On convergence and summability with speed in ultrametric fields

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Abstract. Throughout the present paper, $K$ denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Entries of sequences, infinite series and infinite matrices are in $K$. Following Kangro [2–4], we introduce the concepts of convergence with speed $\lambda$ (or $\lambda$-convergence) and $\lambda$-summability by the infinite matrix $A$ (or $A^\lambda$-summability) in $K$. We then prove a characterization of the matrix class $(c^\lambda,c^\mu)$, where $c^\lambda$ denotes the set of all $\lambda$-convergent sequences.

Keywords. ultrametric (or non-archimedean) field · convergence with speed $\lambda$ (or $\lambda$-convergence) · $\lambda$-summability by the matrix $A$ (or $A^\lambda$-summability) · matrix class $(c^\lambda,c^\mu)$

Mathematics Subject Classification (2010) 40C05 · 40D05 · 40H05 · 46S10

1 Introduction

In the present paper, $K$ denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Sequences, infinite series and infinite matrices have entries in $K$. For a given sequence $x = \{x_k\}$ in $K$ and an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n,k = 0,1,2,\ldots$, we define

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, n = 0,1,2,\ldots,$$

where we suppose that the series on the right converge. $A(x) = \{(Ax)_n\}$ is called the $A$-transform of the sequence $x = \{x_k\}$.

If $X,Y$ are sequence spaces, we write $A = (a_{nk}) \in (X,Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. In the sequel, $c,c_0$ respectively denote the ultrametric Banach spaces of convergent and null sequences.

The following result is well-known (for instance, see [5–7]).
Theorem 1.1 (Kojima-Schur) \( A \in (c,c), \) i.e., \( A \) is convergence preserving or conservative if and only if

\[
\sup_{n,k} |a_{nk}| < \infty; \tag{1.1}
\]

\[
\lim_{n \to \infty} a_{nk} = a_k, k = 0, 1, 2, \ldots; \tag{1.2}
\]

and

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = a. \tag{1.3}
\]

In such a case,

\[
\lim_{n \to \infty} (Ax)_n = sa + \sum_{k=0}^{\infty} (x_k - s)a_k, \tag{1.4}
\]

where \( \lim_{k \to \infty} x_k = s. \)

In the context of Theorem 1.1, it is worthwhile to note that Monna [5] proved Theorem 1.1 using modern tools like the ultrametric version of the Banach-Steinhaus theorem, while Natarajan [6] proved the same result using the “sliding hump method”.

The following result can be easily proved.

**Theorem 1.2** \( A \in (c_0, c) \) if and only if (1.1) and (1.2) hold.

2 Convergence with speed \( \lambda \) (or \( \lambda \)-convergence), \( \lambda \)-summability by the matrix \( A \) (or \( A^\lambda \)-summability), characterization of the matrix class \( (c^\lambda, c^\mu) \)

For a detailed study of \( \lambda \)-convergence and \( A^\lambda \)-summability in the classical case, one can refer to [1]. Following Kangro [2–4], we introduce the concepts of \( \lambda \)-convergence and \( A^\lambda \)-summability in \( K \) as follows.

**Definition 2.1** Let \( \lambda = \{\lambda_n\} \) be a sequence in \( K \) such that

\[ 0 < |\lambda_n| \nearrow \infty, n \to \infty. \]

A sequence \( \{x_n\} \) in \( K \) is said to be convergent with speed \( \lambda \) or \( \lambda \)-convergent if \( \{x_n\} \in c \) with \( \lim_{n \to \infty} x_n = s \) (say) and

\[ \lim_{n \to \infty} \lambda_n(x_n - s) \text{ exists (finitely)}. \]

\( \{x_n\} \) is said to be \( \lambda \)-summable by the matrix \( A = (a_{nk}) \) or \( A^\lambda \)-summable if the \( A \)-transform of \( x = \{x_n\} \), i.e., \( A(x) = \{(Ax)_n\} \) is \( \lambda \)-convergent.
Let $c^\lambda$ denote the set of all $\lambda$-convergent sequences in $K$. By definition, 
\[ c^\lambda \subset c. \]

We note that the sequences 
\[ e_k = \{0, 0, ..., 0, 1, 0, \ldots\}, \text{1 occurring in the } k\text{th place, } k = 0, 1, 2, \ldots; \]
e = \{1, 1, 1, \ldots\};
and 
\[ e^\lambda = \{ \frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \} \]
all belong to $c^\lambda$.
Let $\mu = \{\mu_n\}$ be a sequence in $K$ such that 
\[ 0 \prec |\mu_n| \prec \infty, n \to \infty. \]

We now have the following characterization of the matrix class $(c^\lambda, c^\mu)$.

