Generalized centroid of semirings

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Abstract We define and study the generalized centroid of a semiprime semiring. Let $R$ be a right multiplicatively cancellable semiprime semiring and $C$ the generalized centroid of $R$, then $C$ is semifield.

Keywords Semiring · semiprime ideal · semiprime semiring · quotient semiring

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1 Introduction

Semirings abound in the mathematical world around us. Indeed, the first mathematical structure we encounter - the set of natural numbers - is a semiring. Historically, semirings first appear implicitly in [3] and later [8], [6], [10] and [7] in connection with the study of ideals of a ring. They also appear in [4] and [5] in connection with the axiomatization of the natural numbers and nonnegative rational numbers. Over the years, semirings have been studied by various researchers either in their own right, in an attempt to broaden techniques coming from semigroup theory or ring theory, or in connection with applications ([2], [14]). In [9] Martindale first constructed for any prime ring $R$ a “ring of quotients” $Q$. After, Öztürk and Jun introduced the extended centroid of a prime $Γ$-ring and obtained some results in $Γ$-ring $M$ with derivation which was related to $Q$, and the quotient $Γ$-ring of $M$ ([11], [12], [13]). In this paper, we define and study the generalized centroid of a semiprime semiring.

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2 Preliminaries

A semiring is a nonempty set $R$ on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

(1) $(R, +)$ is a commutative monoid with identity element $0_R$;
(2) $(R, \cdot)$ is a monoid with identity element $1_R$;
(3) Multiplication distributes over addition from either side;
(4) $0_R r = 0_R = r 0_R$ for all $r \in R$;
(5) $1_R \neq 0_R$.

An element $a$ of semiring $R$ is right multiplicatively cancellable if and only if $ba = ca$ only when $b = c$. Left multiplicatively cancellable elements are similarly defined. An element of $R$ is multiplicatively cancellable if and only if it is both left and right multiplicatively cancellable.

An element of a semiring $R$ is a unit if and only if there exists an element $r_1$ of $R$ satisfying $rr_1 = 1_R = r_1 r$. The set of all units of $R$ is denoted by $U(R)$. $U(R)$ is a submonoid of $(R, \cdot)$ which is in fact a group. If $U(R) = R \setminus \{0_R\}$ then $R$ is a division semiring. A commutative division semiring is semifield.

An element $a$ of semiring $R$ is multiplicatively regular if and only if there exists an element $a'$ of $R$ satisfying $aa' a = a$.

A left ideal $U$ of a semiring $R$ is a nonempty subset of $R$ satisfying the following conditions:

(1) If $a, b \in U$ then $a + b \in U$;
(2) If $a \in U$ and $r \in R$ then $ra \in U$;
(3) $U \neq \emptyset$.

Note that if $R$ is semiring then condition (3) is equivalent to the condition that $1_R \notin U$. A right ideal of $R$ is defined in the analogous manner and an ideal of $R$ is a subset which is both a left ideal and right ideal of $R$. Note that ideals are proper, namely $R$ is not an ideal of itself.

**Definition 2.1** If $A$ is a nonempty subset of a semiring $R$, then

$$\text{Ann}_l(A) = \{ r \in R | ra = 0_R \text{ for all } a \in A \}.$$ 

If $A \neq \{0_R\}$ then this is a left ideal of $R$, called the left annihilator ideal of $A$.

Right annihilator ideals are defined similarly. We note that if $H$ is a left ideal of $R$ then $\text{Ann}_l(H)$ is an ideal of $R$.

As in the case of rings, an ideal $U$ of a semiring $R$ is semiprime if and only if for any ideal $H$ of $R$ we have $H^2 \subseteq U$, for ideal $H$ of $R$ only when $H \subseteq U$. Prime ideals are surely semiprime. $R$ is semiprime semiring if ideal $\{0_R\}$ of semiring $R$ is semiprime.

**Proposition 2.2** [1, Proposition 7.4] The following conditions on an ideal $U$ of a semiring $R$ are equivalent:

(1) $U$ is semiprime;
(2) $\{ara | r \in R\} \subseteq U$ if and only if $a \in U$.

