1) = g_1(T_{Z_1} PZ_2, V_1) + \cos^2 \theta_1 g_1(T_{Z_1} QZ_2, V_1) + \cos^2 \theta_2 g_1(T_{Z_1} RZ_2, V_1) + g_1(\nabla Z_1, \omega Z_2, BV_1) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, \omega PZ_2) + (\nabla \pi_*) (Z_1, \phi QZ_2) + (\nabla \pi_*) (Z_1, \phi RZ_2, V_1)) + \frac{1}{\lambda^2} g_2((\nabla \pi_*)(\omega Z_2, \phi Z_1), \pi_*(JCV_1)) - g_1(\omega Z_1, \omega Z_2) g_1(\text{grad} ln \lambda, JCV_1),$$

which completes the proof. $\square$

From Theorems 4.5 and 4.6, we also have the following decomposition result.

**Theorem 4.7** The total space is locally product manifold of the form $N_{1/(\ker \pi^\perp)} \times N_{1/(\ker \pi^\perp)^\perp}$, where $N_{1/(\ker \pi^\perp)}$ and $N_{1/(\ker \pi^\perp)^\perp}$ are leaves of $\ker \pi$ and $(\ker \pi)^\perp$, respectively, if and only if

$$\frac{1}{\lambda^2} \left\{ g_2(\nabla \omega Z_2, \pi_*(\phi Z_1), \pi_*(JCV_1)) - g_2((\nabla \pi_*)(\omega Z_2, \phi Z_1), \pi_*(JCV_1)) + g_2((\nabla \pi_*)(Z_1, \omega PZ_2) + (\nabla \pi_*) (Z_1, \omega \phi QZ_2) + (\nabla \pi_*) (Z_1, \omega \phi RZ_2, V_1)) \right\} = g_1(T_{Z_1} PZ_2, V_1) + \cos^2 \theta_1 g_1(T_{Z_1} QZ_2, V_1) + \cos^2 \theta_2 g_1(T_{Z_1} RZ_2, V_1) + g_1(T_{Z_1} \omega Z_2, BV_1) - g_1(\omega Z_1, \omega Z_2) g_2(\text{grad} ln \lambda, JCV_1),$$

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and
\[
\frac{1}{\lambda^2} \{ g_2(\nabla_{V_1} \pi_*(V_2), \pi_*(\omega \phi PZ_1 + \omega \phi QZ_1 + \omega \phi RZ_1)) - g_2(\nabla_{V_1} \pi_*(V_2), \pi_*(\omega Z_1)) \}
\]
\[
= g_1(A_{V_1} V_2, PZ_1 + \cos^2 \theta_1 QZ_1 + \cos^2 \theta_2 RZ_1) + g_1(A_{V_1} B V_2, \omega Z_1) +
\]
\[
g_1(V_1, \text{grad} \ln \lambda) g_1(V_2, \omega \phi PZ_1 + \omega \phi QZ_1 + \omega \phi RZ_1) +
\]
\[
g_1(V_2, \text{grad} \ln \lambda) g_1(V_1, \omega \phi PZ_1 + \omega \phi QZ_1 + \omega \phi RZ_1) -
\]
\[
g_1(V_1, V_2) g_1(\text{grad} \ln \lambda, \omega \phi PZ_1 + \omega \phi QZ_1 + \omega \phi RZ_1) -
\]
\[
g_1(C V_2, \text{grad} \ln \lambda) g_1(V_1, \omega Z_1) + g_1(V_1, C V_2) g_1(\text{grad} \ln \lambda, \omega Z_1),
\]
for all \(Z_1, Z_2 \in \Gamma(\ker \pi_*)\) and \(V_1, V_2 \in \Gamma(\ker \pi_*)^\perp\).

**Theorem 4.8** The invariant distribution \(D\) defines a totally geodesic foliation on \(N_1\) if and only if
\[
\frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, JPZ_2), \pi_*(\omega X_1)) = g_1(V \nabla_{Z_1} \phi PZ_2, \phi QX_1 + \phi RX_1),
\]
(4.13)
and
\[
g_1(V \nabla_{Z_1} \phi PZ_2, BX_2) = \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, \phi PZ_2), \pi_*(CX_2)),
\]
(4.14)
for all \(Z_1, Z_2 \in \Gamma(D), X_1 \in \Gamma(D_1 \oplus D_2)\) and \(X_2 \in \Gamma(\ker \pi_*)^\perp\).