**Theorem 2.2** $A = (a_{nk}) \in (c^\lambda, c^\mu)$ if and only if 
\[ A(e_k), A(e), A(e^\lambda) \in c^\mu, k = 0, 1, 2, \ldots; \]
\[ \sup_{n,k} \frac{|a_{nk}|}{\lambda_k} < \infty; \]
and 
\[ \sup_{n,k} \left| \frac{\mu_n (a_{nk} - a_k)}{\lambda_k} \right| < \infty. \]

**Proof.** Necessity. Let $A = (a_{nk}) \in (c^\lambda, c^\mu)$. As noted above, 
\[ e_k, e, e^\lambda \in c^\lambda, k = 0, 1, 2, \ldots \]
and so 
\[ A(e_k), A(e), A(e^\lambda) \in c^\mu, k = 0, 1, 2, \ldots, \]
i.e., (2.1) holds.
Since $A(e) \in c^\mu$, \( \{ \sum_{k=0}^{\infty} a_{nk} \}_{n=0}^{\infty} \) converges and so (1.3) holds. Let, now, 
\( x = \{ x_k \} \in c^\lambda \), with \( \lim_{k \to \infty} x_k = s \). Let \( \beta_k = \lambda_k (x_k - s) \) and \( \lim_{k \to \infty} \beta_k = \beta \).
Then, for \( n = 0, 1, 2, \ldots \),

\[
(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k
= \sum_{k=0}^{\infty} a_{nk} \left( \frac{\beta_k}{\lambda_k} + s \right)
= \sum_{k=0}^{\infty} a_{nk} \frac{\beta_k}{\lambda_k} + s \sum_{k=0}^{\infty} a_{nk}.
\]  

(2.4)

Now, \( \{ (Ax)_n \} \in c \) and (1.3) holds. Since \( \{ \beta_k \} \in c \), in view of (2.4), the infinite matrix \( \left( \frac{a_{nk}}{\lambda_k} \right) \in (c,c) \). Now, using Theorem 1.1, we have,

\[
\sup_{n,k} \left| \frac{a_{nk}}{\lambda_k} \right| < \infty,
\]

i.e., (2.2) holds.

Since \( \left( \frac{a_{nk}}{\lambda_k} \right) \in (c,c) \), (1.2) holds and

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} \text{ exists and } = a^\lambda \text{ (say)}.
\]

Further, using (1.4), we have,

\[
y = \lim_{n \to \infty} (Ax)_n
= a^\lambda \beta + as + \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} (\beta_k - \beta).
\]  

(2.5)
In view of (2.4) and (2.5),

\[(Ax)_n - y = \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} \beta_k + s \sum_{k=0}^{\infty} a_{nk} - a^\lambda \beta - as
\]

\[- \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} (\beta_k - \beta) \]

\[= \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} (\beta_k - \beta) + s \sum_{k=0}^{\infty} a_{nk} - a^\lambda \beta - as - \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k} (\beta_k - \beta) \]

\[= \sum_{k=0}^{\infty} \frac{a_{nk} - a_k}{\lambda_k} (\beta_k - \beta)
\]

\[+ \beta \left\{ \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} - a^\lambda \right\}
\]

\[+ s \left\{ \sum_{k=0}^{\infty} a_{nk} - a \right\}. \]

So,

\[\mu_n \{(Ax)_n - y\} = \sum_{k=0}^{\infty} \mu_n \frac{a_{nk} - a_k}{\lambda_k} (\beta_k - \beta)
\]

\[+ \mu_n \beta \left\{ \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} - a^\lambda \right\}
\]

\[+ \mu_n s \left\{ \sum_{k=0}^{\infty} a_{nk} - a \right\}. \tag{2.6} \]

Since \(A(e) \in e^\mu\),

\[\lim_{n \to \infty} \mu_n \left\{ \sum_{k=0}^{\infty} a_{nk} - a \right\} \text{ exists.} \]

