Notes on quasi contact metric manifolds

Y.D. Chai · J.H. Kim · J.H. Park ·
K. Sekigawa · W.M. Shin

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Abstract We study curvature identities on quasi contact metric manifolds based on the geometry of the corresponding quasi Kähler cones and we provide several results derived from the obtained curvature identities.

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1 Introduction

An odd dimensional smooth manifold $M$ is called an almost contact manifold if it admits a triple $(\phi, \xi, \eta)$ of a $(1, 1)$ tensor field, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

(1.1)

for any $X \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ denoting the Lie algebra of all smooth vector fields on $M$. It is well known that $\eta(\xi) = 1$. Further, an almost contact manifold $M = (M, \phi, \xi, \eta)$ equipped with a Riemannian metric $g$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(\xi, X),$$

(1.2)

for any $X, Y \in \mathfrak{X}(M)$ is called an almost contact metric manifold. Especially, an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying

$$d\eta(X, Y) = g(X, \phi Y),$$

(1.3)
for any $X, Y \in \mathfrak{X}(M)$ is called a contact metric manifold. Now, let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $\overline{M} = M \times \mathbb{R}$ be the almost Hermitian structure $(\overline{J}, \overline{g})$ defined by

$$
\overline{J}X = \phi X - \eta(X) \frac{\partial}{\partial t}, \quad \overline{J} \frac{\partial}{\partial t} = \xi,
$$

$$
\overline{g}(X,Y) = e^{-2t}g(X,Y), \quad \overline{g} \left( X, \frac{\partial}{\partial t} \right) = 0, \quad \overline{g} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = e^{-2t},
$$

for any $X, Y \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$.

**Remark 1.1** Let $\tilde{M} = M \times \mathbb{R}_+$ be the product manifold of $M$ and a positive half line $\mathbb{R}_+ = \{ s \in \mathbb{R} | s > 0 \}$ be equipped with the following almost Hermitian structure $(\tilde{J}, \tilde{g})$ defined by

$$
\tilde{J}X = \phi X + s\eta(X) \frac{\partial}{\partial s}, \quad \tilde{J} \frac{\partial}{\partial s} = -\frac{1}{s} \xi,
$$

$$
\tilde{g}(X,Y) = s^2g(X,Y), \quad \tilde{g} \left( X, \frac{\partial}{\partial s} \right) = 0, \quad \tilde{g} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 1,
$$

for any $X, Y \in \mathfrak{X}(M)$ and $s \in \mathbb{R}_+$. Then, it is easily checked that $(\tilde{M}, \tilde{J}, \tilde{g})$ is holomorphically isometric to $(\overline{M}, \overline{J}, \overline{g})$ under the diffeomorphism $F : \overline{M} \to M$ defined by $F(p, t) = (p, e^t)$, for any $(p, t) \in \overline{M}$. Taking account of Remark 1.1, we shall call $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$ the corresponding almost Hermitian cone to the almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$. It is also checked that $M = (M, \phi, \xi, \eta, g)$ is a Sasakian manifold (resp. a contact metric manifold) if and only if the corresponding almost Hermitian cone $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$ is a Kähler manifold (resp. an almost Kähler manifold, see [7]). GRAY and HERVELLA [3] introduced 16 classes of almost Hermitian manifolds. Among their classes, the classes of Kähler manifold $\mathcal{K}$, almost Kähler manifolds $\mathcal{AK}$, nearly Kähler manifolds $\mathcal{NK}$ and quasi Kähler manifolds $\mathcal{QK}$ have been studied extensively by many authors. The inclusion relations between the classes $\mathcal{K}, \mathcal{AK}, \mathcal{NK}$ and $\mathcal{QK}$ are as follows:

$$
\mathcal{K} = \mathcal{NK} \cap \mathcal{AK} \quad \text{and} \quad \mathcal{AK}, \mathcal{NK} \subset \mathcal{QK}.
$$

An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a quasi contact metric manifold if the corresponding almost Hermitian cone $\overline{M} = (\overline{M}, \overline{J}, \overline{g})$ is a quasi Kähler manifold. KIM, PARK and SEKIGAWA [4] discussed basic properties concerning a quasi contact metric manifold and gave a characterization of a quasi contact metric manifold and further a characterization of a contact metric manifold based on the result. It is well-known that quasi Kähler manifold is necessarily an almost Kähler manifold. Thus, a 3-dimensional quasi contact metric manifold is necessarily a contact metric manifold (see [5]). We refer to [5]. In this paper, we will work on the question “Does there exist a $(2n+1)(n \geq 2)$-dimensional quasi contact metric manifold which is not a contact metric manifold?” which was raised on the previous paper [4].
2 Preliminaries

