Some further study on Brück conjecture

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Abstract We study some important problems of relation between a meromorphic function and its differential polynomial sharing a small function which are the continuation of a famous conjecture proposed by Brück. We radically improve a number of recent results to a large extend in a new direction.

Keywords Meromorphic function · Derivative · Small function

Mathematics Subject Classification (2010) 30D35

1 Introduction definitions and results

In the paper, by meromorphic function we always mean a function which is meromorphic in the open complex plane $\mathbb{C}$.

If for some $a \in \mathbb{C} \cup \{\infty\}$, $f$ and $g$ have same set of $a$-points with the same multiplicities, we say that $f$ and $g$ share the value $a$ CM (counting multiplicities) and if we do not consider the multiplicities then $f$, $g$ are said to share the value $a$ IM (ignoring multiplicities). When $a = \infty$ the zeros of $f - a$ means the poles of $f$. Let $m$ be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_m(a; f)$ the set of all $a$-points of $f$ with multiplicities not exceeding $m$, where an $a$-point is counted according to its multiplicity. Also we denote by $\overline{E}_m(a; f)$ the set of distinct $a$-points of $f(z)$ with multiplicities not greater than $m$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_m(a, f) = E_m(a, g)$ ($\overline{E}_m(a, f) = \overline{E}_m(a, g)$) holds for $m = \infty$ we say that $f$, $g$ share the value $a$ CM (IM).

It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $f$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, ($r \to \infty$, $r \notin E$).

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A meromorphic function $a(\neq \infty)$ is called a small function with respect to $f$ provided that $T(r, a) = S(r, f)$ as $(r \to \infty, r \not\in E)$. If $a = a(z)$ is a small function we define that $f$ and $g$ share a IM or a CM according as $f - a$ and $g - a$ share 0 IM or 0 CM respectively.

We use $I$ to denote any set of infinite linear measure of $0 < r < \infty$.

Also it is known to us that the hyper order of $f(z)$, denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$  

The subject on sharing values between entire functions and their derivatives was first studied by RUBEL and YANG [15].

In 1977 they proved that if a non-constant entire function $f$ and $f'$ share two distinct finite numbers $a$, $b$ CM, then $f = f'$.

In 1979, analogous result for IM sharing was obtained by Mues and Steinmetz in the following manner.

**Theorem A ([14])** Let $f$ be a non-constant entire function. If $f$ and $f'$ share two distinct values $a$, $b$ IM then $f' \equiv f$.

Subsequently, similar considerations have been made with respect to higher derivatives and more general (linear) differential expressions as well.

Above theorems motivate the researchers to study the relation between an entire function and its derivative counterpart for one CM shared value. In 1996, in this direction the following famous conjecture was proposed by BRUCK [1]:

**Conjecture.** Let $f$ be a non-constant entire function such that the hyper order $\rho_2(f)$ of $f$ is not a positive integer or infinite. If $f$ and $f'$ share a finite value $a$ CM, then $\frac{f' - a}{f - a} = c$, where $c$ is a non zero constant.

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$, BRÜCK [1] obtained the following result in which additional supposition was required.

**Theorem B ([1])** Let $f$ be a non-constant entire function. If $f$ and $f'$ share the value 1 CM and if $N(r, 0; f') = S(r, f)$ then $\frac{f' - 1}{f - 1}$ is a nonzero constant.

Following example shows the fact that one can not simply replace the value 1 in Theorem B by a small function $a(z)(\neq 0, \infty)$.

**Example 1.1** Let $f = 1 + e^{\pi z}$ and $a(z) = \frac{1}{1-e^z}$.

By Lemma 2.6 of [4] we know that $a$ is a small function of $f$. Also it can be easily seen that $f$ and $f'$ share a CM and $N(r, 0; f') = 0$ but $f - a \neq c (f' - a)$ for every nonzero constant $c$. We note that $f - a = e^{-z} (f' - a)$. So in order to replace the value 1 by a small function more conditions are needed.

However for entire function of finite order, YANG [16] removed the supposition $N(r, 0; f') = 0$ in Theorem B and improved the same in the following way.

