Abstract  In this paper, we consider the Pell-Lucas polynomials. We express these polynomials as complex hyperbolic functions. Using this we obtain general root formula for Pell-Lucas polynomials. Furthermore, we give some interesting identities about images of roots of a polynomial under another member of the family. Finally, we get some amazing relationships between the roots of Pell-Lucas polynomials and the modular group, Hecke groups, generalized Hecke groups with geometric viewpoints.

Keywords  Pell-Lucas polynomials · roots of polynomial equations · the modular group · Hecke groups

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1 Introduction

Fibonacci, Lucas, Pell and Pell-Lucas polynomials are the sequences of orthogonal polynomials, and they are expressed recursively. These polynomials are widely used in the study of many topics such as number theory, combinatorics, algebra, approximation theory, geometry, graph theory (see [6], [12], [20], [24], [25], [39] and [42]). These polynomials satisfy the following properties from [15], [16], [24] and [25]. It is well known that Fibonacci polynomials, $F_n(x)$, by as follows

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), n \geq 3$$  \hspace{1cm} (1.1)

where $F_1(x) = 1$ and $F_2(x) = x$. Lucas polynomials are defined by as

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 3$$  \hspace{1cm} (1.2)

where $L_1(x) = x$ and $L_2(x) = x^2 + 2$. Pell polynomials, $P_n(x)$, by as follows

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), n \geq 2$$  \hspace{1cm} (1.3)
where $P_0(x) = 0$ and $P_1(x) = 1$. Pell-Lucas polynomials are defined by as
\[ Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), n \geq 2 \]  \hspace{1cm} (1.4)
where $Q_0(x) = 2$ and $Q_1(x) = 2x$. The sum of coefficients of $F_n(x)$ is $n^{th}$ Fibonacci number $F_n$. That is, $F_n(1) = F_n$. Similarly, Lucas, Pell and Pell-Lucas numbers obtained via $L_n(1) = L_n$, $P_n(1) = P_n$ and $Q_n(1) = Q_n$. For all $n \geq 1$, $\deg[F_n(x)] = n - 1$, $\deg[L_n(x)] = n$, $\deg[P_n(x)] = n - 1$ and $\deg[Q_n(x)] = n$. Some well-known identities related to these polynomials are as follows:
\[ L_n(x) = F_{n-1}(x) + F_{n+1}(x) \]  \hspace{1cm} (1.5)
\[ (x^2 + 4)F_n^2(x) = L_{n-1}(x) + L_{n+1}(x) \]  \hspace{1cm} (1.6)
\[ Q_n(x) = P_{n-1}(x) + P_{n+1}(x) \]  \hspace{1cm} (1.7)
\[ 4(x^2 + 1)P_n^2(x) = Q_{n-1}(x) + Q_{n+1}(x) \]  \hspace{1cm} (1.8)
Using generating functions and solving recurrences, these polynomials are explicitly given by the Binet-type formulas as
\[ F_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \quad L_n(x) = \alpha^n(x) + \beta^n(x) \]  \hspace{1cm} (1.9)
where $\alpha(x) = \frac{x + \sqrt{x^2 + 1}}{2}$ and $\beta(x) = \frac{x - \sqrt{x^2 + 1}}{2}$.
\[ P_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)}, \quad Q_n(x) = \gamma^n(x) + \delta^n(x) \]  \hspace{1cm} (1.10)
where $\gamma(x) = x + \sqrt{x^2 + 1}$ and $\delta(x) = x - \sqrt{x^2 + 1}$.

The ratio of two consecutive polynomials of Fibonacci and Lucas families converge to the Golden Ratio which appears in many fields in the literature, for example: nature, art, architecture, biology, physics, chemistry, cosmos, theology, finance and so on (see [11], [18], [24], [29], [31] and [32]). Furthermore, the ratio of two consecutive polynomials of Pell and Pell-Lucas families converge to Silver Mean. The ratio is another member of the class of metallic means defined by Spinadel, apart from the Golden Mean. Other metallic means with special naming are Bronze Mean and Cooper Mean (see [38].) There are many interesting studies on different aspects related to the number sequences, polynomials and metallic means mentioned above (see [8], [9], [10], [16], [21], [25], [33], [37] and [40] for more details).