If $R$ and $S$ are semirings then a function $\gamma : R \to S$ is a morphism of semirings if and only if

(1) $\gamma(0_R) = 0_S$
(2) $\gamma(1_R) = 1_S$
A morphism of semirings which is both injective and surjective is called an isomorphism. If there exists an isomorphism between semirings \( R \) and \( S \) we write \( R \cong S \). If \( \gamma : R \to S \) is morphism of semirings then \( \text{Im} (\gamma) = \{ \gamma (r) | r \in R \} \) is a subsemiring of \( S \).

Let \( R \) be a semiring. A left \( R \)-semimodule is a commutative monoid \((M, +)\) with additive identity \( 0_M \) for which we have a function \( R \times M \to M \), defined by \((r, m) \mapsto rm\) and called scalar multiplication, which satisfy the following conditions for all elements \( r, r_1 \) and \( r_2 \) of \( R \) and all elements \( m, m_1 \) and \( m_2 \) of \( M \):

1. \((r_1 r_2) m = r_1 (r_2 m)\)
2. \(r (m_1 + m_2) = rm_1 + rm_2\)
3. \((r_1 + r_2) m = r_1 m + r_2 m\)
4. \(1_M m = m\)
5. \(r 0_M = 0_M = 0_R m\)

If \( R \) is a semiring and \( M \) and \( N \) are left \( R \)-semimodules then a function \( \alpha \) from \( M \) to \( N \) is an \( R \)-homomorphism if the following conditions are satisfied;

1. \((m_1 + m_2) \alpha = m_1 \alpha + m_2 \alpha\) for all \( m_1, m_2 \in M\);
2. \((rm) \alpha = r (m \alpha)\) for all \( m \in M \) and \( r \in R\).

**Definition 2.3** ([15]) Let \( R \) be right multiplicatively cancellable prime semiring and \( Q_r \) the right quotient semiring of \( R \). The set

\[ C := \{ g \in Q_r | gf = fg \text{ for all } f \in Q_r \} \]

is called the extended centroid of semiring \( R \).

**Theorem 2.4** ([15]) The center \( C \) of \( Q_r \) is semifield.

### 3 Results

Let \( R \) is a semiprime semiring. Let us denote set of all nonzero ideals of \( R \) which have zero annihilator in \( R \) and \( R \) by \( F = F(R) \). That is,

\[ F = F(R) = \{ U | \{0_R\} \neq U \text{ is an ideal of } R \text{ such that } \text{Ann}U = \{0_R\} \} \cup \{R\}. \]

In this case the set \( F \) is closed under multiplication. Indeed, let \( U \) and \( V \) be in \( F \). The equality \( UVx = \{0_R\} \), for all \( x \in R \) yields \( Vx \subseteq \text{Ann}U = \{0_R\} \), i.e., \( Vx = \{0_R\} \) and so \( x \in \text{Ann}V = \{0_R\} \) which implies \( x = 0_R \). Then we get that \( UV \in F \).

**Remark 3.1** If \( U, V \in F \), then \( U \cap V \in F \).

Let \( R \) be a semiprime semiring and

\[ \Gamma = \{ (U, f) | f : U \to R \text{ is a right } R \text{ homomorphism for all } U \in F \}. \]

Define a relation “~” on \( \Gamma \) by

\[ (U, f) \sim (V, g) : \iff \text{there exists } K \in F \text{ and } K \subseteq U \cap V \text{ such that } f = g \text{ on } K. \]

Since the set \( F \) is closed under multiplication, it is possible to find such an ideal \( K \in F \) and so “~” is an equivalence relation. This gives a chance for us to get
a partition of \( \Gamma \). We denote the equivalence class by \( \hat{f} = \text{Cl}(U, f) \), where \( \hat{f} := \{ g : V \to R \mid (U, f) \sim (V, g) \} \) and denote by \( Q_r \) set of all equivalence classes. We define an addition “+” on \( Q_r \) as follows:

\[
\hat{f} + \hat{g} := \text{Cl}(U, f) + \text{Cl}(V, g) = \text{Cl}(U \cap V, f + g)
\]

where \( f + g : U \cap V \to R \) is a right \( R \) homomorphism. Assume that \( (U_1, f_1) \sim (U_2, f_2) \) and \( (V_1, g_1) \sim (V_2, g_2) \). Then \( \exists K_1(\in F) \subseteq U_1 \cap U_2 \) such that \( f_1 = f_2 \) on \( K_1 \) and \( \exists K_2(\in F) \subseteq V_1 \cap V_2 \) such that \( g_1 = g_2 \) on \( K_2 \). Taking \( K = K_1 \cap K_2 \) and so \( K \in F \).

