On hyperconnected topological spaces

Vinod Kumar · Devender Kumar Kamboj

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Abstract It is proved that there is only one open ultrafilter iff the space is hyperconnected. Relations of functions demi-open, skeletal and o-ultra, obtained without and with regard to hyperconnectedness, are used to prove that the full subcategory HypTop of Top, consisting of all hyperconnected spaces, is a subcategory of the category skelTOP of all topological spaces with morphisms skeletal continuous functions. HypTop is proved to be mono-epireflective in skelTop by showing that there exists a functor from skelTOP to HypTop turning skelTop into the largest subcategory of Top where the functor is a reflection. Also, skelTop is found to be a maximal subcategory of TOP such that HypTop is reflective in the subcategory with reflection morphisms skeletal functions.

Keywords Hyperconnected space · o-ultrafilter · Skeletal function and o-ultra function

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1 Introduction

There are topological spaces which are much more than being connected (see, e.g., [1,4,6,11]). This paper involves a category of such topological spaces. The term space is used for topological space. A space is hyperconnected (see [11]) if every two non-empty open sets intersect. A filter on a space whose members are open subsets is an open filter. Following that z-filter is used for a zero-set filter; o-filter is used for open filter in [5]. An o-ultrafilter is a maximal o-filter. MATHEW [7] characterized hyperconnected spaces in terms of a filter consisting of all non-empty semi-open sets of the space. In this paper we obtain a characterization of hyperconnectedness in terms of o-ultrafilters. We prove in ZF set theory that the open filter of open dense sets of a topological space is an open ultrafilter iff the space is hyperconnected. It is proved

Vinod Kumar
Centre for Advanced Study in Mathematics
Panjab University
Chandigarh, 160014, India
E-mail: vkvinod96@yahoo.co.in

Devender Kumar Kamboj
Govt. College Nahar (Rewari)
Haryana, 123303, India
E-mail: kamboj.dev81@rediffmail.com
using the Boolean Prime Ideal Theorem that a space is hyperconnected iff there is only one open ultrafilter. Relations of functions demi-open, skeletal and o-ultra, with regard to hyperconnectedness are studied. We characterize monomorphisms, epimorphisms, extremal monomorphisms and extremal epimorphisms in the full subcategory HypTop of Top consisting of all hyperconnected spaces. In [1] a functor is obtained from the category Top to its subcategory HypTop. It is worth noting that this functor cannot be a reflection from Top to HypTop. We use the skeletal functions of [8] and prove that HypTop is mono-epireflective in skelTop the category of all spaces with morphisms skeletal continuous functions. This is proved by showing that there exists a functor from skelTOP to HypTop turning skelTop into the largest subcategory of Top where the functor is a reflection. For a general reflection functor, skelTop is found to be a maximal subcategory of TOP such that HypTop is reflective in the subcategory with reflection morphisms skeletal functions.

2 Notation, definitions and preliminaries

For notation and definitions, we shall mainly follow [10]. For completeness sake, we have included some of the standard notation and definitions.

Let \((X, \tau)\) be a topological space. For \(A \subseteq X\), \(\text{int}_\tau(A)\) and \(\text{cl}_\tau(A)\) denote \(\tau\)-interior and \(\tau\)-closure of \(A\), respectively. An filter \(F\) on \((X, \tau)\) is a prime filter if for open sets \(A, B \subseteq X\), \(A \cup B \in F\) only if \(A \in F\) or \(B \in F\). An ultrafilter on \((X, \tau)\) is a maximal filter. For an ultrafilter \(F\) on \((X, \tau)\), the filter \(\{A \in \tau : \text{cl}_\tau(\text{int}_\tau(A)) \in F\}\) is called the \textit{rounding} of \(F\) (see [5]), and \(F\) is said to be \textit{round} if \(rF = F\). \(\text{od}_X\) is used to denote the filter \(\{G \in \tau : \text{cl}_\tau(G) = X\}\). For \(x \in X\), \(\alpha x\) denotes the filter \(\{G \in \tau : x \in G\}\). A subset \(A\) of \(X\) is an \(\alpha\)-open set (see [10]) if \(A \subseteq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))\).

The collection of all \(\alpha\)-open sets of \(X\) becomes a topology on \(X\) denoted by \(\alpha \tau\). A space \((X, \tau)\) is extremally connected (see [4]) if the closures of every two non-empty open sets intersect.