In this section, we prepare some basic terminologies and review the results from the previous paper [4]. Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $h$ be the $(1, 1)$ tensor field on $M$ defined by

$$h = \frac{1}{2} \mathcal{L}_\xi \phi,$$

(2.1)

here $\mathcal{L}_\xi$ denotes the Lie differentiation with respect to the characteristic vector field $\xi$. The tensor field $h$ plays an important role in the geometry of almost contact metric manifolds. By the definition (2.1), we may easily observe that the tensor field $h$ satisfies the following equalities

$$h \xi = 0, \quad \text{tr} h = 0.$$

(2.2)

We denote by $\nabla$ the Levi-Civita connection of $g$ and $R$ the curvature tensor defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

(2.3)

for any $X, Y \in \mathfrak{X}(M)$. We denote by $\rho$ and $\tau$ the Ricci tensor and the scalar curvature of $M = (M, \phi, \xi, \eta, g)$, respectively. Further, we denote by $\rho^*$ and $\tau^*$ the Ricci $*$-tensor and the scalar $*$-curvature of $M = (M, \phi, \xi, \eta, g)$ defined respectively by

$$\rho^*(X,Y) = \frac{1}{2} \text{tr}(Z \mapsto R(X, \phi Y) \phi Z),$$

(2.4)

for any $X, Y, Z \in \mathfrak{X}(M)$, and

$$\tau^* = \text{tr} Q^*,$$

(2.5)

where $Q^*$ is the $(1,1)$ tensor field on $M$ defined by

$$g(Q^* X, Y) = \rho^*(X,Y),$$

(2.6)

for any $X, Y \in \mathfrak{X}(M)$.

Recently, Kim, Park and Sekigawa [4] gave a following characterization of a contact metric manifold.

**Theorem 2.1** ([4], Theorem 3.2) A contact metric manifold is characterized as an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the following conditions:

$$h$$

is symmetric,

(2.7)

$$(\nabla_X \phi)Y + (\nabla_{\phi X} \phi) \phi Y = 2g(X,Y)\xi - \eta(Y)X - \eta(X)\eta(Y)\xi - \eta(Y)hX,$$

(2.8)

for any $X, Y \in \mathfrak{X}(M)$. They also proved the following.

**Theorem 2.2** ([4], Theorem 4.2) A quasi contact metric manifold is characterized as an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ satisfying the condition (2.8) in Theorem 2.1

We here note that the equality (2.8) is equivalent to the following on any almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$:

$$(\nabla_X \phi)Y + (\nabla_{\phi X} \phi) \phi Y = 2g(X,Y)\xi - 2\eta(Y)X + \eta(Y)\nabla_{\phi X} \xi,$$

(2.9)

for any $X, Y \in \mathfrak{X}(M)$ (see [4], Lemma 4.1).
Remark 2.1 For any almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$, the following equalities (2.10), (2.11) and (2.12) are derived from (2.8) or (2.9):

$$\nabla_X \eta(Y) + (\nabla_{\phi X} \eta) \phi Y + 2g(\phi X, Y) \xi = 0,$$

(2.10)

$$\nabla_{\xi} \phi = 0,$$

(2.11)

$$\nabla_{\xi} \xi = 0,$$

(2.12)

for any $X, Y \in \mathfrak{X}(M)$ (see [4], Proposition 2.6).

3 Fundamental formulas

In this section, we shall deduce several basic formulas on a quasi contact metric manifold based on the discussions in the previous sections. Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi contact metric manifold. Then, from (2.1) and (2.5), we see that the tensor field $h$ takes the following form:

$$hX = \frac{1}{2}(-\nabla_{\phi X} \xi + \phi \nabla_X \xi),$$

(3.1)

for any $X \in \mathfrak{X}(M)$. From (3.1), taking account of (2.12), we have

$$h\phi X = \frac{1}{2}(-\nabla_{\phi^2 X} \xi + \phi^2 \nabla_{\phi X} \xi) = \frac{1}{2}(\nabla_X \xi - (\nabla_{\phi X} \phi) \xi).$$