For any \( x \in K \), we have \( (f_1 + g_1)(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) = (f_2 + g_2)(x) \), and so \( f_1 + g_1 = f_2 + g_2 \) on \( K \). Therefore \( (U_1 \cap V_1, f_1 + g_1) \sim (U_2 \cap V_2, f_2 + g_2) \), which means that the addition “+” in \( Q_r \) is well-defined.

Now we will prove that \( Q_r \) is a commutative monoid. Let \( \hat{f} = \text{Cl}(U, f), \hat{g} = \text{Cl}(V, g), \hat{h} = \text{Cl}(H, h) \) be elements of \( Q_r \). Then one can easily check that \( \hat{f} + (\hat{g} + \hat{h}) = (\hat{f} + \hat{g}) + \hat{h} \) and \( \hat{f} + \hat{g} = \hat{g} + \hat{f} \). Taking \( \hat{\theta} = \text{Cl}(R, \theta) \in Q_r \) where \( \theta : R \to R, x \mapsto 0_R \) for all \( x \in R \), we have \( \hat{f} + \hat{\theta} = \text{Cl}(U, f) + \text{Cl}(R, \theta) = \text{Cl}(U \cap R, f + \theta) = \text{Cl}(U, f) = \hat{f} \) and similarly \( \hat{\theta} + \hat{f} = \hat{f} \) for all \( \hat{f} \in Q_r \). Hence \( \hat{\theta} \) is the additive identity in \( Q_r \). Therefore \( (Q_r, \hat{\cdot}) \) commutative monoid.

Now we define a multiplication “\( \cdot \)” on \( Q_r \) as follows:

\[
\hat{f}\hat{g} := \text{Cl}(U, f).\text{Cl}(V, g) = \text{Cl}(VU, fg)
\]

where \( fg : VU \to R \) is a right \( R \) homomorphism, for all \( \hat{f}, \hat{g} \in Q_r \). Assume that \( (U_1, f_1) \sim (U_2, f_2) \) and \( (V_1, g_1) \sim (V_2, g_2) \). Then \( \exists K_1(\in F) \subseteq U_1 \cap U_2 \) such that \( f_1 = f_2 \) on \( K_1 \) and \( \exists K_2(\in F) \subseteq V_1 \cap V_2 \) such that \( g_1 = g_2 \) on \( K_2 \). Also \( V_1U_1 \cap V_2U_2 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) = (V_1 \cap V_2) \cap (U_2 \cap V_2) \) and there exists \( K \subseteq F \) such that \( K \subseteq V_1U_1 \cap V_2U_2 \). For any \( x \in K \), \( x \in V_1U_1 \cap V_2U_2 \). So that \( x \in V_1U_1 \cap V_2U_2 \).

Then, \( x = \sum_{\text{finite}} a_ib_i, a_i \in V_1 \cap V_2 \) and \( b_i \in U_1 \cap U_2 \). Therefore

\[
(f_1g_1)(x) = f_1(g_1(x)) = f_1\left(g_1\left(\sum_{\text{finite}} a_ib_i\right)\right)
\]

\[
= f_1\left(\sum_{\text{finite}} g_1(a_i)b_i\right) = f_1\left(\sum_{\text{finite}} g_2(a_i)b_i\right)
\]

\[
= f_2\left(\sum_{\text{finite}} g_2(a_i)b_i\right) = f_2\left(g_2\left(\sum_{\text{finite}} a_ib_i\right)\right)
\]

\[
= f_2(g_2(x)) = (f_2g_2)(x)
\]

and so \( f_1g_1 = f_2g_2 \) on \( K \). Hence “\( \cdot \)” is well-defined. Now we will prove that \( (Q_r, \hat{\cdot}) \) is a monoid. Let \( \hat{f}, \hat{g}, \hat{h} \in Q_r \), where \( U, V, H \in F \) are domains of \( f, g \) and \( h \) respectively. Since \( H (VU) = (HV) U \), we get for all \( x \in H (VU) \),
\[ ((fg)h)(x) = (fg)(h(x)) = f(g(h(x))) \]
\[ = f((gh)(x)) = (f(gh))(x). \]

Hence \( (fg)h = f(gh) \) on \( H(VU) \). That is, \( \hat{fg} \hat{h} = \hat{f} \hat{gh} \).