Let \((X, \tau)\) and \((Y, \mu)\) be two spaces and \(f : (X, \tau) \rightarrow (Y, \mu)\) a function. For an ultrafilter \(F\) on \(X\), \(fF\) is the ultrafilter \(\{B \in \mu : f(A) \subseteq B\) for some \(A \in F\}\) on \(Y\). \(f\) is (i) demi-open (see [2]) if for every \(A \in \tau\), \(\text{int}_\tau(\text{cl}_\tau(f(A))) \neq \emptyset\); (ii) skeletal (see [8]) if for every \(B \in \mu\), \(\text{int}_\mu(f^{-1}(\text{cl}_\mu(B))) \subseteq \text{cl}_\mu(f^{-1}(B))\); (iii) ultra (see [5]) if for every ultrafilter \(F\) on \(X\), \(fF\) is an \(\alpha\)-ultrafilter on \(Y\). For a set \(X\), \(id_X\) is used for the \textit{identity function} on \(X\).

It is clear that a space is hyperconnected iff every non-empty open set is dense. Thus:

**Remark 2.1** \((X, \tau)\) is hyperconnected iff \(\tau = \text{od}_X \cup \{\emptyset\}\).

Skeletal, demi-open and \(\alpha\)-ultra functions are known to be equivalent with the assumption of continuity of the function (see [5]). In fact, it can be seen that there is no need of continuity to prove the equivalence of the concepts of demi-open and skeletal functions.

**Remark 2.2** Let \((X, \tau)\) and \((Y, \mu)\) be two spaces and \(f : X \rightarrow Y\) a function:

(a) For an ultrafilter \(F\) on \(X\), \(B\) open in \(Y\), \(B \in fF\) iff \(\text{int}_\tau(f^{-1}(B)) \in F\).

(b) The following are equivalent:

(i) \(f\) is demi-open.

(ii) Inverse image of every open dense subset of \(Y\) is dense in \(X\).

(iii) \(f\) is skeletal.
3 Hyperconnected spaces and o-ultrafilters

Theorem 3.1 The following are equivalent for a space \((X, \tau)\):

(i) \(odX_\tau\) is an o-ultrafilter on \((X, \tau)\).

(ii) \(odX_\tau \supseteq F\), for every o-filter \(F\) on \((X, \tau)\).

(iii) \((X, \tau)\) is hyperconnected.

Proof. (i) \(\Rightarrow\) (ii) For a non-empty open set \(A\) of \(X\), \(odX_\tau\) is contained in the o-filter generated by \(\{G \cap A : G \in odX_\tau\}\). Since \(odX_\tau\) is an o-ultrafilter, \(A \in odX_\tau\). Thus \(F \subseteq odX_\tau\), for every o-filter \(F\) on \((X, \tau)\).

(ii) \(\Rightarrow\) (iii) Let \(F\) be the o-filter generated by a non-empty open set \(A\) of \(X\). Since \(FodX\), \(AodX\). Therefore \((X, \tau)\) is hyperconnected.

(iii) \(\Rightarrow\) (i) By Remark 2.1, if \((X, \tau)\) is hyperconnected, then \(odX_\tau = \tau \setminus \{\emptyset\}\). Since every o-filter \(F\) is a subcollection of \(\tau \setminus \{\emptyset\}\), \(F \subseteq odX_\tau\). Thus \(odX_\tau\) is an o-ultrafilter on \((X, \tau)\). \(\Box\)

Theorem 3.2 A space \((X, \tau)\) is hyperconnected iff it has a largest o-filter.

Proof. If \((X, \tau)\) is hyperconnected, then by Theorem 3.1, \(odX_\tau\) is the largest o-filter of it. Now suppose that \((X, \tau)\) has a largest o-filter \(F\). For non-empty open sets \(A\) and \(B\) of \(X\), let \(F_A\) and \(F_B\) be the o-filters generated by \(A\) and \(B\) respectively. Since \(F_A\) and \(F_B\) are contained in \(F\), \(A \cap B \neq \emptyset\). Therefore \((X, \tau)\) is hyperconnected. \(\Box\)

Proposition 3.3 The following are equivalent for a space \((X, \tau)\):

(i) \((X, \tau)\) is hyperconnected.

(ii) There is only one o-ultrafilter on \((X, \tau)\).

Proof. (i) \(\Rightarrow\) (ii) follows using Theorem 3.1.