(3.2)

Setting $Y = \xi$ in (2.9), we get

$$-\phi \nabla_X \xi = 2\eta(X) \xi - 2X + \nabla_{\phi X} \xi.$$

(3.3)

Operating $\phi$ to the both sides of (3.3), we have

$$(\nabla_{\phi X} \phi) \xi = -2\phi X - \nabla_X \xi.$$

(3.4)

Thus, from (3.2) and (3.4), we obtain

$$h\phi X = \nabla_X \xi + \phi X.$$

(3.5)

On one hand, from (3.1) and (3.4), we obtain

$$\phi hX = \frac{1}{2}(\phi \nabla_{\phi X} \xi + \phi^2 \nabla_{\phi X} \xi) = \frac{1}{2}((\nabla_{\phi X} \phi) \xi - \nabla_X \xi) = -\phi X - \nabla_X \xi.$$

(3.6)

Thus, from (3.5) and (3.6), we have

$$h\phi + \phi h = 0,$$

(3.7)

and

$$\nabla_X \xi = -\phi X - \phi hX,$$

(3.8)

for any $X \in \mathfrak{X}(M)$. From (3.7), we have $\eta(h\phi X) = 0$ for any $X \in \mathfrak{X}(M)$, and hence, also $\eta(h\phi^2 X) = -\eta(hX) = 0$, for any $X \in \mathfrak{X}(M)$. Thus, we have

$$\eta \circ h = 0.$$
From (3.8) and (3.9), taking account of (2.2), we have
\[
\|\nabla \eta\|^2 = (\nabla_i \eta_j)(\nabla^i \xi^j) = (\phi_{ij} + \phi_{ai} h_i^a)(\phi^{ij} + \phi_{aj} h^a_j)
\]
\[
= \phi_{ij} \delta^{ij} + \phi_{ij} \phi_{ak} h^k + \phi_{aj} h_i^a \phi^{ij} + \phi_{aj} h_i^a \phi_{ak} h^k
\]
\[
= 2n + (g_{ib} - \eta_i \eta_b) h^b_i + (\delta^i_{ja} - \eta_a \xi^i) h_i^a + (g_{ab} - \eta_a \eta_b) h^a_i h^b_i
\]
\[
= 2n + h_{ij} h^{ij} = 2n + \|h\|^2.
\]
From (3.10), we see that
\[
\text{From (3.10), we see that, and hence, in this case,}
\]
\[
\text{Now, the equality (2.9) is rewritten in terms of local coordinates as}
\]
\[
\phi_i^a(\nabla_a \phi_b^k)\phi_j^b = 2g_{ij} \xi^k - 2\eta_i \delta^k_j + \eta_j \phi^a \nabla_a \xi^k.
\]
From (3.8), taking account of (3.7), we get
\[
\phi_i^a(\nabla_a \phi_b^k)\phi_j^b = 2g_{ij} \xi^k - 2\eta_i \delta^k_j + \eta_j \phi^a \nabla_a \xi^k.
\]
Thus, from (3.11) and (3.12), we have
\[
\nabla_i \phi_j^i = -2n\eta_j.
\]
From (3.8), taking account of (3.7), we get
\[
\text{div } \xi = \nabla_i \xi^i = -tr \phi - tr(\phi h) = -tr(\phi h) = tr(h \phi) = tr(\phi h) = -div \xi, \text{ and hence}
\]
\[
div \xi = (\nabla_i \xi^i) = 0.
\]
From (3.1), taking account of (2.11) and (2.12), we get
\[
\xi^k \nabla_k h_j^i = \frac{1}{2} \xi^k(-\nabla_k \phi_j^a) \nabla_a \xi^i - \phi_j^a \nabla_k \nabla_a \xi^i
\]
\[
+ (\nabla_k \phi_i^a) \nabla_j \eta_a + \phi_{ja} \nabla_k \nabla_j \xi^a
\]
\[
= -\frac{1}{2} \xi^k \phi_{ja} \nabla_a \nabla_k \xi^i - \frac{1}{2} \xi^k \phi_j^a R_{kab}^i \xi^b + \frac{1}{2} \xi^k \phi_{ja} \nabla_j \nabla_k \xi^a
\]
\[
+ \frac{1}{2} \xi^k \phi_{ja} R_{kja}^b \xi^b
\]
\[
= \frac{1}{2} \phi_j^a (\nabla_a \xi^k) \nabla_k \xi^i - \frac{1}{2} \phi_j^a (\nabla_j \xi^k) \nabla_k \xi^a
\]
\[
- \frac{1}{2} \xi^k \phi_{ja} R_{kab}^i \xi^b + \frac{1}{2} \xi^k \phi_{ja} R_{kja}^b \xi^b.
\]
Here, taking account of (3.7) and (3.8), we get
\[ \frac{1}{2} \phi_j a (\nabla_a \xi^k) \nabla_k \xi^i = \frac{1}{2} \phi_j a (\phi_a k + \phi_b k_h b_h a) (\phi_k i + \phi_c h_k c) \]
\[ = \frac{1}{2} \phi_j a (-\delta_a i + \xi \eta_a - \phi_a k h_c \phi_k c - \phi_k b_h a b_h i \phi_k c) \]
\[ = -\frac{1}{2} \phi_j i + \frac{1}{2} \phi_a h_c a h_j c. \] (3.16)