Taking \( \hat{1} \in Q_r \) where \( 1 : R \rightarrow R, x \mapsto x \) for all \( x \in R \). Let \( \hat{f} = Cl(U, f) \in Q_r \).
Since \( RU \subseteq U \), we get for all \( x \in RU, (f1)(x) = f(1(x)) = f(x) \) and \( (1f)(x) = 1(f(x)) = f(x) \). Thus, \( \hat{1f} = \hat{f} \hat{1} = \hat{f} \). Hence \( \hat{1} \) is the multiplicative identity in \( Q_r \).
Therefore \( (Q_r, \cdot) \) is monoid.

Let \( \hat{f} = Cl(U, f), \hat{g} = Cl(V, g), \hat{h} = Cl(H, h) \) be elements of \( Q_r \). \( (V \cap H)U \subseteq VU \cap HU \), we get for all \( x \in (V \cap H)U \),
\[ [f(g+h)](x) = f((g+h)(x)) = f(g(x) + h(x)) \]
\[ = f(g(x)) + f(h(x)) = (fg + fh)(x). \]

Hence \( f(g+h) = fg + fh \) on \( (V \cap H)U \). That is, \( \hat{f} \hat{g} \hat{h} = \hat{fg} \hat{h} \).

Suppose that \( \hat{\theta} = \hat{1} \). Then, \( Cl(R, \theta) = Cl(R, 1) \). That is, there exists \( K \subseteq F \) and \( r \in R \cap R \) such that \( \theta = 1 \) on \( K \). Therefore, \( \theta(x) = 1(x) \) for all \( x \in K \). This is a contradiction with \( \{0_R\} \neq K \). Thus, \( \hat{\theta} \neq \hat{1} \).

Now we prove that \( \hat{f} \hat{\theta} = \hat{\theta} \hat{f} = \hat{\theta} \hat{f} \) for all \( \hat{f} = Cl(U, f) \in Q_r \). Since \( RU \subseteq RU \cap R \) and \( f\theta = \theta \) on \( RU \), we get \( \hat{f} \hat{\theta} = \hat{\theta} \hat{f} \). Similarly \( \hat{\theta} \hat{f} = \hat{\theta} \).

Thus, \( (Q_r, +, \cdot) \) be a semiring with multiplicative identity \( \hat{1} \).

Let \( R \) be semiprime right multiplicatively cancellable semiring. For a fixed element \( a \in R \), consider a mapping \( \lambda_a : R \rightarrow R \) by \( \lambda_a(r) = ar \) for all \( r \in R \). It is easy to prove that the mapping \( \lambda_a \) is a right \( R \) homomorphism. Define a mapping \( \Psi : R \rightarrow Q_r \) by \( \Psi(a) = \lambda_a = Cl(R, \lambda_a) \) for \( a \in R \). Clearly the mapping \( \Psi \) is injective morphism and so \( R \) is a subring of \( Q_r \), and in this case, we call \( Q_r \) the right quotient semiring of \( R \) and will be denoted by \( Q_r \). One can of course, characterize \( Q_l \), the left quotient semiring of \( R \) in similar manner. For purpose of convenience, we use \( q \) instead of \( \hat{q} \in Q_r \).

**Definition 3.1** The set
\[ C := \{ g \in Q_r \mid gf = fg \text{ for all } f \in Q_r \} \]
is called the generalized centroid of a semiring \( R \).