In [19], V. E. Hoggatt and M. Bicknell obtain the roots of large classes of polynomials Fibonacci and Lucas using hyperbolic trigonometric functions. Therefore, the general root formulas for the polynomials have been achieved. This contribution is quite remarkable considering the Abel-Ruffini theorem. There are many papers on the topic from different aspects (see [5], [13], [15], [24], [30], [35], [36] and [41]). Also, P. F. Byrd studied on hyperbolic function representations of Pell polynomials called Fibonacci polynomials
at that time. Pell and Pell-Lucas polynomials are studied in the literature extensively. (See for more details [16], [21], [25], [39] and [42]).

On the other hand, in [17] while studying Dirichlet series, Hecke introduced the groups $H(\lambda)$ generated by two M"obius transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda$$

where $\lambda$ is a fixed positive real number. Let $S = TU$ i.e.

$$S(z) = -\frac{1}{z+\lambda}$$

By identifying the transformation $\frac{az+b}{cz+d}$ with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is related to a multiplicative group of $2 \times 2$ matrices in which a matrix is identified with its negative. Notice that $T$ and $S$ have matrix representation

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -1 \\ 1 & \lambda \end{bmatrix}$$

respectively.

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2 \cos \frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$ or $\lambda \geq 2$. These groups have come to be known as the Hecke groups and denoted $H(\lambda_q)$, $H(\lambda)$ for $q \geq 3, \lambda \geq 2$, respectively. Hecke group $H(\lambda_q)$ is the Fuchsian group of the first kind when either $\lambda = \lambda_q$ or $\lambda = 2$ and $H(\lambda)$ is the Fuchsian group of the second kind when $\lambda > 2$. In this study, we will focus the case $\lambda = \lambda_q, q \geq 3$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and $q$ and it has a presentation

$$H(\lambda_q) = H_q = \langle T, S \mid T^2 = S^q = I \rangle \cong C_2 * C_q.$$  

The first Hecke groups $H_q$ are $H_3 = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H_4 = H(\sqrt{2})$, $H_5 = H(\frac{1+\sqrt{5}}{2})$ and $H_6 = H(\sqrt{3})$.

Lehner studied in [28], a more general class $H_{p,q}$ of Hecke groups $H_q$, by taking

$$X(z) = -\frac{1}{z+\lambda_p} \quad \text{and} \quad V(z) = z + \lambda_p + \lambda_q$$

where $2 \leq p \leq q, p + q \geq 4$. Here, if we take $Y = XV = \frac{-1}{z+\lambda_q}$ then the group presentation is

$$H_{p,q} = \langle X, Y \mid X^p = Y^q = I \rangle \cong C_p * C_q$$

The group named as generalized Hecke groups $H_{p,q}$. Also, known from [28] $H_{2,q} = H_q$. Furthermore, all Hecke groups $H_q$ are included in generalized Hecke groups $H_{p,q}$. The modular group, Hecke groups and generalized Hecke groups have been extensively studied from many points of view in the literature. (See for more details [3], [4], [7], [17], [22], [23], [26] and [34].) Also, there are many remarkable studies on $2 \cos \frac{\pi}{q}$ and $\cos \frac{2\pi}{q}$ in the literature. Moreover, finding the minimal polynomial of $\cos \frac{2\pi}{q}$ is an old problem due
to its connection to the cyclotomic polynomials. The algebraic numbers are investigated many papers related to Chebyshev polynomials, Gaussian periods, Dickson polynomials, Ramanujan sums and Möbius inversion (see for more details [1], [2], [14], [27]).