**Theorem C ([16])** Let $f$ be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If $f$, $f^{(k)}$ share the value a CM then $\frac{f^{(k)} - a}{f - a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.
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Theorem C somehow can be considered as a solution to the Brück conjecture. Next we consider the following examples which show that in Theorem B one can not simultaneously replace “CM” by “IM” and “entire function” by “meromorphic function”.

Example 1.2 $f(z) = 1 + \tan z$.

Clearly $f(z) - 1 = \tan z$ and $f'(z) - 1 = \tan^2 z$ share 1 IM and $N(r, 0; f') = 0$. But the conclusion of Theorem B ceases to hold.

Example 1.3 $f(z) = \frac{2}{1 - e^{-2z}}$.

Clearly $f'(z) = -\frac{4e^{-2z}}{(1-e^{-2z})^2}$. Here $f - 1 = \frac{1+e^{-2z}}{1-e^{-2z}}$ and $f' - 1 = \frac{(1+e^{-2z})^2}{(1-e^{-2z})^2}$. Here $N(r, 0; f') = 0$ but the conclusion of Theorem B does not hold.

From the above discussion the following question is inevitable.

Question 1.1 Can the conclusion of Theorem B be obtained for a non-constant meromorphic function sharing a small function IM together with its $k$-th derivative counterpart?

ZHANG [18] extended Theorem B to meromorphic function and also studied the value sharing of a meromorphic function with its $k$-th derivative counterpart, but restricted the investigation for CM sharing only.

In the mean time a new notion of scalings between CM and IM known as weighted sharing ([5]-[6]), appeared in the uniqueness literature.

In 2004, with the notion of weighted sharing of values LAHIRI-SARKAR [9] improved the results of ZHANG [18], but neither they considered the case of small function nor they pay their attention to IM sharing.

In 2005, ZHANG [19] further extended the result of Lahiri-Sarkar to a small function and proved the following result for IM sharing.

Theorem D ([19]) Let $f$ be a non-constant meromorphic function and $k(\geq 1)$ be integer. Also let $a = a(z)$ ($\neq 0, \infty$) be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share 0 IM. If

$$4N(r, \infty; f) + 3N_2(r, 0; f^{(k)}) + 2N(r, 0; (f/a)') < (\lambda + o(1)) T(r, f^{(k)}), \quad (1.1)$$

for $r \in I$, where $0 < \lambda < 1$ then $\frac{f^{(k)} - a}{f/a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

Further results in connection with Theorem B in the direction of IM sharing have been obtained by LIU and YANG in the following two theorems.

Theorem E ([11]) Let $f$ be a non-constant meromorphic function. If $f$ and $f'$ share 1 IM and if

$$\frac{N(r, \infty; f) + N(r, 0; f')}{2} < (\lambda + o(1)) T(r, f'), \quad (1.2)$$

for $r \in I$, where $0 < \lambda < \frac{1}{4}$ then $\frac{f' - 1}{f - 1} \equiv c$ for some constant $c \in \mathbb{C}/\{0\}$.

Theorem F ([11]) Let $f$ be a non-constant meromorphic function and $k$ be a positive integer. If $f$ and $f^{(k)}$ share 1 IM and

$$(3k + 6)N(r, \infty; f) + 5N(r, 0; f) < (\lambda + o(1)) T(r, f^{(k)}), \quad (1.3)$$

for $r \in I$, where $0 < \lambda < 1$ then $\frac{f^{(k)} - 1}{f - 1} \equiv c$ for some constant $c \in \mathbb{C}/\{0\}$.
In 2008, in connection with the results of Lahiri-Sarkar [9] and Zhang [19], Zhang and Lü (see [20]) further investigated the analogous problem of Brück conjecture for the $n$-th power of a meromorphic function sharing a small function with its $k$-th derivative. Zhang and Lü (see [20]) obtained the following theorem.

**Theorem G ([20])** Let $f$ be a non-constant meromorphic function and $k(\geq 1)$ and $n(\geq 1)$ be integers. Also let $a \equiv a(z) (\neq 0, \infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $f^{(k)}$ share a small function. Moreover, let $\Gamma_M = \max \{ \Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = \max \{n_1 + 2n_2 + \ldots + kn_k : 1 \leq j \leq t\}$.

$$4N(r, \infty; f) + N(r, 0; f^{(k)}) + 2N_2(r, 0; f^{(k)}) + 2N(r, 0; (f^n/a)') < (\lambda + o(1)) T(r, f^{(k)}),$$  \hspace{1cm} (1.4)

for $r \in I$, where $0 < \lambda < 1$ then $\frac{f^{(k)} - a}{f^n - a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

At the end of [20] the following question was raised by Zhang and Lü [20].

