Korovkin type theorems in weighted $L_p$-spaces via statistical $A$-summability

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Abstract In this paper, we study Korovkin type approximation theorems on weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$, with help of statistical $A$-summability which is stronger than $A$-statistical convergence. Also, we construct examples such that our new approximation result works but its statistical case does not work.

Keywords Statistical $A$-summability · Positive linear operator · Korovkin type approximation theorem

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1 Introduction

Approximation theory has important applications in theory of polynomial approximation, in various areas of functional analysis, in numerical solutions of differential and integral equations, etc [5]. The well-known Korovkin theorem [5,20] on approximation of continuous functions on a compact interval, is mainly based on the existence of the limit $\lim_{n} L_n(f;x) = f(x)$. Many researchers have extended this theorem for various operators on different spaces [1,2,8,12,14,15]. One of the most important paper in these extensions is given by GADJIEV [13].

Recently, some Korovkin type approximation theorems have been studied via statistical convergence [3,4,7,10,17–19]. Also, DIRIK, ARAL and DEMIRCI [6] have studied a Korovkin type approximation theorem on weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$.
using the concept of $I$–convergence. Those results which obtained are stronger than the classical Korovkin theorem.

The purpose of the paper is to give Korovkin type approximation theorems on weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$ using the concept of statistical $A$-summability which is stronger than $A$-statistical convergence.

We now recall some basic definitions and notations used in the paper.

Let $\mathbb{R}$ denote the set of real numbers. The function $\omega$ is called a weight function if it is a positive continuous function on the whole real axis and, for a fixed $p \in [1, \infty)$, satisfying the condition

$$\int_{\mathbb{R}} t^{2p} \omega(t) \, dt < \infty. \quad (1.1)$$

We denote by $L_{p,\omega}(\mathbb{R})$ $(1 \leq p < \infty)$ the linear space of measurable, $p$–absolutely integrable functions on $\mathbb{R}$ with respect to weight function $\omega$, i.e.

$$L_{p,\omega}(\mathbb{R}) = \{ f : \omega \in \mathbb{R}; \| f \|_{p,\omega} = \left( \int_{\mathbb{R}} |f(t)|^p \omega(t) \, dt \right)^{\frac{1}{p}} < \infty \}. \quad (1.2)$$

The analogous of (1.1) and (1.2) in multidimensional space are given as follows. Let $\Omega$ be a positive continuous function in $\mathbb{R}^n$, satisfying the condition

$$\int_{\mathbb{R}^n} |t|^{2p} \Omega(t) \, dt < \infty, \quad (1.3)$$

and

$$L_{p,\Omega}(\mathbb{R}^n) = \{ f : \Omega \in \mathbb{R}^n; \| f \|_{p,\Omega} = \left( \int_{\mathbb{R}^n} |f(t)|^p \Omega(t) \, dt \right)^{\frac{1}{p}} < \infty \}. \quad (1.4)$$

Let $A = \{a_{kj}\}, k, j = 1, 2, \ldots$ be an infinite summability matrix. For a given sequence $x = \{x_j\}$, the $A$–transform of $x$, denoted by $Ax := \{Ax\}_k$ is given by $(Ax)_k = \sum_{j=1}^{\infty} a_{kj}x_j$, provided the series converges for each $k \in \mathbb{N}$. We say that $A$ is regular (see [16]) if $\lim Ax = L$ whenever $\lim x = L$.

Assume that $A$ is a nonnegative regular summability matrix. The $A$-density of a subset $K \subset \mathbb{N}$, denoted by $\delta_A(K)$, is given by

$$\delta_A(K) = \lim_k \sum_{j \in K} a_{kj},$$

provided the limit exists. Then the sequence $x = \{x_j\}$ is called $A$-statistically convergent to $L$ provided that, for every $\varepsilon > 0$,

$$\delta_A(\{ j \in \mathbb{N} : |x_j - L| \geq \varepsilon \}) = 0. \quad (1.5)$$

In this case we write $\text{st}_A \lim x = L$.