In this study, we focus on the roots of Pell-Lucas polynomials. We state Pell-Lucas polynomials in terms of complex hyperbolic functions. Then, we obtain roots of Pell-Lucas polynomials. It is known from W. N. H. Abel that an algebraic equation of degree five or more has no solution. Also considering the D’Alembert-Gauss theorem, we can interpret that the general root formulas for the polynomials are very valuable. At that point, the existence of the formulas is remarkable and significant. Finally, we investigate the image of a root of a polynomial under another member of the family. Finally, we obtain strong relationships between the roots of Pell-Lucas polynomials and the modular group, Hecke groups, generalized Hecke groups with geometric viewpoint.

2 Main results

In this section, we express Pell-Lucas polynomials as complex hyperbolic functions. And we prove the general root formula of Pell-Lucas polynomials. We give some results about roots of these polynomials. Then, we obtain several interesting identities about the images of a root of a Pell-Lucas polynomial under another member of the family.

Theorem 2.1 Let \( x = \sinh z \) then,

\[
Q_{2n}(x) = 2 \cosh 2nz \tag{2.1}
\]

\[
Q_{2n+1}(x) = 2 \sinh (2n + 1)z \tag{2.2}
\]

Proof. Observe that if \( x = \sinh z \) we have

\[
\gamma(x) = x + \sqrt{x^2 + 1} = \sinh z + \cosh z = \frac{e^z - e^{-z}}{2} + \frac{e^z + e^{-z}}{2} = e^z
\]

\[
\delta(x) = x - \sqrt{x^2 + 1} = \sinh z - \cosh z = \frac{e^z - e^{-z}}{2} - \frac{e^z + e^{-z}}{2} = -e^z.
\]

After substituting these to ones in Equation 1.10 Binet formula of Pell-Lucas polynomials we obtain, the desired result as

\[
Q_{2n}(x) = \gamma^{2n}(x) + \delta^{2n}(x) = e^{2nz} + (-1)^{2n}e^{-2nz} = 2 \cosh 2nz
\]

\[
Q_{2n+1}(x) = \gamma^{2n+1}(x) + \delta^{2n+1}(x) = e^{(2n+1)z} + (-1)^{2n+1}e^{-(2n+1)z} = 2 \sinh(2n + 1)z.
\]
Theorem 2.2  Roots of Pell-Lucas polynomials are

\[ Q_{2n}(x) = 0 : x = \pm i \sin \left( \frac{(2k+1)\pi}{4n} \right) \]  

(2.3)

where \( k = 0, 1, ..., n - 1 \)

\[ Q_{2n+1}(x) = 0 : x = \pm i \sin \left( \frac{k\pi}{2n+1} \right) \]  

(2.4)

where \( k = 0, 1, ..., n \).

Proof. We first deal with the roots of the even subscripted Pell-Lucas polynomials. Consider the Theorem 2.1. If \( Q_{2n}(x) = 0 \) then, \( 2 \cosh 2nz = 0 \) which yields

\[ \cosh 2nz = \cos (2na + i2nb) = \cos 2na \cos 2nb + i \sin 2na \sin 2nb = 0 \]

for \( z = a + ib \) where \( a, b \in \mathbb{R} \). Since \( \cosh 2na \geq 1 \) for \( n \in \mathbb{N} \), \( \cos 2nb \) and \( \sin 2na \sin 2nb \) must be zero. Hence, \( b = \frac{(2k+1)\pi}{4n} \) for \( 0 \leq k \leq n - 1 \) and \( k \in \mathbb{N} \). We limit the values of \( k \) as \( 0 \leq k \leq n - 1 \). Because, the values greater than \( n - 1 \) conflict with values in this range \( 0 \leq k \leq n - 1 \). Considering the sine function features, we express the appropriate values of \( b \) as follows.

\[ b = \pm \sin \left( \frac{(2k+1)\pi}{4n} \right) \]  

for \( k = 0, 1, ..., n - 1 \). Now, we examine the real part of the root of \( Q_{2n}(x) \). To do that, we use the imaginary part of the previous equation in the above line.

\[ \sinh 2na \sin 2n \left( \frac{(2k+1)\pi}{4n} \right) = \sinh 2na \sin \left( \frac{(2k+1)\pi}{2n} \right) = 0 \]

Here \( a = 0 \) because, \( \sinh 2na = \frac{e^{2na} - e^{-2na}}{2} \) must be zero. Thus, the general root formula of this polynomial is obtained as follows. \( x = \pm \sin \left( \frac{(2k+1)\pi}{4n} \right) \) where \( k = 0, 1, ..., n - 1 \). Notice that the degree of the even subscripted Pell-Lucas polynomial \( Q_{2n}(x) \) is \( 2n \) and we find exactly \( 2n \) roots via obtained root formula.