**Remark 3.2** Assume that \( q = Cl(U, f) \in C \). For all \( r \in R \), \( Cl(R, \lambda_r)Cl(U, f) = Cl(U, f)Cl(R, \lambda_r) \) and so there exists \( K(\in F) \subseteq RU \) such that \( \lambda_r f = f \lambda_r \) on \( K \).
From here, \( (\lambda_r f)(x) = (f \lambda_r)(x) \) for all \( x \in K \), i.e., \( rf(x) = f(rx) \). Hence \( f \) acts as a \( R \) homomorphism on \( K \).

The following theorem characterizes the quotient semiring \( Q_r \) of \( R \). The proof is same the proof of the corresponding theorem in ring theory, and we omit it.
Theorem 3.2 Let $R$ be a right multiplicatively cancellable semiprime semiring and $Q_r$ the quotient semiring of $R$. Then the semiring $Q_r$ satisfies the following properties:

(i) $Q_r$ is semiprime semiring.

(ii) For any element $q$ of $Q_r$, there exists an ideal of $U_q \in F$ which has zero annihilator with a right $R$ homomorphism $q : U \rightarrow R$, such that $q(U_q) \subseteq R$ (or $q(U_q) \subseteq R$).

(iii) If $q \in Q_r$ and $q(U_q) = \{0_R\}$ for a certain $U_q \in F$ ($q(U_q) = \{0_R\}$ for a certain $U_q \in F$), then $q \in Q_r$.

(iv) If $U \in F$ and $\Psi : U \rightarrow R$ is a right $R$ homomorphism, then there exists an element $q \in Q_r$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = qu$ for all $u \in U$).

(v) Let $W$ be a subsemimodule (an $(R, R)$ subsemimodule) in $Q_r$ and $\Psi : W \rightarrow Q_r$, a right $R$ homomorphism. If $W$ contains the ideal $U$ of $R$ such that $\Psi(U) \subseteq R$ and $AnnU = Ann_rW$, then there is an element $q \in Q_r$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = qb$ for any $b \in W$) and $q(a) = 0$ for any $a \in Ann_rW$ (or $qa = 0_R$ for any $a \in Ann_rW$).

Let $W$ be a nonzero subsemimodule of the right $Q_r R$ semimodule. Then we get that $(0_W \neq )w \in W$ and $W \subseteq F$ such that $wU_w \subseteq R$ (or $w(U_w) \subseteq R$) and so $\{0_R\} \neq wU_w\subseteq W^2$. Hence, let a, b be an arbitrary element of $Q_r$. Then $a^2 = a^2b^2$. Since $a^2 \in C$, $a^2 = y^2$. Let $a^2 = Cl(U_{a^2}, f)$ and so $Cl(R, \lambda x)y^2. Cl(U_{a^2}, f) = Cl(R, \lambda y)x.$ Therefore there exists $K \subseteq F$ such that $K \subseteq U_{a^2} \cap R \cap U_{a^2}$ and $\lambda x = \lambda yx$ on $K$. For all $z \in K$, $(\lambda xz)(z) = (\lambda yx)(z)$ and so $xzf(z) = yf(z)$. Since $R$ be right multiplicatively cancellable semiring, we get $x = y$. Thus $ax = ay$. That is, $\Psi$ is well-defined. It is easy to see that $\Psi$ is right $R$ homomorphism. There exists $a_1 \in Q_r$ such that $a_1 a^2 = ax$ for all $x \in J$. We have that $a_1 a^2 = a$. Let us prove that the element $a_1$ in $C$. Let $q$ be an arbitrary element of $Q_r$. Then $(a_1 a^2)^2 = q(a_1 a^2)^2$ and so $a^2 = q(a_1 a^2)^2$. Multiplying this equality from left by $a_1 a^2$, we get $a_1 a^2 = q(a_1 a^2)$. Assume that $a = Cl(U_{a^2}, f), a_1 a^2 = Cl(V, g)$ and $q(a) = Cl(H, h)$. Therefore $Cl(V, g) Cl(U_{a^2}, f) = Cl(H, h) Cl(U_{a^2}, f)$ and so there exists $L \subseteq U_{a^2} \cap U_{a^2}$ such that $g = h$ on $L$. Since $a \in C$, there exists $W \subseteq F$ such that $d$ is a $R$ homomorphism on $W$. On the other hand $d = (W \setminus L)$ is an ideal which has zero annihilator in $R$, of $R$. Also $g = h$ on $W \setminus L$. That is, $a_1 a^2 = q(a_1 a^2)$. This completes the proof. 