Note that if we take $A = (C, 1)$, which is the Cesáro matrix, then $(C, 1)$–statistical convergence coincides with the notion of statistical convergence, which was introduced in [11]. Finally, if we replace the matrix $A$ by the identity matrix, then $A$–statistical convergence reduces to the usual convergence.
Definition 1.1 ([9]) Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix and $x = \{x_j\}$ be a sequence. We say that $x$ is statistically $A$–summable to $L$ if for every $\varepsilon > 0$, $\delta\{k \in \mathbb{N} : |(Ax)_k - L| \geq \varepsilon\} = 0$, i.e., $\lim_{N \to \infty} \frac{\sum_{k \leq N} |(Ax)_k - L| \geq \varepsilon}{N} = 0$.

Thus $x = \{x_j\}$ is statistically $A$–summable to $L$ if and only if $Ax$ is statistically convergent to $L$. In this case we write $(A)_x \rightarrow x$, $x(L)$, or $\lim_{n \to \infty} (A)_n x = L$.

We note that if we take $A = (C, 1)$ then statistical $A$–summability is reduced to the statistical $(C, 1)$ – summability.

Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix and $\{L_j\}$ be a sequence of positive linear operators from $L_{p, \omega}$ into $L_{p, \omega}$. By $A_k(f; x)$ we denote

$$ A_k(f; x) = \sum_{j=1}^{\infty} a_{kj} L_j (f(t); x). \quad (1.6) $$

2 Main results

Now we first recall the classical and statistical cases of Korovkin type results introduced in [15, 6], respectively.

**Theorem 2.1** ([15]) Let $\{L_j\}_{j \in \mathbb{N}}$ be the sequence of positive linear operators $L_j : L_{p, \omega}(\mathbb{R}) \to L_{p, \omega}(\mathbb{R})$ and let the sequence $\{\|L_j\|\}$ be uniformly bounded. If $\lim_j \|L_j(t^i, x) - x^i\|_{p, \omega} = 0$, $i = 0, 1, 2$, then for any function $f \in L_{p, \omega}(\mathbb{R})$, we have $\lim_j \|L_j(f) - f\|_{p, \omega} = 0$.

**Theorem 2.2** ([6]) Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix. Let $\{L_j\}_{j \in \mathbb{N}}$ be the sequence of positive linear operators $L_j : L_{p, \omega}(\mathbb{R}) \to L_{p, \omega}(\mathbb{R})$ and let the sequence $\{\|L_j\|\}$ be uniformly bounded. If $st \rightarrow \lim_j \|L_j(t^i, x) - x^i\|_{p, \omega} = 0$, $i = 0, 1, 2$, then for any function $f \in L_{p, \omega}(\mathbb{R})$, we have $st \rightarrow \lim_j \|L_j(f) - f\|_{p, \omega} = 0$.

**Theorem 2.3** Let $A = \{a_{kj}\}$ be a non-negative regular summability matrix and $\{L_j\}$ be a sequence of positive linear operators from $L_{p, \omega}$ into $L_{p, \omega}$. Assume that

$$ \sup_k \|A_k\|_{L_{p, \omega} \to L_{p, \omega}} < \infty. \quad (2.1) $$

If

$$ st \rightarrow \lim_k \|A_k(t^i; x) - x^i\|_{p, \omega} = 0, \quad i = 0, 1, 2, \quad (2.2) $$

then for any function $f \in L_{p, \omega}(\mathbb{R})$, we have $st \rightarrow \lim_k \|A_k f - f\|_{p, \omega} = 0$.

Proof. We give the proof of theorem as similarly as the proof of theorem in [15]. Let $\chi^B(t)$ be the characteristic function of the interval $[-B, B]$ and $\chi^B(t) = 1 - \chi^B(t)$ for any $B \geq 0$. We can choose a such large $B$ such that for every $\varepsilon > 0$, $\|f \chi^B\|_{p, \omega} < \varepsilon$.

Using the assumption of the convergence of the series in (1.6) for each $k$, $f$ and the linearity of the operators $L_j$, we get

$$ \|A_k f - f\|_{p, \omega} = \|A_k (\chi^B - \chi^B) f - (\chi^B f)\|_{p, \omega} \leq \|A_k (\chi^B f) - \chi^B f\|_{p, \omega} + \|A_k (\chi^B f) - \chi^B f\|_{p, \omega} \quad (2.3) $$

$$ = I'_k + I''_k. $$

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From the condition \((2.1)\), there exists a constant \(K > 0\) such that

\[
\sup_k \|A_k\|_{p,\omega} \leq K. \tag{2.4}
\]