Roots of odd subscripted Pell-Lucas polynomials can be proved similarly.

\[ \Box \]

\[ \Box \]

Theorem 2.3  Let \( x = i \cosh z \) then,

\[ Q_n(x) = 2^n \cosh nz \]

(2.5)

Proof. Observe that if \( x = i \cosh z \) we have

\[ \gamma(x) = x + \sqrt{x^2 + 1} = i \cosh z + i \sinh z = i \left( \frac{e^z + e^{-z}}{2} \right) + \frac{e^z - e^{-z}}{2} = ie^z \]

\[ \delta(x) = x - \sqrt{x^2 + 1} = i \cosh z - i \sinh z = i \left( \frac{e^z + e^{-z}}{2} \right) - \frac{e^z - e^{-z}}{2} = ie^{-z} \]
After substituting these to ones in Equation 1.10 Binet formula of Pell-Lucas polynomials we obtain, the desired result as

\[ Q_n(x) = \gamma^n(x) + \delta^n(x) = i^n e^{nz} + i^n e^{-nz} = 2i^n \cosh nz. \]

\[ \square \]

**Theorem 2.4** 
Roots of the \( n \)th Pell-Lucas polynomial \( Q_n(x) \) are

\[ x = \pm i \cos \left( \frac{(2k + 1)\pi}{2n} \right) \text{ for } k = 0, 1, ..., n - 1. \]

**Proof.** This can be proven in a similar manner as in Theorem 2.2 using Theorem 2.3.

\[ \square \]

Now we are ready to calculate images of roots of a Pell-Lucas polynomial under other members of the family.

**Theorem 2.5** If \( r \) is a root of the Pell-Lucas polynomial \( Q_{2n-1}(x) \), i.e. \( Q_{2n-1}(r) = 0 \), then \( Q_{2n}(r) = 2\sqrt{r^2 + 1} \) and \( Q_{2n+1}(r) = 4r\sqrt{r^2 + 1} \).

**Proof.** Let \( x = r \) be a root of \( Q_{2n-1}(x) \). Using the Cassini-like formula for Pell-Lucas polynomials, we get

\[ Q_{2n}^2(r) = 4(r^2 + 1) \]

Therefore,

\[ Q_{2n}(r) = 2\sqrt{r^2 + 1} \]

Now we need the image of \( a \) under the polynomial \( Q_{2n+1}(x) \). To do that, we use the recurrence formula

\[ Q_{2n+1}(x) = 2xQ_{2n}(x) + Q_{2n-1}(x) \]

If we put \( x = r \) then, we obtain \( Q_{2n+1}(r) = 2rQ_{2n}(r) = 4r\sqrt{r^2 + 1} \). Thus, we get the desired result.

\[ \square \]

**Corollary 2.6** \( Q_{2n-1}(r) = 0 \), implies \( Q_{2n+1}(r).Q_{2n}(r) = 8r^3 + 8r \).

**Theorem 2.7** If \( r \) is root of the Pell-Lucas polynomial \( Q_{2n-1}(x) \), when \( r \neq 0 \) i.e. \( Q_{2n-1}(r) = 0 \) then, we have \( Q_{2n+1}(r)Q_{2n}(r) \neq 0 \).