What will happen if $f^n$ and $[f^{(k)}]^m$ share a small function?

In the direction of the above question, Liu [10] investigated the possible answer and obtained the following result.

**Theorem H ([10])** Let $f$ be a non-constant meromorphic function and $k(\geq 1)$, $n(\geq 1)$ and $m(\geq 2)$ be integers. Also let $a \equiv a(z) (\neq 0, \infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $(f^{(k)})^m$ share a small function.

$$\frac{4}{m}N(r, \infty; f) + \frac{5}{m}N(r, 0; f^{(k)}) + \frac{2}{m}N(r, 0; (f^n/a)') < (\lambda + o(1)) T(r, f^{(k)}),$$  \hspace{1cm} (1.5)

for $r \in I$, where $0 < \lambda < 1$ then $\frac{(f^{(k)})^m - a}{f^n - a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

So we see that several special forms on the functions and sharing in connection with the Brück conjecture were obtained by many authors.

Next we recall the following definition.

**Definition 1.1** Let $n_{ij}, n_{i1}, \ldots, n_{kj}$ be non-negative integers.

The expression $M_j[f] = f^{(n_{ij})} \cdot f^{(1)}^{n_{i1}} \ldots f^{(k)}^{n_{kj}}$ is called a differential monomial generated by $f$ of degree $d(M_j) = \sum_{i=0}^k n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^k (i + 1)n_{ij}$.

The sum $P[f] = \sum_{j=1}^t b_j M_j[f]$ is called a differential polynomial generated by $f$ of degree $\overline{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.

The numbers $d(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ and $k$ (the highest order of the derivative of $f$ in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P) = d(P)$.

$P[f]$ is called a Linear Differential Polynomial generated by $f$ if $d(P) = 1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We denote by $Q = \max \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = \max\{n_{i1} + 2n_{i2} + \ldots + kn_{ik} : 1 \leq j \leq t\}$.
Noting that \((f^{(k)})^m\) is nothing but a differential monomial generated by \(f\), it will be quite natural to investigate whether Theorems D-H can be extended up to differential polynomial generated by \(f\). In this direction recently Li and Yang [12] improved Theorem D in the following manner.

**Theorem 1** ([12]) Let \(f\) be a non-constant meromorphic function \(P[f]\) be a differential polynomial generated by \(f\). Also let \(a = a(z) (\neq 0, \infty)\) be a small meromorphic function. Suppose that \(f - a\) and \(P[f] - a\) share \(0\) IM and \((t - 1)d(P) \leq \sum j=1 d(M_j)\).

If

\[
4N(r, \infty; f) + 3N_2(r, 0; P[f]) + 2N(r, 0; (f/a)') < (\lambda + o(1)) T(r, P[f]),
\]

(1.6)

for \(r \in I\), where \(0 < \lambda < 1\) then \(\frac{P[f] - a}{f^n - a} = c\) for some constant \(c \in \mathbb{C}/\{0\}\).

As for a differential monomial \(t = 1\), Theorem I always holds without any restriction on the degree of the monomial. But for general differential polynomial one can not eliminate the supposition \((t - 1)d(P) \leq \sum j=1 d(M_j)\) in the above theorem. So it would be interesting to investigate whether in Theorem I, the condition over the degree can be removed, sharing notion can further be relaxed, (1.6) can further be weakened as well so that it will improve Theorems D-H to a large extent. In this paper our main intention is to deal with this problem. We shall improve, unify, generalize and extend all the Theorems D-H. Following theorem is the main result of the paper.

**Theorem 1.2** Let \(f\) be a non-constant meromorphic function and \(n(\geq 1)\) be an integer. Let \(m(\geq 1)\) be a positive integer or infinity and \(a = a(z) (\neq 0, \infty)\) be a small meromorphic function. Suppose that \(P[f]\) be a differential polynomial generated by \(f\) such that \(P[f]\) contains at least one derivative. Suppose further that \(E_m(a, f^n) = E_m(a, P[f])\).