Hence, from \((2.1)\), we have \(I_k^p \leq \|A_k \chi_1^B f\|_{p,\omega} + \|\chi_2^B f\|_{p,\omega} \leq (K + 1)\|\chi_2^B f\|_{p,\omega} < (K + 1)\varepsilon\). For every function \(f \in L_{p,\omega}(\mathbb{R})\) the inequality \(\|\chi_1^B f\|_p \leq \omega_{\min}^{-1/p}\|f\|_{p,\omega}\) implies \(L_{p,\omega}(\mathbb{R}) \subset L_p(-B, B)\). Since the space of continuous functions on \([-B, B]\) is dense in \(L_p(-B, B)\), given \(f \in L_{p,\omega}(\mathbb{R})\), for each \(\varepsilon' > 0\), there exists a continuous function \(\varphi\) on \([-B, B]\) satisfying the condition \(\varphi(x) = 0\) for \(|x| > B\) such that

\[
\|(f - \varphi) \chi_1^B\|_p < \frac{\varepsilon'}{2} \tag{2.5}
\]

Using the inequalities \((2.4)\) and \((2.5)\), we get

\[
I_k = \|A_k (\chi_1^B f) - \chi_1^B f\|_{p,\omega}
\leq \|A_k (f - \varphi) \chi_1^B\|_{p,\omega} + \|A_k (\varphi \chi_1^B) - \varphi \chi_1^B\|_{p,\omega} + \|(f - \varphi) \chi_1^B\|_{p,\omega} \tag{2.6}
\]

On the other hand, since \(\chi_2^B \chi_1^B \varphi = 0\) for some \(B_1 > B\), we get the equality

\[
\|A_k (\varphi \chi_1^B) - \varphi \chi_1^B\|_{p,\omega} = \|\left(\chi_1^{B_1} + \chi_2^{B_1}\right) A_k (\varphi \chi_1^B) - \left(\chi_1^{B_1} + \chi_2^{B_1}\right) \varphi \chi_1^B\|_{p,\omega}
\leq \left\|\left[ A_k (\varphi \chi_1^B) - \varphi \chi_1^B\right] \chi_1^{B_1}\right\|_{p,\omega} + \left\|\chi_2^{B_1} A_k (\varphi \chi_1^B)\right\|_{p,\omega}.
\]

Now, by denoting \(M_\varphi = \max_{t \in \mathbb{R}} |\varphi(t)| \chi_1^B(t)\), we get

\[
\left\|\chi_2^{B_1} A_k (\varphi \chi_1^B)\right\|_{p,\omega} = \left(\int_{|t| > B_1} |A_k (\varphi \chi_1^B; t)|^p \omega(t) \, dt\right)^{\frac{1}{p}}
\leq M_\varphi \left(\int_{|t| > B_1} |A_k (1; t) - 1|^p \omega(t) \, dt\right)^{\frac{1}{p}} + M_\varphi \left(\int_{\mathbb{R}} \chi_2^{B_1} \omega(t) \, dt\right)^{\frac{1}{p}}.
\]

Since \(\omega \in L_1(\mathbb{R})\), we can choose a number \(B_1\) such that

\[
\left(\int_{\mathbb{R}} \chi_2^{B_1} \omega(t) \, dt\right)^{\frac{1}{p}} < \frac{\varepsilon'}{M_\varphi}.
\]

Using this inequality we have \(\left\|\chi_2^{B_1} A_k (\varphi \chi_1^B)\right\|_{p,\omega} \leq M_\varphi \|A_k (1; x) - 1\|_{p,\omega} + \varepsilon'\). As a corollary, we get the following inequality for \(I_k^p\)

\[
I_k^p \leq 2\varepsilon' + M_\varphi \|A_k (1; x) - 1\|_{p,\omega} + \left\|[A_k (\varphi \chi_1^B) - \varphi \chi_1^B] \chi_1^{B_1}\right\|_{p,\omega}.
\]
Since \( \varphi \chi^B_1 \) is a continuous function on \([-B, B]\), for given any \( \varepsilon' > 0 \) there exist a \( \delta > 0 \) such that \( |\varphi(t) \chi^B_1(t) - \varphi(x) \chi^B_1(x)| < \varepsilon' + 2M_\varepsilon \frac{(t-x)^2}{\delta^2} \). So we have