**Proof.** For all \(Z_1, Z_2 \in \Gamma(D), X_1 \in \Gamma(D_1 \oplus D_2)\) and \(X_2 \in \Gamma(\ker \pi_*)^\perp\), using equations (2.4), (2.10), (3.2) and (3.3), we have
\[
g_1(V \nabla_{Z_1} Z_2, X_1)
\]
\[
= g_1(V \nabla_{Z_1} J Z_2, J X_1),
\]
\[
= g_1(V \nabla_{Z_1} J PZ_2, J QX_1 + J RX_1),
\]
\[
= g_1(T_{Z_1} \phi PZ_2, \omega X_1) + g_1(V \nabla_{Z_1} \phi PZ_2, \phi QX_1 + \phi RX_1),
\]
\[
= -\frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, JPZ_2), \pi_*(\omega X_1)) + g_1(V \nabla_{Z_1} \phi PZ_2, \phi QX_1 + \phi RX_1).
\]
Now, again using equations (2.4), (2.10)(3.2), (3.3), (3.7) and Lemma 2.3, we have
\[
g_1(V \nabla_{Z_1} Z_2, X_2)
\]
\[
= g_1(V \nabla_{Z_1} J Z_2, J X_2),
\]
\[
= g_1(V \nabla_{Z_1} \phi PZ_2, BX_2 + CX_2),
\]
\[
= g_1(V \nabla_{Z_1} \phi PZ_2, BX_2) + g_1(T_{Z_1} \phi PZ_2, CX_2),
\]
\[
= g_1(V \nabla_{Z_1} \phi PZ_2, BX_2) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(Z_1, \phi PZ_2), \pi_*(CX_2)),
\]
which completes the proof. \(\square\)
Theorem 4.9 The slant distribution $D_1$ defines a totally geodesic foliation on $N_1$ if and only if

\[
\frac{1}{\lambda^2} g_2((\nabla \pi_*)(V_1, \omega Q V_2), \pi_*(\omega R Z_1)) = -g_1(\mathcal{T}_{V_1} \omega \phi V_2, Z_1) + g_1(\mathcal{T}_{V_1} \omega Q V_2, JP Z_1 + \phi R Z_1),
\]

(4.15)

and

\[
\frac{1}{\lambda^2} \{g_2((\nabla \pi_*)(V_1, \omega V_2), \pi_*(C Z_2)) - g_2((\nabla \pi_*)(V_1, \omega \phi V_2), \pi_*(Z_2))\} = g_1(\mathcal{T}_{V_1} \omega V_2, B Z_2),
\]

(4.16)

for all $V_1, V_2 \in \Gamma(D_1), Z_1 \in \Gamma(D \oplus D_2)$ and $Z_2 \in \Gamma(\ker \pi_*)$.

Proof. For all $V_1, V_2 \in \Gamma(D_1), Z_1 \in \Gamma(D \oplus D_2)$ and $Z_2 \in \Gamma(\ker \pi_*)$, from equations (2.5), (3.2), (3.3) and the Lemma 3.3(i), we have

\[
g_1(\mathcal{N} V_1, V_2, Z_1) = g_1(\nabla \phi V_2, J Z_1) + g_1(\nabla \omega V_2, J Z_1),
\]

\[
= \cos^2 \theta \theta g_1(\nabla \phi V_2, Z_1) - g_1(\mathcal{T}_{V_1} \omega \phi V_2, Z_1)
\]

\[
+ g_1(\mathcal{T}_{V_1} \omega Q V_2, JP Z_1 + \phi R Z_1) + g_1(\mathcal{H} \mathcal{V}_1 \omega Q V_2, \omega R Z_1).
\]

We know that $\pi$ is conformal and from the equation (2.10) and the Lemma 2.3, we get

\[
\sin^2 \theta g_1(\nabla \phi V_2, Z_1)
\]

\[
= -g_1(\mathcal{T}_{V_1} \omega \phi V_2, Z_1) + g_1(\mathcal{T}_{V_1} \omega Q V_2, JP Z_1 + \phi R Z_1)
\]

\[
+ g_1(\mathcal{H} \mathcal{V}_1 \omega Q V_2, \omega R Z_1),
\]

\[
= -g_1(\mathcal{T}_{V_1} \omega \phi V_2, Z_1) + g_1(\mathcal{T}_{V_1} \omega Q V_2, JP Z_1 + \phi R Z_1) -
\]

\[
\frac{1}{\lambda^2} g_2((\nabla \pi_*)(V_1, \omega V_2), \pi_*(\omega R Z_1)).
\]