Lemma 3.4 $C$ is multiplicatively cancellable semiring.

Proof. Let $sp = sq$ for $p, q, s \in C$. Then $Cl(H, h)Cl(U, f) = Cl(H, h)Cl(V, g)$ where $p = Cl(U, f), q = Cl(V, g), s = Cl(H, h)$. Hence there exists $(\{0_R\} \neq )K \subseteq F$ such that $K \subseteq UH \cap VH$ and $hf = Qg$ on $K$. Since $f, g, h \in C$, there exists $W \subseteq F$ such that $f(x)h(y) = g(x)h(y)$ for all $x, y \in K \cap W$. And so $f = g$ on $K \cap W$. That is, $p = q$. Thus $C$ is multiplicatively cancellable semiring.
We have shown that all elements of \( C \) are multiplicatively regular. For any element \( a \in C \), there exists an element \( a_1 \), in \( C \) such that \( a_1a^2 = a \). Since \( C \) is multiplicatively cancellable semiring, \( a_1a = 1 \). Thus all nonzero elements of \( C \) have multiplicative inverse. Thus we have the following result:

**Corollary 3.5** \( C \) is a semifield.

We now let \( S = RC \), a subsemiring of \( Q_r \) containing \( R \). We shall call \( S \) the central closure of \( R \). The same proof used in showing that \( Q_r \) was semiprime may be employed to show that \( S \) is semiprime.

**Proposition 3.6** Let \( R \) be right multiplicatively cancellable semiprime semiring and \( S \) be the central closure of \( R \). Then \( S \) is a right multiplicatively cancellable semiprime semiring.

**Proof.** The proof is similar with the proof of [15, Proposition 2]. \( \square \)

**Theorem 3.7** Let \( f : R \to S \) be an additive map satisfying \( f(xy) = f(x)y \) for all \( x, y \in R \). Then there exists \( q \in Q_r(S) \) such that \( f(x) = qx \) for all \( x \in R \).

**Proof.** Let us extend \( f \) from \( R \) to \( S \) according to \( \bar{f} (\sum x_i \lambda_i) = \sum f(x_i) \lambda_i \), where \( x_i \in R \) and \( \lambda_i \in C \). Let \( \sum x_i \lambda_i = \sum y_i \beta_i \), \( x_i, y_i \in R \), \( \lambda_i, \beta_i \in C \). There exists a nonzero ideal \( \mathcal{I} \) in \( R \) such that \( \lambda_i \mathcal{I} \subseteq R \) for every \( i \). For \( a \in \mathcal{I} \), the sum \( \sum x_i (\lambda_i a) \) in \( R \). Then,

\[
\sum x_i (\lambda_i a) = \sum y_i (\beta_i a)
\]

\[
\sum f(x_i) \lambda_i a = \sum f(y_i) \beta_i a
\]

\[
(\sum f(x_i) \lambda_i) a = (\sum f(y_i) \beta_i) a.
\]

Since \( S \) is a right multiplicatively cancellable semiring, we get \( \sum f(x_i) \lambda_i = \sum f(y_i) \beta_i \). And so, \( \bar{f} (\sum x_i \lambda_i) = \bar{f} (\sum y_i \beta_i) \). That is, \( \bar{f} \) is well-defined. The fact that \( \bar{f}(xy) = \bar{f}(x)y \) for all \( x, y \in S \) can be seen by a direct computation. Thus \( \bar{f} : S \to S \) is a right \( S \) homomorphism, hence there exists \( q \in Q_r(S) \) such that \( \bar{f}(x) = qx \), \( x \in S \). Since \( \bar{f} \) is an extension of \( f \), this proves the lemma. \( \square \)

**Theorem 3.8** Let \( R \) be right multiplicatively cancellable semiprime semiring and \( S \) be the central closure of \( R \). If \( a \) and \( b \) are nonzero elements in \( S \) such that \( axb = bxa \) for all \( x \in R \) and \( S \) is left cancellable then there exists \( q \in C \) such that \( qa = b \).

**Proof.** The proof is similar with the proof of [15, Theorem-3]. \( \square \)

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