Similarly, we get
\[ -\frac{1}{2} \phi a i (\nabla_j \xi^k) \nabla_k \xi a = -\frac{1}{2} \phi_j i - \frac{1}{2} \phi_a h_c a h_j c. \] (3.17)

Thus, from (3.5), (3.15) and (3.17), we have
\[ \xi^k \nabla_k h_j i = -\frac{1}{2} \phi \xi^k \phi_j a R^a k b i \xi^b + \frac{1}{2} \phi \xi^k \phi a i R^a k b i \xi^b. \]

(3.18) is rewritten in index-free form as follows:
\[ (\nabla_\xi h) X = -\frac{1}{2} R(\xi, \phi X) \xi + \frac{1}{2} \phi R(\xi, X) \xi. \]

On one hand, taking account of (2.11), (2.12), (3.7) and (3.8), we get
\[ R(\xi, X) \xi = \nabla_\xi (\nabla_X \xi) - \nabla_X (\nabla_\xi \xi) - \nabla_{[\xi, X]} \xi \]
\[ = -\nabla_\xi (\phi X + \phi h X) - \nabla_{[\xi, X]} \xi \]
\[ = -\phi \nabla_\xi X - \phi (\nabla_\xi h) X - \phi h \nabla_\xi X \]
\[ + \phi \nabla_\xi X - \phi \nabla_X \xi + \phi h (\nabla_X \xi - \nabla_X \xi) \]
\[ = -\phi (\nabla_\xi h) X + \phi (\phi X + \phi h X) + \phi h (\phi X + \phi h X) \]
\[ = -\phi (\nabla_\xi h) X - X + \eta (X) \xi + \phi (\phi h) X + \phi (h \phi) X + h \phi h X \]
\[ = -\phi (\nabla_\xi h) X - X + \eta (X) \xi + h^2 X, \]
and hence
\[ (\nabla_\xi h) X - \eta((\nabla_\xi h) X) \xi - \phi X + \phi h^2 X = \phi R(\xi, X) \xi. \] (3.19)

Here, from (2.12) and (3.9), we get
\[ \eta_h (\xi^k \nabla_k h_j i) = \xi^k \eta_h (\nabla_k h_j i) = -\xi^k (\nabla_k \eta_h) h_j i = 0. \] (3.20)

Thus, from (3.19) and (3.20), we have
\[ (\nabla_\xi h) X = \phi X - h^2 \phi X - \phi R(X, \xi) \xi, \]
for any \( X \in \mathfrak{X}(M). \) Thus, from (3.18)' and (3.21), we have also
\[ \phi X - h^2 \phi X - \phi R(X, \xi) \xi = -\frac{1}{2} R(\xi, \phi X) \xi + \frac{1}{2} \phi R(\xi, X) \xi. \] (3.22)
for any $X \in \mathfrak{X}(M)$. From (3.21), taking account of (3.7), we have also
\[
\phi^2 X - \phi^2 h^2 X - \phi^2 R(X, \xi) = -\frac{1}{2} \phi R(\xi, \phi X) + \frac{1}{2} \phi^2 R(\xi, X) \xi, \quad \text{and hence} \quad -X + \eta(X) \xi + h^2 X + R(X, \xi) - \eta(R(X, \xi) = -\frac{1}{2} \phi R(\xi, \phi X) - \frac{1}{2} R(\xi, X) + \frac{1}{2} \eta(R(\xi, X) \xi), \tag{3.23}
\]
for any $X \in \mathfrak{X}(M)$. Here, (3.23) is rewritten in terms of local coordinates as
\[
\delta_j^i + \eta_j^i \xi^j + h^a h^b + R_{ijab} \xi^a \xi^b = \frac{1}{2} \phi_{ab} R_{ac} \phi_j^c \xi^a \xi^j - \frac{1}{2} R_{ijab} \xi^a \xi^b,
\]
and hence
\[
-2n + tr h^2 + \rho_{ij} \xi^a \xi^b = -\frac{1}{2} \rho_{ij} \xi^a \xi^j + \frac{1}{2} \rho_{ab} \xi^a \xi^b = 0, \quad \text{namely}
\[
\rho(\xi, \xi) = 2n - tr h^2. \tag{3.24}
\]