**Proof.** We know from Corollary 2.6 that if \( Q_{2n-1}(r) = 0 \) then, \( Q_{2n+1}(a).Q_{2n}(a) = 8r^3 + 8r \). Suppose that \( 8r^3 + 8r = 0 \). This equation is only realized when \( r = 0 \) or \( r^2 + 1 = 0 \). Now, considering the hypothesis of the theorem we examine only the second case \( r^2 + 1 = 0 \). Also, we consider the root interval from the root formula of the Pell-Lucas polynomial for \( Q_{2n-1}(x) \). The roots of \( Q_{2n-1}(x) \) can be calculated by Theorem 2.4 as:

\[ r = i \cos \left( \frac{(2k + 1)\pi}{4n - 2} \right) \]
for $k = 0, 1, ..., 2n - 2$. For that purpose must be $4n - 2k = 3$ or $2k + 1 = 0$. But considering all possible values of $k$ and elementary calculations we get contradictions. Thus, we obtain the desired result as $8r^3 + 8r \neq 0$.

It is able to verify the following theorems using the same techniques. So, we leave the proofs to the readers.

**Theorem 2.8** If $r$ is a root of the Pell-Lucas polynomial $Q_{2n+1}(x)$, when $r \neq 0$ i.e. $Q_{2n+1}(r) = 0$ then, $Q_{2n}(r) = 2\sqrt{r^2 + 1}$ and $Q_{2n-1}(r) = -4r\sqrt{r^2 + 1}$.

**Proof.** It can be proven using recurrence relation and Cassini formula for Pell-Lucas polynomials.

**Corollary 2.9** $Q_{2n+1}(r) = 0$, implies $Q_{2n}(r)Q_{2n-1}(r) = -8r^3 - 8r$.

**Theorem 2.10** If $r$ is a root of the Pell-Lucas polynomial $Q_{2n}(x)$, i.e. $Q_{2n}(r) = 0$ then, $Q_{2n+1}(r) = Q_{2n-1}(r) = -2i\sqrt{r^2 + 1}$.

**Proof.** This can be proven using the recurrence relation and Cassini formula for Pell-Lucas polynomials as like the same technique in Theorem 2.5.

**Corollary 2.11** $Q_{2n}(r) = 0$, implies $Q_{2n+1}(r)Q_{2n-1}(r) = -4r^2 - 4$.

Now we need some identities expressed in [21] and [25].

**Theorem 2.12**

\[
\sum_{r=1}^{n} Q_{2r}(x) = \frac{Q_{2n+1}(x) - 2x}{2x} \quad (2.6)
\]

\[
\sum_{r=1}^{n} Q_{2r-1}(x) = \frac{Q_{2n}(x) - 2}{2x} \quad (2.7)
\]

\[
Q_{4n}(x) - 2 = 4(x^2 + 1)P_{2n}^2(x) \quad (2.8)
\]

**Proof.** This identities explained in [21] and [25].

We obtain some amazing properties about the roots of Pell and Pell-Lucas polynomials via the above theorem.

**Theorem 2.13**

\[
Q_{2n+1}(r) = 0 \text{ then, } Q_2(r) + Q_4(r) + ... + Q_{2r}(a) = -1.
\]

\[
Q_{2n}(r) = 0 \text{ then, } Q_1(r) + Q_3(r) + ... + Q_{2n-1}(r) = \frac{1}{r} = \pm \frac{i}{\sin \left( \frac{(2k+1)r}{4n} \right)}.
\]

where $k = 0, 1, ..., n - 1$

\[
P_{2n}(r) = 0 \text{ then, } Q_{4n}(r) = 2.
\]

**Proof.** The proof can be seen by using Theorem 2.2 and Theorem 2.12.
3 Relationships between the roots of Pell-Lucas polynomials and the modular group & Hecke groups & generalized Hecke groups

In this section, we consider the complex numbers as vectors in the complex plane. All the roots of Pell-Lucas polynomials are pure imaginary complex numbers. Each norm of the roots of a Pell-Lucas polynomial is smaller than one. We interpret the roots in the complex plane as related to the parameter of the modular group, Hecke groups and generalized Hecke groups.