(ii) \(\Rightarrow\) (i). By Remark 4 (d) of \([5]\), \(odX_\tau = \bigcap\{F : F\) is an o-ultrafilter on \((X, \tau)\}\). Since there is only one o-ultrafilter on \((X, \tau)\), \(odX_\tau\) is an o-ultrafilter. \(\Box\)

Remark 3.1 It is worth noting that Theorems 3.1 and 3.2, and (i) implies (ii) in Proposition 3.3 are proved in \(ZF\) set theory as Zorn’s lemma or any of its equivalent or weaker forms is not used while proving these. To prove (ii) implies (i) in Proposition 3.3, we have used Remark 4 (d) of \([5]\) the proof of which uses: An o-filter on a space is contained in an o-ultrafilter. We also note that (ii) implies (i) in Proposition 3.3 is not possible without the use of the Boolean Prime Ideal Theorem, in fact, is equivalent to the Boolean Prime Ideal Theorem in \(ZF\) set theory.

For every function, the concepts of demi-open and skeletal are equivalent by Remark 2.2 (b). Now, we find their relation with the concept of o-ultra functions.

Lemma 3.4 Let \((X, \tau)\) and \((Y, \mu)\) be two spaces. If \(f : (X, \tau) \rightarrow (Y, \mu)\) is o-ultra, then it is skeletal.

Proof. In view of Remark 2.2 (b), to prove that \(f : (X, \tau) \rightarrow (Y, \mu)\) is skeletal, it is sufficient to show that inverse image of every open dense subset of \((Y, \mu)\) is dense in \((X, \tau)\). For an open and dense subset \(B\) of \(Y\), let \(A = X \setminus cl_\tau(f^{-1}(B))\). If \(A \neq \emptyset\), we have an o-ultrafilter \(F\) on \((X, \tau)\) containing \(A\). Then \(fF\) is an o-ultrafilter as \(f\) is o-ultra. Since \(B\) is open and dense in \(Y\), \(B \in fF\). By Remark 2.2 (a), \(int_\tau((f^{-1}(B))) \in F\). So \(A \cap int_\tau((f^{-1}(B))) \neq \emptyset\) as both \(A\) and \(int_\tau((f^{-1}(B)))\) are in \(F\). Therefore, \(A \cap cl_\tau(f^{-1}(B)) \neq \emptyset\), which is not possible as \(A = X \setminus cl_\tau(f^{-1}(B))\). Thus \(A = \emptyset\). Hence \(X = cl_\tau(f^{-1}(B))\). \(\Box\)
For a space \((X, \tau)\), \(odX_\tau \cup \{\emptyset\}\) is a topology on \(X\) to be denoted by \(\tau D\). The converse of Lemma 3.4 is not true. For this, we have the following:

**Proposition 3.5** Let \((X, \tau)\) be a space. Then:

(i) \(id_X : (X, \tau D) \rightarrow (X, \tau)\) is skeletal.
(ii) \(id_X : (X, \tau D) \rightarrow (X, \tau)\) is o-ultra iff \((X, \tau)\) is hyperconnected.
(iii) If \((X, \tau)\) is not hyperconnected, then \(id_X : (X, \tau D) \rightarrow (X, \tau)\) is skeletal but it is not o-ultra.

**Proof.** (i) follows using Remark 2.2 (b) and the definition of \(\tau D\).

(ii) If \((X, \tau)\) is hyperconnected, then \(\tau = \tau D\) by Remark 2.1, and therefore \(id_X : (X, \tau D) \rightarrow (X, \tau)\) is o-ultra. Now suppose \(id_X : (X, \tau D) \rightarrow (X, \tau)\) is o-ultra. Let \(\mu = \tau D\). Since \((X, \tau D)\) is hyperconnected, by Theorem 3.1, \(odX_\mu\) is o-ultrafilter on \((X, \tau D)\).

It can be seen that \(id(\muD) = odX_\tau\). Therefore, \(odX_\tau\) is an o-ultrafilter on \((X, \tau)\) as \(id_X : (X, \tau D) \rightarrow (X, \tau)\) is o-ultra. Now, by Theorem 3.1, \((X, \tau)\) is hyperconnected.

(iii) follows from (i) and (ii). □

**Theorem 3.6** For every space \((X, \tau)\), \((X, \tau D)\) is a hyperconnected space.

**Proof.** Let \(\mu = \tau D\). So in view of Remark 2.1, and the definition of \(\mu D\) and \(\tau D\), it is sufficient to prove that \(odX_\mu = odX_\tau\). Since \(\mu D \subset \tau D\), it follows that \(odX_\mu \subset odX_\tau\).