**Remark 3.1** From (3.18'), we have the following equality:
\[
g(\nabla \xi h)X, Y) - g(\nabla \xi h)Y, X) = -\frac{1}{2} R(\xi, \phi X, \xi, Y) - \frac{1}{2} g(R(\xi, X) \xi, \phi Y)
+ \frac{1}{2} R(\xi, \phi Y, \xi, X) + \frac{1}{2} g(R(\xi, Y) \xi, \phi X) = 0, \tag{3.25}
\]
for any $X \in \mathfrak{X}(M)$.

(3.25) is rewritten in terms of local coordinates as:
\[
\xi^k (\nabla_{h_{ij}} - h_{ji}) = 0. \tag{3.25'}
\]

From (3.25)', we have further
\[
\xi || H_- ||^2 = 0, \tag{3.26}
\]
where $H_-$ is the skew-symmetric part of $H$, where $H(X, Y) = g(hX, Y)$ for any $X \in \mathfrak{X}(M)$. (3.26) is trivial in the case where $M = (M, \phi, \xi, \eta, g)$ is a contact metric manifold.

**4 Curvature identities and results**

In this section, we shall establish several curvature identities on a quasi contact metric manifold based on [4] the formulas obtained and also the ones in the previous sections of the present paper and further provide several results concerning the question introduced in § 1. Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi contact metric manifold and $\mathcal{M} = (M \times \mathbb{R}, \tilde{J}, \tilde{g})$ be the corresponding quasi Kähler cone endowed with the quasi Kähler structure $(\tilde{J}, \tilde{g})$ defined by (1.4). First, we recall the following curvature identity on a quasi Kähler manifold $\mathcal{M} = (\mathcal{M}, \tilde{J}, \tilde{g})$ (see [2]):
\[
\begin{align*}
R_{\alpha\mu\nu\kappa} &+ J^\alpha_\lambda J^\beta_\mu J^\gamma_\nu J^\rho_\kappa R_{\alpha\beta\gamma\rho} \\
- J^\alpha_\lambda J^\beta_\mu R_{\alpha\beta\nu\kappa} &- J^\gamma_\nu J^\rho_\kappa R_{\alpha\mu\nu\gamma} + J^\alpha_\lambda J^\rho_\kappa R_{\alpha\mu\gamma\rho} + J^\beta_\mu J^\rho_\kappa R_{\lambda\nu\mu\sigma} \\
&+ J^\alpha_\lambda J^\rho_\kappa R_{\alpha\mu\sigma\nu} + J^\beta_\mu J^\rho_\kappa R_{\lambda\beta\gamma\nu} \\
&= -2g_{\sigma\alpha}(\nabla_{J^\rho_\kappa} J^\alpha_\mu) - 2g_{\sigma\nu}(\nabla_{J^\alpha_\lambda} J^\rho_\kappa), \tag{4.1}
\end{align*}
\]
where we suppose that the Latin indices 1 ≦ λ, μ, ν, α, β, γ, ⋯ ≦ 2n + 2 = Δ and
a, b, c, i, j, k, l, ⋯ run over the range 1, 2, ⋯, 2n + 1.
Setting λ = i, μ = j, ν = k, κ = l in (4.1), and taking account of the formulas (see [8],
(2.26)), we have