Again, from equations (2.5), (3.2), (3.3) and the Lemma 3.3(i), we have

\[
g_1(\nabla V_1, V_2, Z_2) = g_1(\nabla \phi V_2, J Z_2) + g_1(\nabla \omega V_2, J Z_2),
\]

\[
= \cos^2 \theta g_1(\nabla \phi V_2, Z_2) - g_1(\mathcal{H} \mathcal{V}_1 \omega \phi V_2, Z_2)
\]

\[
+ g_1(\mathcal{H} \mathcal{V}_1 \omega V_2, C Z_2) + g_1(\mathcal{T}_{V_1} \omega V_2, B Z_2).
\]

We know that $\pi$ is conformal and from the equation (2.10) and the Lemma 2.3, we get

\[
\sin^2 \theta g_1(\nabla \phi V_2, Z_2)
\]

\[
= -g_1(\mathcal{H} \mathcal{V}_1 \omega \phi V_2, Z_2) + g_1(\mathcal{H} \mathcal{V}_1 \omega V_2, C Z_2) + g_1(\mathcal{T}_{V_1} \omega V_2, B Z_2)
\]

\[
= \frac{1}{\lambda^2} g_2((\nabla \pi_*)(V_1, \omega \phi V_2), \pi_*(Z_2)) - \frac{1}{\lambda^2} g_2((\nabla \pi_*)(V_1, \omega V_2), \pi_*(C Z_2)) +
\]

\[
g_1(\mathcal{T}_{V_1} \omega V_2, B Z_2),
\]

which completes the proof. 

\[\square\]
Theorem 4.10 The slant distribution \( D_2 \) defines a totally geodesic foliation on \( N_1 \) if and only if

\[
\frac{1}{\lambda^2} g_2((\nabla \pi_*)(U_1, \omega QU_2), \pi_*(\omega RW_1)) = -g_1(\mathcal{T}_{U_1} \omega \phi U_2, W_1) + g_1(\mathcal{T}_{U_1} \omega QU_2, J P W_1 + \phi RW_1),
\]

for all \( U_1, U_2 \in \Gamma(\ker \pi^*) \) and \( V_1, V_2 \in \Gamma(ker \pi^*) \).

Proof. The proof of the above Theorem is similar to the proof of Theorem 4.9. So we omit it. \( \square \)

From Theorems 4.8, 4.9 and 4.10, we have the following theorem for fiber separation.

Theorem 4.11 The fibres of \( \pi \) are local product Riemannian manifold of the form \( (N_1)_D \times (N_1)_D \times (N_1)_D \), where \( (N_1)_D, (N_1)_D, (N_1)_D \) are leaves of \( D, D_1 \) and \( D_2 \), respectively if and only if the conditions (4.13), (4.14), (4.15), (4.16) and (4.17) hold.

Theorem 4.12 \( \pi \) is a totally geodesic if and only if

\[
\frac{1}{\lambda^2} \{ g_2(\nabla \pi_*)(V_1, R \phi QV_2, \pi_*(U_1)) - g_2(\nabla \pi_*)(V_1, \omega \Phi QV_2, \pi_*(U_1)) + g_2(\nabla \pi_*)(J P V_2, \omega V_1), \pi_*(J C V_1)) - g_2(\nabla \pi_*)(J C V_1, \pi_*(J C V_1)) \}
\]

\[
= -g_1(\mathcal{V} \nabla V_1, JP V_2 + \mathcal{T}_{V_1} \omega QV_2 + \mathcal{T}_{V_1} \omega RV_2, U_1) - g_1(\cos \theta_1 \mathcal{V} \nabla V_1, QV_2 + \cos \theta_2 \mathcal{V} \nabla V_1, RV_2, U_1) +
\]

\[
g_1(\mathcal{T}_{JP V_2} \phi V_1 + A_{\omega QV_2} \phi V_1 + A_{\omega R V_2} \phi V_1, J C V_1) -
\]

\[
g_1(\omega RV_2, \omega V_1) g_1(\nabla \ln \lambda, J C V_1) + g_1(\omega RV_2, \omega V_1) g_1(\nabla \ln \lambda, J C V_1),
\]

and

\[
\frac{1}{\lambda^2} \{ g_2(\nabla \pi_*)(U_1, \omega \Phi QV_1, \pi_*(U_2)) + g_2(\nabla \pi_*)(U_1, \omega \Phi RV_1, \pi_*(U_2)) -
\]