Let \(A \in odX_\tau\). Let \(G \in \tau D\). \(G \neq \emptyset\). Since \(\tau D \subset \tau\), by \(\tau\)-denseness of \(A\) we have \(A \cap G \neq \emptyset\). Therefore \(A \in odX_\mu\). The proof is complete. □

**Corollary 3.7** \((X, \tau)\) is hyperconnected iff \(\tau = \tau D\).

**Lemma 3.8** Let \((X, \tau)\) and \((Y, \mu)\) be two spaces and \(f : (X, \alpha\tau) \rightarrow (Y, \mu)\) continuous. If \(F\) is an o-ultrafilter on \((X, \tau)\), then, with \(f : (X, \tau) \rightarrow (Y, \mu)\), \(fF\) is a prime o-filter on \((Y, \mu)\).

**Proof.** Let \(G \cup H \in fF\) with \(G, H\) open in \(Y\). Then \(f(V) \subset G \cup H\) for some \(V\) in \(F\), and so \(V \subset f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)\).

As \(f : (X, \alpha\tau) \rightarrow (Y, \mu)\) is continuous, therefore \(V \subset (int_r(cl_r(int_r(f^{-1}(G)))) \cup (int_r(cl_r(int_r(f^{-1}(H))))).

Since \(V \in F\), \((int_r(cl_r(int_r(f^{-1}(G)))) \cup (int_r(cl_r(int_r(f^{-1}(H)))) \in F\). But \(F\) being o-ultrafilter is prime o-filter (by Proposition 3.3 of [5]), either \((int_r(cl_r(int_r(f^{-1}(G)))) \in F\) or \((int_r(cl_r(int_r(f^{-1}(H)))) \in F\). Also \(F\) being o-ultrafilter is round o-filter, by Proposition 3 (c) of [5]). Therefore either \(int_r(f^{-1}(G)) \in F\) or \(int_r(f^{-1}(H)) \in F\) (by definition of round o-filter). Now by Remark 2.2 (a), either \(G \in fF\) or \(H \in fF\). Thus \(fF\) is a prime o-filter. □

**Theorem 3.9** Let \((X, \tau)\) and \((Y, \mu)\) be two spaces and \(f : (X, \alpha\tau) \rightarrow (Y, \mu)\) continuous. The following are equivalent:

(i) \(f : (X, \tau) \rightarrow (Y, \mu)\) is skeletal.
(ii) For every round o-filter \(F\) on \((X, \tau)\), \(fF\) is a round o-filter with \(f : (X, \tau) \rightarrow (Y, \mu)\).
(iii) \(f : (X, \tau) \rightarrow (Y, \mu)\) is o-ultra.

**Proof.** (i) \(\implies\) (ii). For a round o-filter \(F\) on \(X\), let \(B\) be open in \(Y\) such that \(int_r(cl_r(B)) \in fF\). Then by Remark 2.2 (a), \(int_r(f^{-1}(int_r(cl_r(B)))) \in F\).
Since \( f : (X, \tau) \to (Y, \mu) \) is skeletal, \( \text{int}_\tau((f^{-1}(\overline{B}))) \subset \overline{f^{-1}(B)} \); so \( \text{int}_\tau((f^{-1}(\overline{B}))) \subset \text{int}_\tau(f^{-1}(B)) \).

Therefore, \( \text{int}_\tau((f^{-1}(\overline{B}))) \subset \text{int}_\tau(\overline{f^{-1}(B)}) \) as \( f : (X, \alpha\tau) \to (Y, \mu) \) is continuous. This implies that \( \text{int}_\tau(\overline{f^{-1}(B)}) \) is an o-filter, so \( \text{int}_\tau(f^{-1}(B)) \in F \). But \( F \) is round o-filter, so \( \text{int}_\tau(f^{-1}(B)) \in \text{int}_\tau(f^{-1}(B)) \). Again by Remark 2.2 (a), \( B \in fF \). Hence \( fF \) is a round o-filter.

(ii) \( \implies \) (iii). If \( F \) is an o-ultrafilter, then by Proposition 3 (c) of [5], \( F \) is a round o-filter and a prime o-filter. Therefore \( fF \) is round o-filter. Thus, by Proposition 3 (c) of [4], \( fF \) is an o-ultrafilter.

(iii) \( \implies \) (i) follows by Lemma 3.4. \( \square \)

We note that Proposition 6 of [5] follows as a corollary of Theorem 3.9.