If

\[
4N(r, \infty; f) + N_2(r, 0; P[f]) + 2N(r, 0; P[f]) + N(r, 0; (f^n/a)') + N(r, 0; (f^n/a)') | (f^n/a) \neq 0 < (\lambda + o(1)) T(r, P[f]),
\]

(1.7)

for \(r \in I\), where \(0 < \lambda < 1\) then \(\frac{P[f] - a}{f^n - a} = c\) for some constant \(c \in \mathbb{C}/\{0\}\).

**Remark 1.1** Clearly in Theorem 1.2 when \(m = \infty\) we have \(f^n - a\) and \(P[f] - a\) share \(0\) IM and we obtain the improved, extended and generalized version of Theorem I in the direction of Question 1.1.

Though we use the standard notations and definitions of the value distribution theory available in [4], we explain some definitions and notations which are used in the paper.

**Definition 1.3** ([9]) Let \(p\) be a positive integer and \(a \in \mathbb{C} \cup \{\infty\}\).

(i) \(N(r, a; f | \geq p) (N(r, a; f | \geq p))\) denotes the counting function (reduced counting function) of those \(a\)-points of \(f\) whose multiplicities are not less than \(p\).

(ii) \(N(r, a; f | \leq p) (N(r, a; f | \leq p))\) denotes the counting function (reduced counting function) of those \(a\)-points of \(f\) whose multiplicities are not greater than \(p\).
Definition 1.4 ([17]) For \( a \in \mathbb{C} \cup \{\infty\} \) and a positive integer \( p \) we denote by \( N_p(r,a;f) \) the sum \( \overline{N}(r,a;f) + \overline{N}(r,a;f \geq 2) + \ldots + \overline{N}(r,a;f \geq p) \). Clearly \( N_1(r,a;f) = \overline{N}(r,a;f) \).

Definition 1.5 Let \( k \) be a positive integer and for \( a \in \mathbb{C} \cup \{0\} \), \( E_k(a;f) = E_k(a;g) \). Let \( z_0 \) be a zero of \( f(z) - a \) of multiplicity \( p \) and a zero of \( g(z) - a \) of multiplicity \( q \). We denote by \( \overline{N}_L(r,a;f) \) the counting function of those \( a \)-points of \( f \) and \( g \) where \( p > q \geq 1 \), by \( \overline{N}_{f>s}(r,a;g) (\overline{N}_{g>s}(r,a;f)) \) the counting functions of those \( a \)-points of \( f \) and \( g \) for which \( p > q = s(q > p = s) \), by \( \overline{N}^1_E(r,a;f) \) the counting function of those \( a \)-points of \( f \) and \( g \) where \( p = q \geq 1 \) and by \( \overline{N}^2_E(r,a;f) \) the counting function of those \( a \)-points of \( f \) and \( g \) where \( p = q \geq 2 \), each point in these counting functions is counted only once. In the same way we can define \( \overline{N}_L(r,a;f), N^1_E(r,a;g), \overline{N}^2_E(r,a;g) \). We denote by \( \overline{N}_{f\geq k+1}(r,a;f) (\overline{N}_{g\geq k+1}(r,a;g) \mid f \neq a) \) the reduced counting functions of those \( a \)-points of \( f \) and \( g \) for which \( p \geq k + 1 \) and \( q = 0 \) \((q \geq k + 1 \text{ and } p = 0)\).

Definition 1.6 ([7]) Let \( a, b \in \mathbb{C} \cup \{\infty\} \). We denote by \( N(r,a;f \mid g \neq b) \) the counting function of those \( a \)-points of \( f \), counted according to multiplicity, which are not the \( b \)-points of \( g \).

Definition 1.7 ([5,6]) Let \( f, g \) share a value \( a \) IM. We denote by \( \overline{N}_s(r,a;f,g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \).

Clearly \( \overline{N}_s(r,a;f,g) = \overline{N}_s(r,a;g) \) and \( \overline{N}_s(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g) \).

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let \( F, G \) be two non-constant meromorphic functions. Henceforth we shall denote by \( H \) the following function.