\[
\|A_k(\varphi \chi^B_1) - \varphi \chi^B_1 \|_{p,\omega} \leq \|A_k((\varphi(t) \chi^B_1(t) - \varphi(x) \chi^B_1(x))(x)) \|_{p,\omega} \\
+ \|\varphi(x) \chi^B_1(x)(A_k(1; x) - 1)\|_{p,\omega} \\
\leq (\varepsilon' + \frac{2M_\varepsilon}{\delta^2}B^2 + M_\varepsilon) \|A_k(1; x) - 1\|_{p,\omega} \\
+ \frac{4M_\varepsilon}{\delta^2} B \|A_k(t; x) - x\|_{p,\omega} + \frac{2M_\varepsilon}{\delta^2} \|A_k(t^2; x) - x^2\|_{p,\omega}.
\]

Using (2.7), we can write

\[
I_k' \leq 2\varepsilon' + \left( \varepsilon' + \frac{2M_\varepsilon}{\delta^2}B^2 + 2M_\varepsilon \right) \|A_k(1; x) - 1\|_{p,\omega} \\
+ \frac{4M_\varepsilon}{\delta^2} B \|A_k(t; x) - x\|_{p,\omega} + \frac{2M_\varepsilon}{\delta^2} \|A_k(t^2; x) - x^2\|_{p,\omega}.
\]

Then we obtain the following equality for (2.3) as \( \|A_k f - f\|_{p,\omega} \leq 2\varepsilon' + (K+1)\varepsilon + C\{\|A_k(1; x) - 1\|_{p,\omega} + \|A_k(t; x) - x\|_{p,\omega} + \|A_k(t^2; x) - x^2\|_{p,\omega}\} \), where \( C := \max\{\varepsilon' + \frac{2M_\varepsilon}{\delta^2}B^2 + 2M_\varepsilon, \frac{4M_\varepsilon}{\delta^2} B, \frac{2M_\varepsilon}{\delta^2}\} \). Let \( r > 0 \) be a number such that \( (2\varepsilon' + (K+1)\varepsilon) < r \).

Then, let

\[
D := \left\{ k \leq N : \sum_{i=0}^{2} \|A_k(t^i; x) - x^i\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{C} \right\},
\]

\[
D_1 := \left\{ k \leq N : \|A_k(1; x) - 1\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{3C} \right\},
\]

\[
D_2 := \left\{ k \leq N : \|A_k(t; x) - x\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{3C} \right\},
\]

\[
D_3 := \left\{ k \leq N : \|A_k(t^2; x) - x^2\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{3C} \right\}.
\]

It is easy to see that \( D \subset D_1 \cup D_2 \cup D_3 \) and we have

\[
\left| \left\{ k \leq N : \|A_k f - f\|_{p,\omega} \geq r \right\} \right| \\
\leq \left| \left\{ k \leq N : \|A_k(1; x) - 1\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{3C} \right\} \right| \\
+ \left| \left\{ k \leq N : \|A_k(t; x) - x\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{3C} \right\} \right| \\
+ \left| \left\{ k \leq N : \|A_k(t^2; x) - x^2\|_{p,\omega} \geq \frac{r - (2\varepsilon' + (K+1)\varepsilon)}{3C} \right\} \right|,
\]

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where \(|A|\) denotes the cardinality of the set \(A\). Then taking the limit \(N \to \infty\), using the hypothesis of theorem, we obtain
\[
\lim_{N \to \infty} \frac{1}{N} \left\lfloor k \leq N : \|A_k f - f\|_{p, \omega} \geq r \right\rfloor = 0
\]
which is the desired result. \(\square\)

Now we give an example of a sequence of positive linear operators which satisfy the conditions of Theorem 2.3 in the weighted space \(L_{p, \omega}(\mathbb{R})\).