**Theorem 3.10** Let \((X, \tau)\) and \((Y, \mu)\) be two spaces and \( f : (X, \alpha\tau) \to (Y, \mu) \) continuous. Then \( f : (X, \tau D) \to (Y, \mu D) \) is continuous iff \( f : (X, \tau) \to (Y, \mu) \) is skeletal.

**Proof.** Since \( f : (X, \tau) \to (Y, \mu) \) is continuous, using the definition of \( \mu D \) and \( \tau D \), \( f : (X, \tau D) \to (Y, \mu D) \) is continuous iff inverse image of every open dense subset in \((Y, \mu)\) is dense in \((X, \tau)\). Now, the result follows using Remark 2.2 (b). \( \square \)

**Corollary 3.11** Let \((X, \tau)\) be a space. Then \( \text{id}_X : (X, \tau) \to (X, \tau D) \) is continuous and skeletal.

**Corollary 3.12** Every continuous function with hyperconnected domain is skeletal.

**Proof.** If \( f : (X, \tau) \to (Y, \mu) \) is continuous, then \( f : (X, \tau) \to (Y, \mu D) \) is continuous. Therefore, for a hyperconnected space \((X, \tau)\), \( f : (X, \tau D) \to (Y, \mu D) \) is continuous. \( \square \)

**Remark 3.2** For a space \((X, \tau)\), \( \text{id}_X : (X, \tau) \to (X, \tau D) \) is continuous, skeletal and dense, but the space \((X, \tau D)\) is not an extension of \((X, \tau)\). If \((Y, \mu)\) is a hyperconnected space and there exists a continuous and skeletal function, \( j : (X, \tau) \to (Y, \mu) \). Then, using Corollary 3.7 and Theorem 3.10, there exists a continuous function \( k : (X, \tau D) \to (Y, \mu D) \) such that \( k \circ \text{id}_X = j \); \( k \) is skeletal by Corollary 3.12. Thus \((X, \tau D)\) can be said to be the “largest” hyperconnected space obtained from \((X, \tau)\).

Even a closed subspace of a hyperconnected space need not be hyperconnected (see [1]). However if a subspace of a hyperconnected space is either regularly closed or open or dense, then it is hyperconnected. The following result gives necessary and sufficient conditions for a subspace of a hyperconnected space to be hyperconnected.

**Theorem 3.13** A subspace \((Y, \mu)\) of a hyperconnected space \((X, \tau)\) is hyperconnected iff the inclusion function \( i : (Y, \mu) \to (X, \tau) \) is skeletal.

**Proof.** If \((Y, \mu)\) is hyperconnected, then by Corollary 3.12, \( i \) is skeletal. Conversely, let \( i \) be skeletal. To prove that \((Y, \mu)\) is hyperconnected, let \( A \) be non-empty open set in \((Y, \mu)\). Then \( A = B \cap Y \) for some non-empty set \( B \) open in \((X, \tau)\). Since \((X, \tau)\) is hyperconnected, so \( B \) is dense in \((X, \tau)\). Since \( i \) is skeletal, by Remark 2.2 (b), \( i^{-1}(B) = B \cap Y = A \) is dense in \((Y, \mu)\). \( \square \)
4 Some categorical aspects of hyperconnected spaces

For the definitions in this section we follow [10].

**Definition 4.1** We say that a full subcategory $\mathcal{C}$ of Top is a Sierpinsky subcategory of Top if:

(i) for an object $X$ of $\mathcal{C}$ and $x \in X$, $\{x\}$ is an object in $\mathcal{C};$

(ii) for every object $X$ in $\mathcal{C}$, $X \times \{0,1\}$ is an object in $\mathcal{C}$, where $\{0,1\}$ has the Sierpinsky topology;

(iii) continuous image of an object in $\mathcal{C}$ is an object in $\mathcal{C}.$

**Theorem 4.2** (i) The monomorphisms in a Sierpinsky subcategory $\mathcal{C}$ of Top are precisely the one-one continuous functions.

(ii) The epimorphisms in a Sierpinsky subcategory $\mathcal{C}$ of Top are precisely the onto continuous functions.

**Proof.** (i) It is obvious that one to one continuous function between objects of $\mathcal{C}$ is a monomorphism. Conversely, let $m : X \to Y$ be a monomorphism in $\mathcal{C}$ which is not one to one. Then, there are two distinct elements $x_1$ and $x_2$ in $X$ such that $m(x_1) = m(x_2).$ Let $Z$ be the singleton space $\{z = x_1\}.$ For $i = 1, 2,$ define $f_i : Z \to X$ by $f_i(z) = x_i.$ Then $Z,$ $f_1$ and $f_2$ belong to the category $\mathcal{C},$ $f_1 \neq f_2$ and $m \circ f_1 = m \circ f_2.$ This shows that $m$ is not a monomorphism. Thus monomorphisms in $\mathcal{C}$ are precisely the one to one continuous maps.