\[ R_{ijkl} + φ_i^a φ_j^b φ_k^c φ_l^d R_{abcd} + φ_i^a φ_k^c R_{ajcd} + φ_j^b φ_l^d R_{abcd} - φ_i^a φ_j^b R_{abkl} \]
\[ - φ_k^c φ_l^d R_{ijkl} + φ_i^a φ_k^c R_{ajkl} + φ_j^b φ_l^d R_{abcd} + 4φ_j^b φ_l^d R_{jkl} - 4φ_i^a φ_j^b R_{abkl} \]
\[ = 2(∇_j φ_i^a)∇_a φ_{kl} + 2(∇_j φ_i^a)∇_a φ_{kl} - 2η_k (∇_j φ_i^a - ∇_j φ_{iL}) \]
\[ + 2η_l (∇_i φ_j^k - ∇_j φ_{ik}) - 2η_l (∇_j φ_{ik} + 2η_i ∇_j φ_{ik}). \]

Similarly, setting λ = Δ, μ = j, ν = k, κ = l in (4.1), we have

\[ ξ^a φ_j^b φ_k^c φ_l^d R_{abcd} + ξ^a φ_k^c R_{ajcd} - ξ^a φ_j^b R_{abkl} + ξ^a φ_l^d R_{ajkl} \]
\[ = 2(∇_j ξ^i)∇_a φ_{ki} + 2η_k (∇_j ξ^i) - 2η_l (∇_j ξ^i) - 2η_l (∇_j ξ^i) - 2η_l (∇_j ξ^i) - 2η_l (∇_j ξ^i). \]

Furthermore, setting λ = i, μ = Δ, ν = k, κ = Δ in (4.1), we have

\[ φ_i^a φ_j^b φ_k^c φ_l^d R_{abcd} + φ_i^b φ_j^c R_{abkl} + 4φ_i^a φ_j^b R_{abkl} \]
\[ = 2φ_i^a ∇_a φ_{ki} + 2φ_i^a ∇_a φ_{ki} + 2η_i (∇_j φ_{ki} - 2ξ^i ∇_j φ_{ki}). \]

Transvecting (4.2) with \( g^{il} \) and taking account of (3.11), we have

\[ ρ_{jk} + (g^{ad} - ξ^a ξ^d) φ_j^b φ_k^c R_{abcd} + φ_i^a φ_k^c R_{ajcd} + φ_j^b φ_l^d R_{abcd} - φ_i^a φ_j^b R_{abkl} \]
\[ - φ_k^c φ_l^d R_{ijkl} + (g^{ad} - ξ^a ξ^d) R_{ajkl} + ρ_{bc} φ_j^b φ_k^c - 8(n - 1)(g_{jk} - η_j η_k) \]
\[ = 2(∇_j φ_i^a)∇_a φ_{ki} - 2(∇_j φ_i^a)∇_a φ_{ki} - 2η_k (∇_i φ_j^k - 2ξ^i ∇_j φ_{ki}) \]
\[ - 2η_l (∇_i φ_j^k + 2η_i (∇_j φ_{ki} - 2ξ^i ∇_j φ_{ki})). \]

Here, taking account of (3.8), we get

\[ -2ξ^i ∇_j φ_{ki} + 2η_i (∇_j φ_{ki}) = 2ξ^i ∇_j φ_{ki} + 2η_i (∇_j φ_{ki}) \]
\[ = 4η_k (∇_j φ_{ki}) = -4(∇_j η_k) φ_{ki} = -4(∇_j η_k) φ_{ki} \]
\[ = 4φ_{ki}(φ_{ij} + φ_{ik} h_{ij}^a). \]

Thus, taking account of (3.13), the right-hand side of (4.5) reduces to

\[ -2(∇_j φ_i^a)∇_a φ_{ki} + 2(∇_j φ_i^a)∇_a φ_{ki} + 4(2n - 1)η_j η_k + 4g_{jk} + 4h_{jk}. \]