Observation 3.1 We investigate the relationship between parameter of the modular group and roots of Pell-Lucas polynomial in the complex plane geometrically. The parameter of the modular group is $\lambda_3 = 2 \cos \frac{\pi}{3}$. The roots of Pell-Lucas polynomial $Q_n(x)$ known as $i \cos \frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ..., n-1$ from Theorem 2.4. Firstly, we consider these roots as vectors and we rotate the roots 270 degrees counterclockwise around the origin in the complex plane. Thus, we get $\cos \frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ..., n-1$. Later, we apply to double the norm of these vectors. Consequently, here $2 \cos \frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ..., n-1$ are obtained. Now, we examine whether these vectors can coincide with the parameter of the modular group parameter. For any of these vectors to coincide with the modular group parameter must be $2n - 6k = 3$. However, it is clear that the equality is not possible considering the Theorem 2.4. Therefore, we can state the Pell-Lucas polynomials $Q_n(x)$ do not generate a parameter for the modular group, unlike Fibonacci polynomials as a geometric interpretation.

Observation 3.2 The parameter of Hecke group as Fuchsian group of first kind is $\lambda_q = 2 \cos \frac{\pi}{q}$ for $q \geq 3$ and all roots of Pell-Lucas polynomial $Q_q(x)$ known as $i \cos \frac{(2k+1)\pi}{2q}$ for $k = 0, 1, ..., q-1$ from Theorem 2.4. If the first root $i \cos \frac{\pi}{q}$ of the Pell-Lucas polynomial $Q_q(x)$ is rotated 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors the parameter of the Hecke group $H(\lambda_{2q})$ is obtained. Therefore, we can state geometrically that the Pell-Lucas polynomial $Q_q(x)$ generates a parameter for the Hecke group $H(\lambda_{2q})$ as Fuchsian group of first kind.

Observation 3.3 Parameters of generalized Hecke groups $H_{p,q}$ are $\lambda_p = 2 \cos \frac{\pi}{p}$ and $\lambda_q = 2 \cos \frac{\pi}{q}$. Also, all roots of Pell-Lucas polynomial $Q_p(x)$ known as $i \cos \frac{(2k+1)\pi}{2p}$ for $k = 0, 1, ..., p-1$. And all roots of Pell-Lucas polynomial $Q_q(x)$ known as $i \cos \frac{(2k+1)\pi}{2q}$ for $k = 0, 1, ..., q-1$. If the first roots $i \cos \frac{\pi}{2p}$ of the Pell-Lucas polynomial $Q_p(x)$ and $i \cos \frac{\pi}{2q}$ of the Pell-Lucas polynomial $Q_q(x)$ are rotated 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors, the parameters of the generalized Hecke groups $H_{2p,2q}$ are obtained. Therefore, we can state geometrically that the Pell-Lucas polynomial $Q_p(x)$ and $Q_q(x)$ generate parameters for the generalized Hecke groups $H_{2p,2q}$.
Remark 3.1 Every Pell-Lucas polynomial $Q_n(x)$ for $n \geq 2$ generates at least one parameter for the Hecke group. For example, the Pell-Lucas polynomial $Q_5(x)$ generates one parameter as $2 \cos \frac{\pi}{10}$ via the root $i \cos \frac{\pi}{10}$ rotate 270 degrees counterclockwise around the origin in the complex plane and double the norm of the vector. The Pell-Lucas polynomial $Q_6(x)$ generates two parameter as $2 \cos \frac{\pi}{12}$ and $2 \cos \frac{\pi}{4}$ via the roots $i \cos \frac{\pi}{12}$ and $i \cos \frac{3\pi}{12}$ rotate 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors. The Pell-Lucas polynomial $Q_{18}(x)$ generates three parameters as respectively $2 \cos \frac{\pi}{36}$, $2 \cos \frac{\pi}{12}$ and $2 \cos \frac{\pi}{4}$ via the roots $i \cos \frac{\pi}{36}$, $i \cos \frac{3\pi}{36}$ and $i \cos \frac{9\pi}{36}$ rotate 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors.