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G-1} \right). \tag{2.1}
\]

Lemma 2.1 Let \( E_m(1;F) = E_m(1;G); F, G \) share \( \infty \) IM and \( H \neq 0 \). Then

\[
N(r,\infty;H) \leq \overline{N}(r,0;F \geq 2) + \overline{N}(r,0;G \geq 2) + \overline{N}_s(r,\infty;F,G) + \overline{N}_{F \geq m+1}(r,1;F \mid G \neq 1) + \overline{N}_{G \geq m+1}(r,1;G \mid F \neq 1) + \overline{N}_L(r,1;F') + \overline{N}_L(r,1;G') + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),
\]

where \( \overline{N}_0(r,0;F') \) is the reduced counting function of those zeros of \( F' \) which are not the zeros of \( F(F-1) \) and \( \overline{N}_0(r,0;G') \) is similarly defined.
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Proof. We can easily verify that possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) poles of $F$ and $G$ with different multiplicities, (iii) the common zeros of $F - 1$ and $G - 1$ with different multiplicities, (iv) zeros of $F - 1$ ($F - 1$) which are not the zeros of $G - 1$ ($F - 1$), (v) those 1-points of $F$ ($G$) which are not the 1-points of $G$ ($F$), (v) zeros of $F'$ which are not the zeros of $F(F - 1)$, (v) zeros of $G'$ which are not zeros of $G(G - 1)$. Since $H$ has simple pole the lemma follows from above. □

Lemma 2.2 ([19]) Let $f$ be a non-constant meromorphic function and $k$ be a positive integer, then $N_p(r, 0; f^{(k)}) \leq N_{p+k}(r, 0; f) + kN(r, \infty; f) + S(r, f)$.

Lemma 2.3 ([8]) If $N(r, 0; f^{(k)} | f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then $N(r, 0; f^{(k)} | f \neq 0) \leq kN(r, \infty; f) + N(r, 0; f | < k) + kN(r, 0; f | \geq k) + S(r, f)$.

Lemma 2.4 ([13]) Let $f$ be a non-constant meromorphic function and let

$$R(f) = \sum_{n=0}^{\infty} a_n f^n$$

be an irreducible rational function in $f$ with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then $T(r, R(f)) = dT(r, f) + S(r, f)$, where $d = \max\{n, m\}$.

Lemma 2.5 ([2]) Let $f$ be a meromorphic function and $P[f]$ be a differential polynomial. Then

$$m \left( r, \frac{P[f]}{\bar{d}(P)} \right) \leq (\bar{d}(P) - d(P))m \left( r, \frac{1}{f} \right) + S(r, f).$$

Lemma 2.6 Let $f$ be a meromorphic function and $P[f]$ be a differential polynomial. Then we have

$$N \left( r, \infty; \frac{P[f]}{\bar{d}(P)} \right) \leq (\bar{d}(P) - d(P)) \bar{N}(r, \infty; f) + (\bar{d}(P) - d(P)) N(r, 0; f | \geq k + 1) + Q \bar{N}(r, 0; f | \geq k + 1) + \bar{d}(P)N(r, 0; f | \leq k) + S(r, f).$$

Proof. Let $z_0$ be a pole of $f$ of order $r$, such that $b_j(z_0) \neq 0, \infty (1 \leq j \leq t)$. Then it would be a pole of $P[f]$ of order at most $r\bar{d}(P) + \bar{d}(P) P$. Since $z_0$ is a pole of $\bar{d}(P)$ of order $\bar{d}(P)$, it follows that $z_0$ would be a pole of $P[f]/\bar{d}(P)$ of order at most $r\bar{d}(P)$. Next suppose $z_1$ is a zero of $f$ of order $s(> k)$, such that $b_j(z_1) \neq 0, \infty (1 \leq j \leq t)$. Clearly it would be a zero of $M_j(f)$ of order $s_n_0 + (s - 1)n_1 + \ldots + (s - k)n_k = s.d(M_j) - (\Gamma_{M_j} - d(M_j))$. Hence $z_1$ be a pole of $P[f]/\bar{d}(P)$ of order

$$s.d(P) - s.d(M_j) + (\Gamma_{M_j} - d(M_j)) = s.d(P) - d(M_j) + (\Gamma_{M_j} - d(M_j)).$$