\[
g_2(\nabla \pi_*)(U_1, J P V_1), \pi_*(C V_2)) - g_2(\nabla \pi_*)(U_1, \omega QV_1, \pi_*(C V_2)) +
\]

\[
g_2(\nabla \pi_*)(U_1, \omega RV_1, \pi_*(C V_2)) \}
\]

\[
= -g_1(\mathcal{V} \nabla U_1, JP V_1 + A_{U_1} \omega QV_1 + A_{U_1} \omega RV_1, B U_2) -
\]

\[
g_1(\cos \theta_1 \mathcal{V} \nabla U_1, QV_1 + \cos \theta_2 \mathcal{V} \nabla U_1, RV_1, U_2) +
\]

\[
g_1(\omega RV_1, \nabla \ln \lambda) g_1(U_1, C V_2) + g_1(\omega RV_1, \nabla \ln \lambda) g_1(U_1, C V_2) -
\]

\[
g_1(U_1, \omega QV_1) g_1(\nabla \ln \lambda, C V_2) - g_1(U_1, \omega RV_1) g_1(\nabla \ln \lambda, C V_2),
\]

for all \( V_1, V_2 \in \Gamma(ker \pi_*) \) and \( U_1, U_2 \in \Gamma(ker \pi_*)^\perp \).
Proof. Since $\pi$ is a Riemannian submersion, we get
\[ (\nabla \pi_*) (U_1, U_2) = 0, \]
for all $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

For all $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$, using equations (2.4), (2.5), (2.10), (3.2), (3.3) and Lemmas 3(i) and 4(i), we have
\[
g_1((\nabla \pi_*) (V_1, V_2), \pi_* U_1) \\
= -g_1(\nabla V_1, V_2, U_1) \\
= -g_1(\nabla V_1, J PV_1, JU_1) \\
= -g_1(\nabla V_1, J PV_1, JU_1) - g_1(\nabla V_1, J RV_1, JU_1), \\
= -g_1(\nabla V_1, J PV_1, JU_1) - g_1(\nabla V_1, \phi QV_1, JU_1) - g_1(\nabla V_1, \phi RV_1, JU_1) \\
- g_1(\nabla V_1, \omega QV_1, JU_1) - g_1(\nabla V_1, \omega RV_1, JU_1),
\]
and
\[
g_2((\nabla \pi_*) (V_1, V_2), \pi_* U_1) \\
= -g_1(\nabla V_1, J PV_1 + T V_1, \omega QV_2 + T V_1, \omega RV_2, U_1) \\
- g_1(T V_1, J PV_1 + H \nabla V_1, \omega QV_2 + H \nabla V_1, \omega RV_2, CU_1) \\
- g_1(\cos^2 \theta_1 V \nabla V_1, QV_2 + \cos^2 \theta_2 V \nabla V_1, RV_2 - H \nabla V_1, \omega \phi QV_2 - H \nabla V_1, \omega \phi RV_2, U_1).
\]

Since the equation (2.10) and the Lemma 2.3, we get
\[
g_2((\nabla \pi_*) (V_1, V_2), \pi_* U_1) \\
= -g_1(\nabla V_1, J PV_1 + T V_1, \omega QV_2 + T V_1, \omega RV_2, U_1) - g_1(\cos^2 \theta_1 V \nabla V_1, QV_2 + \cos^2 \theta_2 V \nabla V_1, RV_2, U_1) - \frac{1}{\lambda^2} g_2((\nabla \pi_*) (V_1, R \phi QV_2), \pi_* (U_1)) + g_1(\nabla V_1, J PV_1 + A_{\omega QV_2} \phi V_1 + A_{\omega RV_2} \phi V_1, JCU_1) \\
- \frac{1}{\lambda^2} g_2((\nabla \pi_*) (J PV_1, \omega V_1), \pi_* (JCU_1)) + \frac{1}{\lambda^2} g_2((\nabla \omega QV_2, \pi_* (V_1), \pi_* (JCU_1)) + \frac{1}{\lambda^2} g_2((\nabla \omega RV_2, \pi_* (V_1), \pi_* (JCU_1)) + g_1(\omega QV_2, \omega V_1) g_1(\nabla \omega, JCU_1) + g_1(\omega RV_2, \omega V_1) g_1(\nabla \omega, JCU_1).
\]