(ii) It is easy to verify that onto continuous functions between objects of $\mathcal{C}$ are epimorphisms. Conversely let $f : X \to Y$ be an epimorphism in $\mathcal{C}.$ Suppose $f(X) \neq Y.$ The product space $Y \times \{0,1\}$ is an object in $\mathcal{C}.$ The set $Z = \{(f(X) \times \{0,1\}), ((Y \setminus f(X)) \times \{0\}), ((Y \setminus f(X)) \cup \{1\})\}$ being a partition of $Y \times \{0,1\}$ is the set of equivalence classes of an equivalence relation on $Y \times \{0,1\}.$ Let $q : Y \times \{0,1\} : \to Z$ be the quotient function. For $i \in \{0,1\},$ $q(y, i) = (\{Y \setminus f(X)\} \times \{i\})$ if $y \notin f(X)$ and $q(y, i) = f(X) \times \{0,1\}$ otherwise. Then $Z$ is an object in $\mathcal{C}.$ Let $i \in \{0,1\}, j_i : Y \to Y \times \{0,1\}$ defined as $j_i(y) = (y, i)$ is continuous; therefore $f_j = q \circ j_i$ is continuous. For $x$ in $X,$ $(f_0 f)(x) = (f_1 f)(x) = q(j_i((f(x)))) = q(f(x), i)) = f(X) \times \{0,1\}.$ Thus $f_0 \circ f = f_1 \circ f.$ But for $y \in Y \setminus f(X),$ $f_0(y) \neq f_1(y);$ therefore $f_0 \neq f_1.$ This gives a contradiction as $f$ is an epimorphism. Hence $f$ is onto. $\Box$

**Theorem 4.3** The extremal monomorphisms in a Sierpinsky subcategory $\mathcal{C}$ of Top are precisely the embeddings.

**Proof.** First suppose that $m : X \to Y$ is an extremal monomorphism in $\mathcal{C}.$ Let $Z = m(X).$ Then $Z$ is an object in $\mathcal{C}$ and $m : X \to Z$ being an onto morphism is an epimorphism in $\mathcal{C}.$ The inclusion function $i : Z \to Y$ is a morphism in $\mathcal{C}$ and $m = i \circ m.$ Since $m$ is extremal monomorphism, so $m$ is an isomorphism. But isomorphisms are homeomorphism in $\mathcal{C}.$ Thus $m$ is an embedding. Conversely suppose that $m : X \to Y$ is an embedding in $\mathcal{C}.$ $m$ is a monomorphism in $\mathcal{C}.$ For an object $Z$ in $\mathcal{C},$ let $f : X \to Z$ be an epimorphism in $\mathcal{C}$ and $h : Z \to Y$ a morphism in $\mathcal{C}$ such that $m = h \circ f.$ Then by Theorem 4.2 (ii), $f$ is an onto function, that is $f(X) = Z.$ So $h(Z) = m(X).$ For $z \in Z,$ we have a unique $x_z \in X$ such that $h(z) = m(x_z);$ we define $s : Z \to X$ by $s(z) = m(x_z).$ Since $m = h \circ f,$ it follows that, for $x$ in $X,$ $(s \circ f)x = s(f(x)) = x = id_X(x).$ Thus $f$ has a left inverse. Since $f$ is an epimorphism and has a left inverse, therefore $f$ is an isomorphism by 9.4 (f)(3) of [10]. This shows that $m$ is an extremal monomorphism in $\mathcal{C}.$ $\Box$

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Theorem 4.4 The extremal epimorphisms in a Seierpinsky subcategory \( \varsigma \) of \( \text{Top} \) are precisely the quotient functions.

Proof. First suppose that \( m : X \to Y \) is an extremal epimorphism in \( \varsigma \). Then by Theorem 4.2 (ii), \( m \) is onto continuous function. \( Y^* = Y \), the space with quotient topology w.r.t. \( m \) is an object in \( \varsigma \). \( id_Y : Y^* \to Y \) being continuous is a monomorphism in \( \varsigma \). Also, \( id_Y \circ m = m \). Since \( m \) is extremal epimorphism, \( id_Y \) is an isomorphism. Therefore \( m \) is a quotient function.