Since \(A(e^\lambda) \in e^\mu\),

\[\lim_{n \to \infty} \mu_n \left\{ \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} - a^\lambda \right\} \text{ exists.} \]

Since \(\{(Ax)_n\} \in e^\mu\),

\[\lim_{n \to \infty} \mu_n \{(Ax)_n - y\} \text{ exists.} \]
Noting that \( \{ \beta_k - \beta \} \in c_0 \), using (2.6) and Theorem 1.2, the infinite matrix
\[
\left( \mu_n \frac{a_{nk} - a_k}{\lambda_k} \right) \in (c_0, c)
\]
and so
\[
\sup_{n,k} \left| \mu_n \frac{a_{nk} - a_k}{\lambda_k} \right| < \infty,
\]
i.e., (2.3) holds.

Sufficiency. Let (2.1), (2.2) and (2.3) hold. Let \( x = \{ x_k \} \in c^\lambda \). Since \( A(e_k) \in c^\mu \), (1.2) holds. Also, since \( A(e^\lambda) \in c^\mu \), \( \lim_{n \to \infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} \) exists.

These, along with (2.2) and Theorem 1.2, imply that the infinite matrix
\[
\left( \frac{a_{nk}}{\lambda_k} \right) \in (c, c).
\]
Since \( A(e) \in c^\mu \), (1.3) holds. Now, using (2.4), since \( \{ \beta_k \} \in c \),
\[
\lim_{n \to \infty} (Ax)_n \text{ exists.}
\]
Since \( A(e_k) \in c^\mu \),
\[
\lim_{n \to \infty} \mu_n(a_{nk} - a_k) \text{ exists, } k = 0, 1, 2, \ldots
\]
These, along with (2.3) and Theorem 1.2, imply that the infinite matrix
\[
\left( \mu_n \frac{a_{nk} - a_k}{\lambda_k} \right) \in (c_0, c)
\]
so that
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} \mu_n(a_{nk} - a_k) \frac{(\beta_k - \beta)}{\lambda_k}
\]
exists, since \( \{ \beta_k - \beta \} \in c_0 \). Now, using (2.6) and the fact that \( A(e), A(e^\lambda) \in c^\mu \), it follows that
\[
\lim_{n \to \infty} \mu_n \{(Ax)_n - y\} \text{ exists,}
\]
i.e., \( \{(Ax)_n\} \in c^\mu \).

This completes the proof of the theorem.

\[ \square \]

**Definition 2.3** If \( A = (a_{nk}) \in (c^\lambda, c^\lambda) \), \( A \) is said to be \( \lambda \)-conservative or \( \lambda \)-convergence preserving.
Definition 2.4 If \( A \in (c^\lambda, c^\mu) \), where,
\[
\lim_{n \to \infty} \left| \frac{\mu_n}{\lambda_n} \right| = \infty,
\]
(2.7)
\( A \) is said to improve \( \lambda \)-convergence.

Theorem 2.5 Any matrix \( A \), which improves \( \lambda \)-convergence, is \( \lambda \)-conservative.

Proof. Let \( A \in (c^\lambda, c^\mu) \), where (2.7) holds. Let \( x = \{x_k\} \in c^\lambda \), i.e.,
\[
\lim_{k \to \infty} \lambda_k (x_k - s) \text{ exists, where } \lim_{k \to \infty} x_k = s.
\]
Since \( \{(Ax)_n\} \in c^\mu \),
\[
\lim_{n \to \infty} \mu_n \{(Ax)_n - y\} \text{ exists, where } \lim_{n \to \infty} (Ax)_n = y.
\]
We now claim that \( \{(Ax)_n\} \in c^\lambda \), i.e., we have to prove that
\[
\lim_{n \to \infty} \lambda_n \{(Ax)_n - y\} \text{ exists.}
\]
Now,
\[
|\lambda_n \{(Ax)_n - y\}| = |\lambda_n|(Ax)_n - y| = \left| \frac{\lambda_n}{\mu_n} \right| |\mu_n \{(Ax)_n - y\}|
\]
\[
\to 0, \ n \to \infty,
\]
since (2.7) holds and \( \lim_{n \to \infty} \mu_n \{(Ax)_n - y\} \) exists.
This completes the proof of the theorem. \( \square \)

References
Received: 23.XI.2021 / Revised: 21.XII.2021 / Accepted: 22.XII.2021

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