**Example 2.1** We choose \(\omega(x) = e^{-x}\). Note that this selection of \(\omega\) satisfies the condition (1.1). Also note that for \(1 \leq p < \infty\), \(L_{p, \omega}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : e^{-x} f(x) \in L_p(\mathbb{R})\}\). Also, \(A = (C, 1)\) is the Cesáro matrix, i.e.,
\[
c_{kj} = \begin{cases} \frac{1}{k}, & 1 \leq j \leq k, \\ 0, & \text{otherwise}. \end{cases}
\]
and \(\alpha = \{\alpha_j\}\) is defined by \(\alpha_j = (-1)^j\), then we can easily see that 
\(s t - \lim k ((C, 1) \alpha) = 0\). However, the sequence \((\alpha_j)\) does not converge in usual and statistical sense. The Kantorovich variant of the Szasz-Mirakyan operators [21] by replacing \(f(b_j x)\) with an integral mean of \(f(x)\) over the interval \([s+1]b_j / j, sb_j / j\) is as follows:
\[
S_j (f; x) := \frac{j}{b_j} \sum_{s=0}^{\infty} P_{j, s}(x) \int_{sb_j / j}^{(s+1)b_j / j} f(t) \, dt, \quad j \in \mathbb{N}, \quad x \in [0, b_j),
\]
where \(\{b_j\}\) is a sequence of positive real numbers satisfying the conditions \(\lim_{j \to \infty} \frac{b_j}{j} = 0\) and \(\lim_{j \to \infty} b_j = \infty\) and \(P_{j, s}(x) := e^{-jx/b_j} \left(jx/b_j\right)^s / s!\), \(s = 0, 1, 2, \ldots\). It is known that \(S_j(1; x) = 1\), \(S_j (t; x) = x + \frac{b_j}{j} \) and \(S_j (t^2; x) = x^2 + \frac{2b_j}{j} x + \frac{b_j^2}{3j}\). Then using the operators \(S_j\) and the sequence \(\alpha = (\alpha_j)\), we define the sequence of positive linear operators \(L_j(f; x) = (1 + \alpha_j)S_j(f; x)\) for \(f \in L_{p, \omega}(\mathbb{R})\) and \(j \in \mathbb{N}\). By some simple calculations, we obtain
\[
\|C_k (1; x) - 1\|_{p, \omega} = \left\lfloor \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right\|1\|_{p, \omega},
\]
\[
\|C_k (t; x) - x\|_{p, \omega} \leq \frac{1}{k} \sum_{j=1}^{k} \frac{b_j}{j} \|1\|_{p, \omega} + \left\lfloor \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right\|x\|_{p, \omega},
\]
\[
\|C_k (t^2; x) - x^2\|_{p, \omega} \leq \frac{4}{k} \sum_{j=1}^{k} \frac{b_j}{j} \|x\|_{p, \omega} + \frac{2}{3k} \sum_{j=1}^{k} \frac{b_j^2}{j^2} \|1\|_{p, \omega} + \left\lfloor \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right\|x^2\|_{p, \omega},
\]
where \(C_k (f; x) = \sum_{j=1}^{\infty} c_{kj} L_j (f(t); x)\).
Also, \(\sup_k \|C_k\|_{L_{p, \omega} \to L_{p, \omega}} = \sup_k \sup_{\|f\|_{p, \omega}=1} \|C_k (f; x)\|_{p, \omega} < \infty\). Hence (2.1), (2.2) conditions are provided. For any function \(f \in L_{p, \omega}(\mathbb{R})\), we have \(st - \lim k \|C_k f - f\|_{p, \omega} = 0\).
Also, an analogue of Theorem 2.3 for the space of function of several variables can be obtained. Now we establish this theorem.

**Theorem 2.4** Let $A = \{a_{kj}\}$ be an infinite matrix with non-negative real entries and $\{L_j\}$ be a sequence of positive linear operators from $L_{p,\Omega}(\mathbb{R}^n)$ into $L_{p,\Omega}(\mathbb{R}^n)$. Assume that

$$\sup_k \|A_k\|_{L_{p,\Omega}} < \infty. \quad (2.9)$$

If

$$\begin{align*}
st - \lim_k \|A_k(1;x) - 1\|_{p,\Omega} &= 0, \quad i = 0, 1, 2, \\
st - \lim_k \|A_k(t^i;x) - x^i\|_{p,\Omega} &= 0, \quad i = 1, 2, ..., n, \\
st - \lim_k \|A_k(|t|^2;x) - |x|^2\|_{p,\Omega} &= 0, \quad i = 0, 1, 2,
\end{align*} \quad (2.10)$$

then for any function $f \in L_{p,\Omega}(\mathbb{R}^n)$, we have $st - \lim_k \|A_k f - f\|_{p,\Omega} = 0$.