Conversely, suppose that \( m : X \to Y \) is a quotient function, where \( X \) and \( Y \) are objects in \( \varsigma \). But \( m \) being onto it is an epimorphism in \( \varsigma \). Let \( f : Z \to Y \) be a monomorphism in \( \varsigma \) and \( h : X \to Z \) a morphism in \( \varsigma \) such that \( m = f \circ h \). Then by Theorem 4.2 (i), \( f \) is one-one. Since \( m = f \circ h, Y = f(h(X)) \). For \( y \in Y \), we have the unique \( x_y \in X \) such that \( y = f(h(x_y)) \); we define a function \( s : Y \to Z \) by \( s(y) = h(x_y) \) for \( y \in Y \). Now, for any \( y \in Y \), we have \( (f \circ s)y = f(s(y)) = f(h(x_y)) = y = id_Y(y) \). Thus, \( f \) has a right inverse. Since \( f \) is a monomorphism and has a right inverse, so by Theorem 9.4 (f)(2) of [10], \( f \) is an isomorphism. This shows that \( m \) is an extremal epimorphism in \( \varsigma \). \( \square \)

Remark 4.1 It is proved in [6] that continuous image of hyperconnected space is hyperconnected and product of hyperconnected spaces is hyperconnected. So HypTop is a Seierpinsky subcategory of \( \text{Top} \). Let \( \text{ECTop} \) be the full subcategory of \( \text{Top} \) consisting of all extremally connected spaces. We prove that \( \text{ECTop} \) is one more example of a Seierpinsky subcategory of \( \text{Top} \). For this we need the following proposition, which strengthens Proposition 2.12 of [4].

Proposition 4.5 Let \( f : X \to Y \) be a continuous map. If \( X \) is extremally connected, then \( f(X) \) is extremally connected.

Proof. We suppose that \( Y = f(X) \). Let \( H \) and \( K \) be two non-empty open subsets of \( Y \). Since \( f \) is continuous, it follows \( f^{-1}(H) \) and \( f^{-1}(K) \) are non-empty open sets of \( X \). As \( X \) is an extremally connected space, \( \text{cl}_H f^{-1}(H) \cap \text{cl}_X f^{-1}(K) \neq \emptyset \). This implies using continuity of \( f \) that \( f^{-1}(\text{cl}_Y H) \cap f^{-1}(\text{cl}_Y K) \neq \emptyset \). This shows that \( \text{cl}_Y H \cap \text{cl}_Y K \neq \emptyset \). Hence the result. \( \square \)

Corollary 4.6 \( \text{ECTop} \) is a Seierpinsky subcategory of \( \text{Top} \).

Remark 4.2 By Theorem 4.2 (ii) and Corollary 4.6, the epimorphisms in \( \text{ECTop} \) are precisely the onto continuous functions. This strengthens Theorem 4.1 (ii) of [4]. Also by Theorem 4.3 and Corollary 4.6, the extremal monomorphisms in \( \text{ECTop} \) are precisely the embeddings and thus Theorem 4.2 of [4] is strengthened.

Remark 4.3 In [1], Ajmal and Kohli find a functor from the category \( \text{Top} \) to its subcategory \( \text{HypTop} \) by obtaining one point hyperconnectification of every space. It be noted that this functor can not be a reflection because their one point hyperconnectification of a hyperconnected space is different from it, a necessary condition for the functor to be a reflection. Here we find a functor from the category \( \text{skelTop} \) to the category \( \text{HypTop} \) which turns out to be a mono-epireflection.

Remark 4.4 We define a correspondence \( H_y \) from \( \text{Top} \) to \( \text{HypTop} \). For an object \( (X, \tau) \) of \( \text{Top} \), by Theorem 3.6, \( (X, \tau_D) \) is an object of \( \text{HypTop} \). We say that \( H_y \) takes \( (X, \tau) \) to \( (X, \tau_D) \), and for a morphism \( f : (X, \tau) \to (Y, \mu) \) in \( \text{Top} \), \( H_y(f) \) is the function \( f : (X, \tau_D) \to (Y, \mu_D) \).
Theorem 4.7 HypTop is a subcategory of skelTop, $H_y$ is a functor from skelTop to HypTop and HypTop is a mono-epireflective in skelTop.