We have also

\[ (g^{ad} - ξ^a ξ^d) φ_j^b φ_k^c R_{abcd} = ρ_{bc} φ_j^b φ_k^c - φ_j^b φ_k^c R_{abcd} = ξ^a ξ^d, \]
\[ φ_i^a φ_k^c R_{ajcd} = \frac{1}{2} φ_i^a φ_k^c (R_{ajlc} - R_{jalc}) \]
\[ = -\frac{1}{2} φ_i^a φ_k^c R_{jcal} = -\frac{1}{2} φ_i^a φ_k^c R_{jcal} = -ρ_{jk}^a, \]
\[ φ_j^b φ_l^d R_{abcd} = \frac{1}{2} φ_j^b φ_l^d (R_{bdik} - R_{bdik}) = -\frac{1}{2} φ_j^b φ_l^d R_{bdik}. \]
Thus, (4.5) ~ (4.8), taking account of (2.4), we have
\[ 2\rho_{jk} - \phi_j^b \phi_k^c R_{abcd} \xi^a \xi^d - \xi^a \xi^d R_{ajkd} + 2 \rho_{bc} \phi_j^b \phi_k^c - 2 \rho^*_{jk} \]
\[ - 2\rho_{jk} - 8(n - 1) (g_{jk} - \eta_j \eta_k) = -2(\nabla^i \phi_j^a) \nabla_a \phi_{ki} + 2(\nabla_j \phi^a_i) \nabla_a \phi^{ji} + 8m \eta_j \eta_k + 4(g_{jk} - \eta_j \eta_k) + 4h_{jk}. \]  
Thus, transvecting (4.9) with \( g^{jk} \), we have
\[ 2\tau - 2\rho_{ab} \xi^a \xi^b + 2\tau - 2\rho_{bc} \xi^b \xi^c - 2\tau^* - 2\tau^* - 16(n - 1) \]
\[ = 2(\nabla^i \phi_j^a) \nabla_a \phi^i + 2(\nabla_j \phi^a_i) \nabla_a \phi^{ij} + 8m \eta_j \eta_k + 4(n - 1)(g_{jk} - \eta_j \eta_k) + 4h_{jk}. \]  
and hence, from (4.10) taking account of (3.24), we have
\[ 4(\tau - \tau^*) - 8n + 4trh^2 - 16n^2 = 2(\nabla_i \phi_{jk} - \nabla_j \phi_{ik}) \nabla^k \phi^{ij}. \]  
Here, we assume especially that \( M \) is a contact metric manifold. Then, \( d\Phi = 0 \) holds, where \( \Phi(X, Y) = g(X, \phi Y) \). Then, in this case, the right hand side of (4.11) reduces to
\[ 2(\nabla_i \phi_{ja} - \nabla_j \phi_{ia}) \nabla^a \phi^{ij} - 2(\nabla_i \phi_{ai} - \nabla^a \phi^{ij}) = -2(\nabla_a \phi_{ij}) \nabla^a \phi^{ij} = -2||\nabla \phi||^2. \]  
Thus, (4.11) reduces to
\[ \tau^* - \tau + 4n^2 - trh^2 = \frac{1}{2} ||\nabla \phi||^2 - 4n. \]  
This is nothing but the Olszak result [6]. Therefore, we see that (4.11) is a generalization of the Olszak formula.

**Theorem 4.1** Let \( M = (M, \phi, \xi, \eta, g) \) be a \((2n + 1)\)-dimensional quasi contact metric manifold with \( d\Phi = 0 \). Then, we have the equality (4.11). Further, the equality \( \tau^* - \tau + 4n^2 - trh^2 = 0 \) holds if and only if \( M \) is Sasakian.

**Proof.** From direct calculation, we have
\[ 0 \leq q(\nabla_i \phi_{jk} - g_{ij} \eta_k + \eta_j g_{ik})(\nabla^i \phi^{jk} - g^{ij} \xi^k + \xi^j g^{ik}) \]
\[ = ||\nabla \phi||^2 - (g^{ij} \nabla_i \phi_{jk}) \xi^k + (g^{ik} \nabla_i \phi_{jk}) \xi^j - (g_{ij} \nabla^i \phi^{jk}) \eta_k + (2n + 1) \]
\[ - (g_{ij} \xi^j g^{ik} \eta_k) + (g_{ik} \nabla^i \phi^j \eta_j) - (g_{ij} \xi^k \eta_j g_{ik}) + (2n + 1) \]  
\[ = ||\nabla \phi||^2 + (\nabla_i \phi^i) \xi^k + (\nabla_i \phi^i) \xi^j \]
\[ + (\nabla_i \phi^i) \xi^k + (\nabla_k \phi^i_j) \xi^j + 4n + 2 - 2 \]
\[ = ||\nabla \phi||^2 + 4(-2n) + 4n = ||\nabla \phi||^2 - 4n. \]

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Thus, from (4.11') and (4.12), taking account of the result (see [1], Theorem 6.3), we have immediately the required conclusion. □

Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi contact metric manifold. Then, from (3.10), it follows immediately that the characteristic vector field $\xi$ is never parallel. We here assume that the characteristic vector field $\xi$ of $M$ is a Killing vector field. Then we have

$$\nabla_i \eta_j + \nabla_j \eta_i = 0.$$  
(4.13)

From (4.13), we have easily

$$\nabla_i \nabla_i \eta_j + \rho_{jk} \eta_k = 0.$$  
(4.14)

Thus, transvecting (4.14) with $\xi^j$, we have

$$-\|\nabla \eta\|^2 + \rho_{jk} \xi^i \xi^k = 0.$$  
(4.15)

Thus, from (4.15), taking account of (3.10) and (3.24), we have

$$\|h\|^2 + tr h^2 = 0.$$  
(4.16)

Thus, we have the following.

**Theorem 4.2** Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi contact metric manifold with the Killing characteristic vector field $\xi$. Then, the tensor field $h$ is skew-symmetric.

**Proof.** We set

$$h_+ = \frac{1}{2} (h + \cdot h), \quad h_- = \frac{1}{2} (h - \cdot h),$$  
(4.17)

where $\cdot h$ is the transpose of $h$ with respect to the metric $g$. Then, from (4.17), we immediately get

$$h = h_+ + h_-, \quad \cdot h_+ = h_+, \quad \cdot h_- = -h_-$$  
(4.18)

Further, we get

$$\|h\|^2 = tr (h(\cdot h)) = tr ((\cdot h)h), \quad tr (h_+ h_-) = tr (h_- h_+) = 0.$$  
(4.19)

Thus, from (4.17), (4.18) and (4.19), we get

$$\|h\|^2 = tr (h(\cdot h)) = tr ((h_+ + h_-)(h_+ - h_-)) = tr (h_+)^2 - tr (h_-)^2,$$  
(4.20)

and

$$tr (h^2) = tr h_+^2 + tr h_-^2.$$  
(4.21)

Thus, from (4.16), taking account of (4.20) and (4.21), we have

$$tr h_+^2 = 0,$$  
(4.22)

and hence $h_+ = 0$ since $h_+$ is symmetric with respect to the metric $g$ and therefore, $h$ is skew-symmetric. This complete the proof of Theorem 4.2. □
Now it is well-known that a contact metric manifold \( M = (M, \phi, \xi, \eta, g) \) is Sasakian if and only if
\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y,
\]
holds for any \( X, Y \in \mathfrak{X}(M) \) (see [1], Proposition 7.6).

We have the following which is a generalization of the fact.

**Theorem 4.3** A quasi contact metric manifold \( M = (M, \phi, \xi, \eta, g) \) is Sasakian if and only if the equality (4.23) holds for any \( X, Y \in \mathfrak{X}(M) \).

**Proof.** The equality (4.23) is rewritten in terms of local coordinates
\[
R_{ijkl}^k \xi^l = \eta_j \delta^k_i - \eta_i \delta^k_j.
\]
Transvecting (4.2) with \( \xi^l \) and taking account of (4.23'), we have
\[
\nabla_i \phi_{jk} - \nabla_j \phi_{ik} + \eta_i \eta_{jk} - \eta_j \eta_{ik} = 0.
\]
Thus, transvecting (4.24) with \( \xi^k \) and taking account of (2.2), (3.8) and (3.9), we have
\[
\xi^k \nabla_i \phi_{jk} - \xi^k \nabla_j \phi_{ik} = 0.
\]
Here, taking account of (3.8) and (3.9), we get
\[
\xi^k \nabla_i \phi_{jk} = -\phi_{jk} \nabla_i \xi^k = \phi_{jk} (\phi_i^k + \phi_i^a h_{ai})
= g_{ji} - \eta_j \eta_i + (g_{ja} - \eta_j \eta_a) h_{ai} = g_{ji} - \eta_j \eta_i + h_{ij}.
\]
Thus, from (4.25) and (4.26), we have finally
\[
\eta_j \eta_i + h_{ij} = g_{ji} - (g_{ij} - \eta_j \eta_i) - h_{ji} = 0,
\]
and hence
\[
h_{ij} - h_{ji} = 0.
\]
Thus, it follows that \( M \) is a contact metric manifold by virtue of Theorem 4.1, and therefore, \( M \) is Sasakian.

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**References**