**Proof.** Let $\chi_j^B$ be the characteristic function of the ball $|x| \leq B$ and $\chi_j^B(t) = 1 - \chi_j^B(t)$. Then, it is possible to choose a sufficient large $B$ such that

$$\|f \chi_j^B(t)\|_{p,\Omega} < \varepsilon. \quad (2.11)$$

By the condition (2.9) there exists a positive constant $K'$ such that $\sup_k \|A_k\|_{p,\Omega} \leq K'$ and so, for given $\varepsilon' > 0$ there exists a continuous function $\theta$ on $|x| \leq B$ satisfying the condition $\theta(x) = 0$, for $|x| > B$ and such that

$$\|f - \theta \chi_1^B\|_{p,\Omega} < \frac{\varepsilon'}{(K' + 1) \left(\max_{|t| \leq B} \Omega(t)\right)^{1/p}}. \quad (2.12)$$

Since the series (1.6) is convergent for each $k$, $f$ and using the linearity of the operators $L_j$, which means the linearity of $A_k$, we obtain

$$\|A_k f - f\|_{p,\Omega} \leq \|A_k(\chi_1^B \theta) - \chi_1^B \theta\|_{p,\Omega} + (K' + 1) \varepsilon + \varepsilon'. \quad (2.12)$$

Let $B_1 > B$, so we also have $\|A_k(\chi_1^B \theta) - \chi_1^B \theta\|_{p,\Omega} \leq \|A_k(\chi_1^B \theta) - \chi_1^B \theta||\chi_1^B\|_{p,\Omega} + M_\theta \|A_k(1) - 1\|_{p,\Omega} + M_\theta \|\chi_1^B\|_{p,\Omega}$, where $M_\theta := \max_{t \in \mathbb{R}^n} \|\theta(t)\|_{p,\Omega}$. Furthermore, we can choose $B_1$ such that $\|\chi_1^B\|_{p,\Omega} < \varepsilon'/M_\theta$, and for sufficiently large $k$, we estimate $\|A_k(1) - 1\|_{p,\Omega} < \varepsilon'/M_\theta$. Substituting these estimates in (2.12), we obtain

$$\|A_k f - f\|_{p,\Omega} \leq \|A_k(\chi_1^B \theta) - \chi_1^B \theta\|_{p,\Omega} + (K' + 1) \varepsilon + 3 \varepsilon'. \quad (2.12)$$

Since $|\chi_1^B(t)\theta(t) - \chi_1^B(x)\theta(x)| < \varepsilon' + 2M_\theta \sqrt{|t - x|^2}$ we can write

$$\|A_k f - f\|_{p,\Omega} \leq (K' + 1) \varepsilon + 4 \varepsilon' K' \|\Omega\|_1^{1/p} + C \left\{ \|A_k(|t|^2;x) - |x|^2\|_{p,\Omega} + \|A_k(1) - 1\|_{p,\Omega} \right\}.$$
where $C = 2M_\theta \frac{(1+B)^2}{3}$. Now for a given $r' > 0$ such that $(2\varepsilon' + (K' + 1) \varepsilon) < r'$. Then, let us define the following sets

$$D := \left\{ k \leq N : \|A_k f - f\|_{p,\Omega} \geq r' \right\},$$

$$D_1 := \left\{ k \leq N : \|A_k (1; x) - 1\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K' + 1) \varepsilon)}{3C} \right\},$$

$$D_2 := \left\{ k \leq N : \sum_{i=1}^{n} \|A_k (t^i; x) - x^i\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K' + 1) \varepsilon)}{3C} \right\},$$

$$D_3 := \left\{ k \leq N : \|A_k (|t|^2; x) - |x|^2\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K' + 1) \varepsilon)}{3C} \right\}.$$ 

It is easy to see that $D \subset D_1 \cup D_2 \cup D_3$ and we have

$$\left| \left\{ k \leq N : \|A_k f - f\|_{p,\Omega} \geq r' \right\} \right| \leq \left| \left\{ k \leq N : \|A_k (1; x) - 1\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K' + 1) \varepsilon)}{3C} \right\} \right| + \left| \left\{ k \leq N : \sum_{i=1}^{n} \|A_k (t^i; x) - x^i\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K' + 1) \varepsilon)}{3C} \right\} \right| + \left| \left\{ k \leq N : \|A_k (|t|^2; x) - |x|^2\|_{p,\Omega} \geq \frac{r' - (2\varepsilon' + (K' + 1) \varepsilon)}{3C} \right\} \right|,$$

where $|A|$ denotes the cardinality of the set $A$. Then taking the limit as $N \to \infty$, using the hypothesis of theorem, we obtain $\lim_{N \to \infty} \frac{1}{N} |\{ k \leq N : \|A_k f - f\|_{p,\Omega} \geq r' \}| = 0$ which is desired. \qed