Proof. By Corollary 3.11, every continuous function between two hyperconnected spaces is skeletal. Therefore HypTop is a subcategory of skelTop. If $f : (X, \tau) \to (Y, \mu)$ is a morphism in skelTop, then, by Theorem 3.10, $f : (X, \tau D) \to (Y, \mu D)$ is a morphism in HypTop. Thus $H_y$ is a functor from skelTop to HypTop. Also $id_X : (X, \tau) \to (X, \tau D)$ is a morphism in HypTop by Corollary 3.11. Now, in view of Corollary 3.7, the functor $H_y$ becomes a reflection. The reflection morphism $id_X : (X, \tau) \to (X, \tau D)$ being one-one and onto is a monomorphism and an epimorphism in HypTop. □

Remark 4.5 Now there arises a question of finding the largest subcategory of Top in which HypTop is reflective. What follows gives two partial answers to this question.

Lemma 4.8 Let $B$ be a subcategory of Top such that the correspondence $H_y$ is a functor from $B$ to HypTop, then $B$ is a subcategory of skelTop.

Proof. Since the objects of Top and skelTop are same, let $f : (X, \tau) \to (Y, \mu)$ be a morphism in $B$. $H_y$ is a functor from $B$ to HypTop, therefore $H_y(f) = f : (X, \tau D) \to (Y, \mu D)$ is continuous. Now by Theorem 3.10, $f : (X, \tau) \to (Y, \mu)$ is a morphism in skelTop. □

In view of the above result, we have

Proposition 4.9 skelTop is the largest subcategory of Top in which HypTop is reflective if the reflection is given by the correspondence $H_y$.

To obtain another answer of the question posed in the Remark 4.7, first we note the following observations:

Remark 4.6 Let $f : (X, \tau) \to (Y, \mu)$ be a continuous function, where $(X, \tau), (Y, \mu) \in Top$. If there exists a continuous function $g : (X, \tau D) \to (Y, \mu D)$ such that $g \circ id_X = id_Y \circ f$, then $g = f$. Therefore by Theorem 3.10, $f$ is a skeletal function.

Remark 4.7 If $B$ is the largest subcategory of Top in which HypTop is reflective, then by Theorem 4.10, $B$ contains skelTop. Therefore to know about the largest subcategory, say $B$, of Top in which HypTop is reflective, we can suppose that $B$ contains skelTop.

Theorem 4.10 Let $B$ be a subcategory of Top containing skelTop such that HypTop is reflective in $B$. Then $B = skelTop$ iff the reflection morphisms of $B$ are skeletal.

Proof. Let $H : B \to HypTop$ be the reflection, and for $(X, \tau) \in B$, $h_X : (X, \tau) \to H(X, \tau)$ the reflection morphism. If $B = skelTop$, then $H = H_y$ and therefore the reflection morphisms are skeletal. Conversely, for $(X, \tau) \in B$, $h_X : (X, \tau) \to H(X, \tau)$ is skeletal, therefore by Theorem 4.10, there exists a continuous function $j : (X, \tau D) \to H(X, \tau)$ such that $j$ is skeletal and $j \circ id_X = h_X$. Since $id_X : (X, \tau) \to (X, \tau D)$ is a morphism in $B$ and $H$ is a reflection, therefore there exists a continuous function $k : H(X, \tau) \to (X, \tau D)$ such that $k$ is a morphism in $B$ and $k \circ h_X = id_X$. Hence $k$ is skeletal by Corollary 3.11. Thus $k \circ j$ is skeletal. By reflectivity of $H_y$, $k \circ j = id_{H_y}(X, \tau)$. Since $B$ contains skelTop, $j \circ k$ is a morphism in $B$. By reflectivity of
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H, j \circ k = id_H(X, \tau). Now it follows that H(X, \tau) and (X, \tau_D) are isomorphic. The objects of skelTop and B are same, so it is enough to prove the result for morphisms of B. For objects (X, \tau) and (Y, \mu) of B, let f : (X, \tau) \to (Y, \mu) be a morphism in B. f : (X, \tau) \to (Y, \mu) is continuous. H(f) : H(X, \tau) \to H(Y, \mu) is continuous. Taking H(X, \tau) = (X, \tau_D), h_X = id_X. Thus H(f) : (X, \tau_D) \to (Y, \mu_D) is such that g \circ id_X = id_Y \circ f. Now, by Remark 4.6, it follows that f is skeletal. This proves that B is contained in skelTop. □

Corollary 4.11 skelTop is a maximal subcategory of TOP such that HypTop is reflective in the subcategory with reflection morphisms skeletal functions.

References