Next, using equations (2.4), (2.5), (2.10), (3.2), (3.3) and Lemmas 2.3 and 3.4(i), we have
\[
g_2((\nabla \pi_*) (U_1, V_1), \pi_* U_2) \\
= -g_1(\nabla U_1, V_1, U_2) \\
= -g_1(\nabla U_1, J PV_1, JU_2) \\
= -g_1(\nabla U_1, J PV_1, JU_2) - g_1(\nabla U_1, J QV_1, JU_2) - g_1(\nabla U_1, J RV_1, JU_2), \\
= -g_1(\nabla U_1, J PV_1, JU_2) - g_1(\nabla U_1, \phi QV_1, JU_2) - g_1(\nabla U_1, \phi RV_1, JU_2) \\
- g_1(\nabla U_1, \omega QV_1, JU_2) - g_1(\nabla U_1, \omega RV_1, JU_2),
\]

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\[ g_2((\nabla \pi^*)(U_1, V_1), \pi^* U_2) = -g_1(\nabla U_1, JP V_1 + A_1, \omega Q V_1 + A_1, \omega R V_1, B U_2) - g_1(\cos^2 \theta_1 \nabla_U_1, Q V_1 + \cos^2 \theta_2 \nabla_U_1, R V_1 - H \nabla_U_1, \omega Q V_1 - H \nabla_U_1, \omega R V_1, U_2) + g_1(U_1, \omega Q V_1 + \omega R V_1, U_2). \]

Since \( \pi \) is conformal, the equation (2.10) and the Lemma 2.3, we get
\[ g_2((\nabla \pi^*)(U_1, V_1), \pi^* U_2) = -g_1(\nabla U_1, JP V_1 + A_1, \omega Q V_1 + A_1, \omega R V_1, B U_2) - \]
\[ - \frac{1}{\lambda^2} g_2(\nabla \pi^*(U_1, \omega Q V_1), \pi^*(U_2)) - \frac{1}{\lambda^2} g_2(\nabla \pi^*(U_1, \omega R V_1), \pi^*(U_2)) + \]
\[ \frac{1}{\lambda^2} g_2(\nabla \pi^*(U_1, JP V_1), \pi^*(C U_2)) - \frac{1}{\lambda^2} g_2(\nabla_U_1, \pi^*(\omega Q V_1), \pi^*(C U_2)) - \]
\[ \frac{1}{\lambda^2} g_2(\nabla_U_1, \pi^*(\omega R V_1), \pi^*(C U_2)) + g_1(\omega Q V_1, grad\ln \lambda) g_1(U_1, C U_2) + \]
\[ g_1(\omega R V_1, grad\ln \lambda) g_1(U_1, C U_2) - g_1(U_1, \omega Q V_1) g_1(\omega R V_1, g_1(\omega Q V_1, C U_2) - \]
\[ g_1(U_1, \omega Q V_1) g_1(\omega R V_1, g_1(\omega Q V_1, C U_2). \]

Hence the proof is completed. \( \square \)

5 Example

Note that given an Euclidean space \( \mathbb{R}^{2s} \) with coordinates \((x_1, x_2, ..., x_{2s-1}, x_{2s})\)
we can canonically choose an almost complex structure \( J \) on \( \mathbb{R}^{2s} \) as follows:
\[ J(a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + ... + a_{2s-1} \frac{\partial}{\partial x_{2s-1}} + a_{2s} \frac{\partial}{\partial x_{2s}}) \]
\[ = -a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + ... - a_{2s-1} \frac{\partial}{\partial x_{2s-1}} + a_{2s} \frac{\partial}{\partial x_{2s}}, \]
where \( a_1, a_2, ..., a_{2s} \) are smooth functions defined on \( \mathbb{R}^{2s} \). We will use this notation throughout this section.

Example 5.1 Define a map \( \pi : \mathbb{R}^{10} \to \mathbb{R}^4 \) by \( \pi(x_1, x_2, ..., x_{10}) = e^5(x_1 \sin \alpha - x_3 \cos \alpha, x_4, x_6 \sin \beta + x_8 \cos \beta, x_7) \), which is a conformal quasi bi-slant submersion such that
\[ X_1 = \cos \alpha \frac{\partial}{\partial x_1} + \sin \alpha \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \frac{\partial}{\partial x_5}, X_4 = \cos \beta \frac{\partial}{\partial x_6} - \sin \beta \frac{\partial}{\partial x_8}, \]
\[ X_5 = \frac{\partial}{\partial x_9}, X_6 = \frac{\partial}{\partial x_{10}}, \]
\[ \ker \pi_* = D \oplus D_1 \oplus D_2, \]