Now we give an example such that:

**Example 2.2** We choose $\Omega(x, y) = e^{-x-y}$. Note that this selection of $\Omega$ satisfies the condition (1.3). Also note that for $1 \leq p < \infty$, $L_{p,\Omega}(\mathbb{R}^2) = \{ f : \mathbb{R}^2 \to \mathbb{R} : \Omega(x, y)f(x) \in L_p(\mathbb{R}^2) \}$. Also, let $A = (C, 1)$ and $\alpha = (\alpha_j)$ be as the Example 2.1. The Kantorovich variant of the double Szasz-Mirakyan operators by replacing $f(\frac{tb_j}{j}, \frac{sb_j}{j})$ with an integral mean of $f(x, y)$ over the interval $[(t+1)b_j/j, tb_j/j] \times [(s+1)b_j/j, sb_j/j]$ is as follows: for $j \in \mathbb{N}$, $x, y \in [0, b_j)$,

$$S_j(f; x, y) := \frac{j^2}{b_j^2} \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} P_{j,t,s}(x, y) \int_{tb_j/j}^{(t+1)b_j/j} \int_{sb_j/j}^{(s+1)b_j/j} f(u, v) dudv, \quad (2.13)$$

where $\{b_j\}$ is a sequence of positive real numbers satisfying the conditions $\lim_{j \to \infty} \frac{b_j}{j} = 0$ and $\lim_{j \to \infty} b_j = \infty$ and

$$P_{j,t,s}(x, y) := e^{-\frac{j(x+y)}{k_j}} \frac{(jx)^t(jy)^s}{t!s!b_j^{t+s}}, \quad t, s = 0, 1, 2, ....$$
It is known that
\[ S_j (1; x, y) = 1, \]
\[ S_j (u; x, y) = x + \frac{b_j}{2^j}, \]
\[ S_j (v; x, y) = y + \frac{b_j}{2^j}, \]
\[ S_j (u^2 + v^2; x, y) = x^2 + y^2 + \frac{2b_j}{j} (x + y) + \frac{2b_j^2}{3j^2}. \]

Then using the operators \( S_j \) and the sequence \( \alpha = (\alpha_j) \), we define the sequence of positive linear operators \( L_j(f; x, y) = (1 + \alpha_j) S_j(f; x, y) \) for \( f \in L_{p,\Omega}(\mathbb{R}^2) \) and \( j \in \mathbb{N} \).

By some simple calculations, we obtain
\[
\| C_k (1; x, y) - 1 \|_{p,\Omega} = \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \| 1 \|_{p,\Omega},
\]
\[
\| C_k (u; x, y) - x \|_{p,\Omega} \leq \frac{1}{k} \sum_{j=1}^{k} \frac{b_j}{j} \| 1 \|_{p,\Omega} + \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \| x \|_{p,\Omega},
\]
\[
\| C_k (v; x, y) - y \|_{p,\Omega} \leq \frac{1}{k} \sum_{j=1}^{k} \frac{b_j}{j} \| 1 \|_{p,\Omega} + \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \| y \|_{p,\Omega},
\]
\[
\| C_k (u^2 + v^2; x, y) - (x^2 + y^2) \|_{p,\Omega} \leq \frac{4}{k} \sum_{j=1}^{k} \frac{b_j}{j} \| x + y \|_{p,\Omega} + \left| \frac{1}{k} \sum_{j=1}^{k} \alpha_j \right| \| x^2 + y^2 \|_{p,\Omega},
\]

where \( C_k (f; x, y) = \sum_{j=1}^{\infty} c_{kj} L_j (f(u, v); x, y) \). Also,
\[
\sup_k \| C_k \|_{L_{p,\Omega} \rightarrow L_{p,\Omega}} = \sup_k \sup_{\| f \|_{p,\Omega} = 1} \| C_k (f; x, y) \|_{p,\Omega} < \infty.
\]

Hence, (2.9), (2.10) conditions are provided which means that for any function \( f \in L_{p,\Omega}(\mathbb{R}^2) \), we have \( st \lim_k \| C_k f - f \|_{p,\Omega} = 0 \).

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References