Observation 3.4 We set a general way to get the relationship between the parameter of Hecke group as Fuchsian group of first kind and Pell-Lucas polynomial $Q_n(x)$. All roots of Pell-Lucas polynomial $Q_n(x)$ known as $i \cos \left(\frac{2k+1}{2n}\pi\right)$ for $k = 0, 1, ..., n-1$. $Q_n(x)$ generates parameter for Hecke group every provided condition that $2k+1$ divides $n$ except for $2k+1 = n$. For instance, $Q_5(x) = 32x^5 + 40x^3 + 10x$ generates exactly one parameter for Hecke group denoted $H_{10}$.

Remark 3.2 The parameter of Hecke group can not be derived from a unique Pell-Lucas polynomial. For instance, the parameter of the Hecke group $H_6$ is obtained from the Pell-Lucas polynomial $Q_3(x)$. Here the first root of the polynomial $Q_3(x)$ obtained as $i \cos \frac{\pi}{6}$ from Theorem 2.4 for $k = 0$. Also, the parameter is obtained from another Pell-Lucas polynomial $Q_9(x)$ using Theorem 2.4 for $k = 1$. Therefore, we can state the parameter of the Hecke group $H_6$ related to the Pell-Lucas polynomials $Q_{6m+3}(x)$ when $m \in \mathbb{N}$. More generally, the parameter of the Hecke group $H_s$ can be derived from Pell-Lucas polynomials $Q_{ks+\frac{s}{2}}$ when $k$ is a whole number and $s$ is an even positive integer.

Theorem 3.1 (Turan & Hecke & Pell-Lucas Theorem) The number of parameters for Hecke groups generated by $Q_n(x)$ calculates via the formula

$$R(n) = \begin{cases} \prod_{i=1}^{t} (a_i + 1) & \text{if } n \text{ even} \\ \prod_{i=1}^{t} (a_i + 1) - 1 & \text{if } n \text{ odd} \end{cases}$$

where $n = \prod_{i=1}^{t} 2^b p_i^{a_i}$ for $p_i$ distinct odd prime numbers, $a_i$ positive integers and $b$ a nonnegative integer, for $n \geq 2$.

Proof. It can be proven using the fundamental theorem of arithmetic, the formula for the total number of odd divisors of a number considering the root formula of Pell-Lucas polynomials and the parameter of Hecke group as $\lambda_q = 2 \cos \frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$.

\[\Box\]
**Corollary 3.2** Considering the polynomial space, the \( \{Q_n(x) : n \geq 2\} \) set of Pell-Lucas polynomial is a relation with the ability to generate any common parameter for Hecke groups. This relation has reflection and symmetry properties.

**Remark 3.3** We call the above relation as \( \nu \). Note that \( \nu \) is not reflexive relation. We give a counterexample to prove that. \( (Q_5(x), Q_{35}(x)) \in \nu \) via \( H_{10} \) and \( (Q_{35}(x), Q_7(x)) \in \nu \) via \( H_{14} \) but \( (Q_5(x), Q_7(x)) \notin \nu \). Although \( Q_5(x) \) and \( Q_7(x) \) generate one parameter for the Hecke groups, these polynomials do not generate a common parameter for any Hecke group. \( Q_5(x) \) and \( Q_7(x) \) generate a parameter for the Hecke groups \( H_{10} \) and \( H_{14} \) respectively.

**Definition 3.3** (Turan & Hecke & Pell-Lucas Number Sequence) We define a new number sequence derived from the Theorem 3.1. This sequence shows the relationship between the root of the Pell-Lucas polynomial and the Hecke groups interestingly. The twelfth, the thirteenth, and the fourteenth terms of the number sequence are obtained as 1, 2 and 3 from Pell-Lucas polynomials \( Q_{13}(x), Q_{14}(x) \) and \( Q_{15}(x) \) respectively. This sequence is as follows.

\[ 1, 1, 1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 2, 3, 1, 1, 3, 1, 2, 3, 2, \ldots \]

**Remark 3.4** Notice that each term of Turan & Hecke & Pell-Lucas sequence coincides the number of odd proper divisors of \( n \) for \( n \geq 2 \).

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