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where

\[ D = <X_5 = \frac{\partial}{\partial x_9}, X_6 = \frac{\partial}{\partial x_{10}}>, D_1 = <X_1 = \cos \alpha \frac{\partial}{\partial x_1} + \sin \alpha \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2}>, \]

\[ D_2 = <X_3 = \frac{\partial}{\partial x_5}, X_4 = \cos \beta \frac{\partial}{\partial x_6} - \sin \beta \frac{\partial}{\partial x_8}>, \]

\[(\ker \pi_*)^\perp = <H_1 = \sin \alpha \frac{\partial}{\partial x_1} - \cos \alpha \frac{\partial}{\partial x_3}, H_2 = \frac{\partial}{\partial x_4}, H_3 = \sin \beta \frac{\partial}{\partial x_6} + \cos \beta \frac{\partial}{\partial x_8}, H_4 = \frac{\partial}{\partial x_7}>, \]

\[\pi_* H_1 = e^5 \frac{\partial}{\partial v_1}, \pi_* H_2 = e^5 \frac{\partial}{\partial v_2}, \pi_* H_3 = e^5 \frac{\partial}{\partial v_3}, \pi_* H_4 = e^5 \frac{\partial}{\partial v_4}.\]

with quasi bi-slant angle \( \theta_1 = \alpha \) and \( \theta_2 = \beta \).

Hence, we have

\[ g_2(\pi_* H_1, \pi_* H_2) = (e^5)^2 g_1(H_1, H_2), \]
\[ g_2(\pi_* H_3, \pi_* H_4) = (e^5)^2 g_1(H_3, H_4). \]

Therefore, \( \pi \) is a conformal quasi bi-slant submersion with \( \lambda = e^5 \).

**Example 5.2** Let \( \pi : (R^{10}, g_{10}) \rightarrow (R^4, g_4) \) where \( (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2 + dx_{10}^2) \) define by

\[ \pi(x_1, \ldots, x_{10}) = \left( \frac{x_3 + x_5}{\sqrt{2}}, x_6, \frac{\sqrt{3} x_7 - x_9}{2}, x_{10} \right), \]

which is a conformal quasi-bi-slant submersion such that

\[ X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}, X_3 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} \right), X_4 = \frac{\partial}{\partial x_4}, \]
\[ X_5 = \frac{1}{2} \frac{\partial}{\partial x_7} + \sqrt{3} \frac{\partial}{\partial x_9}, X_6 = \frac{\partial}{\partial x_8} \]

\[ \ker \pi_* = D \oplus D_1 \oplus D_2, \]

where

\[ D = <X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2}>, D_1 = <X_3 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5} \right), X_4 = \frac{\partial}{\partial x_4}>, \]
\[ D_2 = <X_5 = \frac{1}{2} \left( \frac{\partial}{\partial x_7} + \sqrt{3} \frac{\partial}{\partial x_9} \right), X_6 = \frac{\partial}{\partial x_8}>, \]

\[(\ker \pi_*)^\perp = <H_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} \right), H_2 = \frac{\partial}{\partial x_6}, H_3 = \frac{1}{2} \left( \sqrt{3} \frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9} \right), H_4 = \frac{\partial}{\partial x_{10}}>, \]

\[\pi_* H_1 = \frac{\partial}{\partial v_1}, \pi_* H_2 = \frac{\partial}{\partial v_2}, \pi_* H_3 = \frac{\partial}{\partial v_3}, \pi_* H_4 = \frac{\partial}{\partial v_4}.\]
with conformal quasi bi-slant angle \( \theta_1 = \frac{\pi}{4} \) and \( \theta_2 = \frac{\pi}{3} \).

Hence, we have

\[
g_2(\pi_*H_1, \pi_*H_1) = e^{-2x_3}g_1(H_1, H_1), \quad g_2(\pi_*H_2, \pi_*H_2) = e^{-2x_3}g_1(H_2, H_2),
\]

\[
g_2(\pi_*H_3, \pi_*H_3) = e^{-2x_2}g_1(H_3, H_3), \quad g_2(\pi_*H_4, \pi_*H_4) = e^{-2x_3}g_1(H_4, H_4).
\]

Thus \( \pi \) is conformal quasi-bi-slant submersion with \( \lambda = e^{-x_3} \).

References


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