So $z_1$ would be a pole of $P[f]/\bar{d}(P)$ of order at most

$$\max\{s.d(P) - d(M_j) + (\Gamma_{M_j} - d(M_j)) : 1 \leq j \leq t\} = s.d(P) - d(P) + Q.$$
Since the poles of $\frac{P[f]}{P[d]}$ comes from the poles or zeros of $f$ and poles or zeros of $b_j(z)$ ($1 \leq j \leq t$) only, it follows that $N(r, \infty; \frac{P[f]}{P[d]}) \leq (\Gamma_P - \mathcal{d}(P)) N(r, \infty; f) + (\mathcal{d}(P) - \mathcal{d}(P)) N(r, 0; f \mid \geq k + 1) + Q \overline{N}(r, 0; f \mid \geq k + 1) + \mathcal{d}(P) N(r, 0; f \mid \leq k) + S(r, f)$. □

Lemma 2.7 ([3]) Let $P[f]$ be a differential polynomial. Then $T(r, P[f]) \leq \Gamma_P T(r, f) + S(r, f)$.

Lemma 2.8 Let $f$ be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then $S(r, P[f])$ can be replaced by $S(r, f)$.

Proof. From Lemma 2.7 it is clear that $T(r, P[f]) = O(T(r, f))$ and so the lemma follows. □

3 Proof of the theorem

Proof of Theorem 1.2. Let $F = \frac{f^n}{a}$ and $G = \frac{P[f]}{P[d]}$. Then $F - 1 = \frac{f^n - a}{a}$, $G - 1 = \frac{P[f] - a}{a}$. Since $E_m(a, f^n) = E_m(a, P[f])$, it follows that $E_m(1, F) = E_m(1, G)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

Case 1. Let $H \neq 0$.

Let $z_0$ be a simple zero of $F - 1$. Then by a simple calculation we see that $z_0$ is a zero of $H$ and hence

$$N_E^1(r, 1; F) = N_E^1(r, 1; G) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, F) \quad (3.1)$$

Using (3.1), Lemmas 2.1, 2.8 and noting that $N(r, \infty; F) = N(r, \infty; G) + S(r, f) = N(r, \infty; f) + S(r, f)$ and $N_{F>1}(r, 1; G) = N(r, 1; G) \mid \geq 2 = N_E^2(r, 1; G) + N_L(r, 1; G) + N_E(r, 1; F) + N_{G \geq m+1}(r, 1; G \mid F \neq 1) + S(r, f)$, we get from the second fundamental theorem that

$$T(r, G) \leq N(r, \infty; G) + N(r, 0; G) + N_E^1(r, 1; G) + N_{F>1}(r, 1; G)$$
$$+ N(r, 1; G \mid \geq 2) - N_0(r, 0; G') + S(r, G)$$
$$\leq 2N(r, \infty; F) + N(r, 0; G) + N(r, 0; G \mid \geq 2)$$
$$+ N(r, 0; F \mid \geq 2) + 2N_L(r, 1; F)$$
$$+ 2N_L(r, 1; G) + N_{F \geq m+1}(r, 1; F \mid G \neq 1)$$
$$+ 2N_{G \geq m+1}(r, 1; G \mid F \neq 1)$$
$$+ N_E^2(r, 1; G) + N_0(r, 0; F') + S(r, f). \quad (3.2)$$

Using Lemmas 2.2, 2.3 we see that

$$N(r, 0; G \mid \geq 2) + 2N_{G \geq m+1}(r, 1; G \mid F \neq 1)$$
$$+ 2N_L(r, 1; G) + N_E^2(r, 1; G) \quad (3.3)$$
$$\leq N(r, 0; G' \mid G \neq 0) + N(r, 0; G') + S(r, f)$$
$$\leq 2N(r, \infty; f) + N(r, 0; P[f]) + N_2(r, 0; P[f]) + S(r, f)$$

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and

\[ \overline{N}(r, 0; F |\geq 2) + \overline{N}_{F \geq m+1}(r, 1; F | G \neq 1) + 2\overline{N}_L(r, 1; F) + \overline{N}_0(r, 0; F') \leq \overline{N}(r, 0; F | F \neq 0) + \overline{N}(r, 0; F') + S(r, f) \]

\[ \leq \overline{N}(r, 0; (f^n/a) | (f^n/a) \neq 0) + \overline{N}(r, 0; (f^n/a)') + S(r, f). \] (3.4)

Using (3.3) and (3.4) in (3.1) we have

\[ T(r, P[f]) \leq 4\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; P[f]) + N_2(r, 0; P[f]) + \overline{N}(r, 0; (f^n/a)') + \overline{N}(r, 0; (f^n/a) | (f^n/a) \neq 0) + S(r, f). \]

This contradicts (1.7).

**Case 2.** Let \( H \equiv 0 \).

On integration we get from (2.1)

\[ \frac{1}{F - 1} \equiv \frac{C}{G - 1} + D, \] (3.5)

where \( C, D \) are constants and \( C \neq 0 \). From (3.5) it is clear that \( F \) and \( G \) share 1 CM.

We claim that \( D = 0 \). If \( \overline{N}(r, \infty; f) \neq S(r, f) \), then by (3.5) we get \( D = 0 \).

So we assume that

\[ \overline{N}(r, \infty; f) = S(r, f) \] (3.6)

and \( D \neq 0 \). Clearly \( \overline{N}(r, \infty; G) = \overline{N}(r, \infty; f) + S(r, f) \).

From (3.5) we get

\[ \frac{1}{F - 1} = \frac{D(G - 1 + \frac{C}{G})}{G - 1}. \] (3.7)

Clearly from (3.7) we have

\[ \overline{N}\left(r, 1 - \frac{C}{D}; G\right) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) = S(r, f). \] (3.8)

If \( \frac{C}{D} \neq 1 \), by the second fundamental theorem, Lemma 2.8 and (3.8) we have

\[ T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1 - \frac{C}{D}; G) + S(r, G) \leq \overline{N}(r, 0; G) + S(r, f) \leq N_2(r, 0; G) + S(r, f) \leq T(r, G) + S(r, f). \]

So \( T(r, G) = N_2(r, 0; G) + S(r, f) \) that is, \( T(r, P[f]) = N_2(r, 0; P[f]) + S(r, f) \), which contradicts (1.7).

If \( \frac{C}{D} = 1 \) we get from (3.5)

\[ \left(F - 1 - \frac{1}{C}\right) G \equiv -\frac{1}{C}. \] (3.9)

From (3.9) it follows that

\[ N(r, 0; f | \geq k + 1) \leq N(r, 0; P[f]) \leq N(r, 0; G) \leq N(r, 0; a) = S(r, f). \] (3.10)
Again from (3.9) we see that \( \frac{1}{I_r((f^n - (1 + 1/C)a)} \equiv - \frac{C}{a} \frac{P[f]}{f^a} \). Hence by the first fundamental theorem, (3.6), (3.10), Lemmas 2.4, 2.5 and 2.6 we get that

\[
(n + \bar{d}(P))T(r, f) = T \left( r, \frac{1}{f^{\bar{d}(P)}}(f^n - (1 + \frac{1}{C})a) \right) + S(r, f) \\
= T \left( r, \frac{P[f]}{f^{\bar{d}(P)}} \right) + S(r, f) \\
\leq m \left( r, \frac{P[f]}{f^{\bar{d}(P)}} \right) + N \left( r, \frac{P[f]}{f^{\bar{d}(P)}} \right) + S(r, f) \\
\leq (\bar{d}(P) - \bar{d}(P))T(r, f) - \{N(r, 0; f |\leq k) \\
+ N(r, 0; f |\geq k + 1)\} + (\bar{d}(P) - \bar{d}(P)) \\
N(r, 0; f |\geq k + 1) + \frac{Q N(r, 0; f |\geq k + 1)}{\bar{d}(P)}N(r, 0; f |\leq k) + S(r, f) \\
\leq (\bar{d}(P) - \bar{d}(P))T(r, f) + d(P)N(r, 0; f |\leq k) + S(r, f). 
\]

From (3.11) it follows that \( nT(r, f) \leq S(r, f) \), which is absurd. Hence \( D = 0 \) and so \( C = \frac{G-1}{T} = C \) or \( \frac{P[f] - a}{f^a} = C \). This proves the theorem. \( \square \)

Acknowledgements The authors wish to thank the referee for his/her valuable remarks and suggestions to-wards the improvement of the paper.

References

1. BRÜCK, R. – On entire functions which share one value CM with their first derivative, Results Math., 30 (1996), 21-24.